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Model reduction for nonlinear control systems

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Contents

5. Nonlinear Hilbert adjoints

The notion of an *adjoint map* or *adjoint operator* can be found in a wide variety of mathematical contexts: functional analysis, differential geometry, differential algebra, representation theory for Lie algebras and topological vector spaces. These concepts appear primarily in a linear setting, i.e., linear maps on linear spaces, and thus are closely related to one another. In linear systems theory, the important notion of an adjoint *state-space system* is usually defined in terms of signal sets that form Hilbert spaces, either L_2 or H_2. From an input-output point of view, the corresponding transfer function follows directly from the familiar Hilbert adjoint in functional analysis.

Once one departs from the context of linear operators, there are some extensions of the adjoint operator definition. It can not be assumed a priori that the existing notions are in any way directly related. For example, in [6] the notion of an adjoint map is defined in terms of a dual map on a topological vector space. In a nonlinear state-space context, the adjoint system has appeared as a useful concept for inner-outer factorizations, but it can also be given an input-output interpretation using the nonlinear Hilbert adjoint operator. In this chapter, the basic objective is to fully develop the idea of a nonlinear Hilbert adjoint in order to use it for Hankel singular value analysis in the next chapters.

5.1 Inner products and Hilbert spaces

5.1.1 The definition of the nonlinear Hilbert Adjoint

In the most general setting, let F be a topological vector space over $\mathbb{R}$ with dual space F'. Let E be a nonempty set, and $\mathcal{A}$ a collection of nonempty subsets of E. Let E^β be a linear space of real-valued functions x^β on E with the property that the restriction x^β_A to every $A \in \mathcal{A}$ is bounded. A mapping $\mathcal{T} : E \to F$ is called $\mathcal{A}$-*bounded* if $\mathcal{T}$ maps the sets of $\mathcal{A}$ into bounded subsets of F. For any $\mathcal{A}$ -bounded mapping $\mathcal{T} : E \to F$, the *dual map* of $\mathcal{T}$ is defined as

$$\begin{aligned}\mathcal{T}' &: F' \to E^{\beta}\\ &: y' \to (\mathcal{T}'(y'))(u) = (y' \circ \mathcal{T})(u), \quad \forall u \in E.\end{aligned}$$

Now if F is a Hilbert space with an inner product $\langle \cdot, \cdot \rangle_F$ then it follows from the Riesz Lemma that for any $y' \in F'$ there exists a unique $y \in F$ such that $y'(\cdot) = \langle y, \cdot \rangle_F$. Hence one can write the identity

$$(\mathcal{T}'(y'))(u) = \langle y, \mathcal{T}(u) \rangle_F, \quad \forall u \in E.$$

If, in addition, E is an inner product space with inner product $\langle \cdot, \cdot \rangle_E$ and $y \in F$ is fixed, then the problem is to determine a corresponding $\tilde{u}_y \in E$ such that

$$\langle \mathcal{T}(u), y \rangle_F = \langle u, \tilde{u}_y \rangle_E, \quad \forall u \in E. \tag{5.1}$$

If $\mathcal{T}$ were a linear operator then such an $\tilde{u}_y$ is known to always exist and be unique, i.e., $\tilde{u}_y = \mathcal{T}^*(y)$, where $\mathcal{T}^*$ is the Hilbert adjoint of $\mathcal{T}$. But in this more general context, the existence and uniqueness of $\tilde{u}_y$ are not automatic. In fact, the identity (5.1) is meaningful in most cases only when $\tilde{u}_y$ is also a function of u. (Defining the domain of $\mathcal{T}^*$ to have the form $F \times E$ also agrees with the state-space notion of adjoint *systems* based on the Hamiltonian extension given in the next sections.) So in this context, consider the following definition.

Definition 5.1.1. *Given two Hilbert spaces E and F, an operator $\mathcal{T} : E \mapsto F$ has a global* nonlinear *Hilbert* adjoint when there exists an operator $\mathcal{T}^* : F \times E \to E$ such that

$$\langle \mathcal{T}(u), y \rangle_F = \langle u, \mathcal{T}^*(y, u) \rangle_E, \quad \forall u \in E, \quad \forall y \in F, \tag{5.2}$$

where $\mathcal{T}^*(y, u)$ is linear in y. △

Linearity in y seems to be rather natural in light of the bi-linearity of inner products, i.e.,

$$\begin{aligned}\langle u, \mathcal{T}^*(\alpha_1 y_1 + \alpha_2 y_2, u) \rangle_E &= \langle \mathcal{T}(u), \alpha_1 y_1 + \alpha_2 y_2 \rangle_F\\ &= \alpha_1 \langle \mathcal{T}(u), y_1 \rangle_F + \alpha_2 \langle \mathcal{T}(u), y_2 \rangle_F\\ &= \langle u, \alpha_1 \mathcal{T}^*(y_1, u) + \alpha_2 \mathcal{T}^*(y_2, u) \rangle_E.\end{aligned}$$

Linearity in y, however, does not follow directly from this argument. This is because there often exists a collection of nontrivial mappings (linear and nonlinear in y) of the form $\mathcal{B} : F \times E \mapsto E$ such that $\langle u, \mathcal{B}(y, u) \rangle_E = 0$, $\forall u \in E$, $\forall y \in F$. In which case, any adjoint mapping $\mathcal{T}^*$ is not uniquely defined since $\mathcal{T}^* + \mathcal{B}$ will also satisfy equation (5.2). In these circumstances, an adjoint operator should be viewed as a member of an equivalence class where two such operators $\mathcal{T}^*$ and $\mathcal{T}^{*'}$ are equivalent if

$$\langle u, \mathcal{T}^*(y, u) \rangle_E = \langle u, \mathcal{T}^{*'}(y, u) \rangle_E, \quad \forall u \in E, \quad \forall y \in F. \tag{5.3}$$

A shorthand notation for (5.3) is simply $\mathcal{T}^*(y,u) \cong \mathcal{T}^{*'}(y,u)$. Thus, any equality involving adjoint operators really means that both expressions belong to the same equivalence class. The following example demonstrates this phenomenon.

Example 5.1.1. Consider the operator

$$\begin{aligned}\mathcal{B} &: L_2[0,T] \times L_2^2[0,T] \mapsto L_2^2[0,T] \\ &: (y,u) \mapsto c(u)A(u)b(y),\end{aligned}$$

where

$$A(u) = \begin{pmatrix} -u_2 & u_1 u_2 \\ u_1 & -u_1^2 \end{pmatrix}$$

and c and b are any suitable mappings on $L_2^2[0,T]$ and $L_2[0,T]$, respectively. It can be verified directly that

$$\langle u, \mathcal{B}(y,u)\rangle_{L_2^2} = 0, \quad \forall u \in L_2^2[0,T], \;\; \forall y \in L_2[0,T],$$

and thus any $\mathcal{T}^*(y,u) + \mathcal{B}(y,u)$ fulfills (5.2) when $\mathcal{T}^*(y,u)$ fulfills (5.2), even if $b(y)$ is a nonlinear mapping. If linearity is required, then setting $b(y) = vy$ for any fixed $v \in \mathbb{R}^2$ implies that $\mathcal{T}^*(y,u) + \mathcal{B}(y,u)$ is an adjoint operator in the sense of Definition 5.1.1. △

It is not necessary in many applications to have a globally defined $\mathcal{T}^*$. The following theorem will lead to a sufficient condition for the existence of a locally defined adjoint operator.

Theorem 5.1.1. *Assume H is a Hilbert space and $U \subset H$ is any convex neighborhood of 0. Let $L : U \mapsto \mathbb{R}$ be a continuously Fréchet differentiable mapping on U with $L(0) = 0$. Then L has a factorization of the form*

$$L(u) = \langle a(u), u\rangle_H,$$

where $a : U \mapsto H$ is continuous on U, and for each $u \in U$ the dual mapping (from the Riesz representation) is

$$\begin{aligned}(a(u))^* &: H \mapsto \mathbb{R} \\ &: \xi \mapsto \int_0^1 (DL(tu))(\xi)dt.\end{aligned}$$

△

Proof: Given that L is continuously Fréchet differentiable on V, and V is convex, then for any fixed $x \in V$ and $h \in H$ the mapping

$$t \mapsto (DL(tx))(h)$$

is a well-defined, continuous real-valued function on $[0,1]$. (Here $DL(x) : H \mapsto \mathbb{R}$ denotes the Fréchet derivative of L at x. That is, $DL(x)$ is the unique bounded linear functional such that for any $h \in H$ with $x +$

$h \in V$, $L(x+h) = L(x) + (DL(x))(h) + o(x,h)$ where $o(x,0) = 0$ and $\lim_{h\to 0} |o(x,h)|/|h| = 0$, e.g., [8].) Observe that

$$\begin{aligned}\int_0^1 (DL(tx))(x)dt &= \int_0^1 (DL)(xdt) \\ &= \int_0^1 L(tx + xdt) - L(tx) + o(tx, xdt) \\ &= \int_0^1 L((t+dt)x) - L(tx) \\ &= \int_0^1 \frac{d}{dt} L(tx)dt \\ &= L(x).\end{aligned}$$

Furthermore, note that the mapping

$$\begin{aligned}(a(x))^* &: H \mapsto \mathbb{R} \\ &: h \mapsto \int_0^1 (DL(tx))(h)dt\end{aligned}$$

is clearly continuous and linear. Hence, from the Riesz Lemma, there exists for each $x \in V$ an element $a(x) \in H$ such that

$$(a(x))^*(h) = \langle a(x), h\rangle_H, \quad \forall h \in H.$$

So it follows directly that for every $x \in V$

$$\begin{aligned}L(x) &= \int_0^1 (DL(tx))(x)dt \\ &= (a(x))^*(x) \\ &= \langle a(x), x\rangle_H.\end{aligned}$$

The continuity of a follows from the continuity of $DL(\cdot)$. ■

This theorem can be viewed as a kind of infinite-dimensional version of the Fundamental Theorem of Integral Calculus. Its application in the nonlinear Hilbert adjoint existence theorem is as follows.

Theorem 5.1.2. *Suppose H_1 and H_2 are two Hilbert spaces and $U \subset H_1$ is any convex neighborhood of 0. Let $\mathcal{T} : U \mapsto H_2$ be a continuously Fréchet differentiable mapping on U such that $\mathcal{T}(0) = 0$. Then there exists a continuous mapping $\mathcal{T}^* : H_2 \times U \mapsto H_1$ with*

$$\langle \mathcal{T}(u), y\rangle_{H_2} = \langle u, \mathcal{T}^*(y,u)\rangle_{H_1}, \quad \forall u \in U, \quad \forall y \in H_2.$$

Specifically, $\mathcal{T}^*(y,u) = a_y(u)$ is such a mapping, where $a_y(\cdot)$ is defined for any fixed $y \in H_2$ by Theorem 5.1.1 with $L_y(u) = \langle \mathcal{T}(u), y\rangle_{H_2}$. △

Different characterizations of adjoint operators have been given in the literature. We refer to the Notes and References of this chapter.

5.1.2 Adjoint properties and examples

While a useful device in many circumstances, a nonlinear Hilbert adjoint operator does not share all of the familiar properties associated with linear adjoint operators. Consider any normed set of linear operators B defined on $L_2[0,\infty)$ as a Banach algebra with composition product $(\mathcal{S},\mathcal{T}) \mapsto \mathcal{S}\mathcal{T}$. B is said to constitute a C^*-algebra if it is equipped with an *adjoint map* (or *involution*) $\mathcal{T} \mapsto \mathcal{T}^*$ such that for all $\mathcal{S},\mathcal{T} \in B$ and any $\alpha \in \mathbb{R}$, the following properties are satisfied:

i. (linearity) $(\alpha\mathcal{S}+\mathcal{T})^* = \alpha\mathcal{S}^* + \mathcal{T}^*$;
ii. (product-reversal) $(\mathcal{S}\mathcal{T})^* = \mathcal{T}^*\mathcal{S}^*$;
iii. (double adjoint) $(\mathcal{T}^*)^* = \mathcal{T}$; and
iv. (C^*-identity) $\|\mathcal{T}\|^2 = \|\mathcal{T}^*\mathcal{T}\|$.

In this section the appropriate extensions of these fundamental properties are presented for the nonlinear Hilbert adjoint. Interspersed in the presentation is a collection of simple examples meant to illustrate the main ideas.

The linearity property (i) is an immediate result which follows from the bi-linearity of the inner product and the interpretation that equality here implies belonging to the same equivalence class. An interesting side issue is that when an adjoint operator exists, i.e., fulfills identity (5.2), it follows from Theorem 5.1.2 that there always exists an explicit form which is linear in the first argument.

Theorem 5.1.3. *Suppose H_1 and H_2 are two Hilbert spaces and $U \subset H_1$ is any convex neighborhood of 0. Let $\mathcal{T} : U \mapsto H_2$ be a continuously Fréchet differentiable mapping on U such that $\mathcal{T}(0) = 0$. Then the mapping*

$$\mathcal{T}^*(y,u) = \int_0^1 (D\mathcal{T}(tu))^*(y)\, dt$$

is a suitable Hilbert adjoint of $\mathcal{T}$ on $H_2 \times U$.

Proof. For any $y \in H_2$, define the scalar-valued mapping on U:

$$L_y(u) = \langle \mathcal{T}(u), y\rangle_{H_2} \equiv \langle u, \mathcal{T}^*(y,u)\rangle_{H_1}.$$

Next observe that for any fixed $u \in U$ and $t \in [0,1]$ it follows that

$$\begin{aligned} DL_y(tu)(\xi) &= \langle D\mathcal{T}(tu)(\xi), y\rangle_{H_2} \\ &= \langle \xi, (D\mathcal{T}(tu))^*(y)\rangle_{H_1}, \quad \forall \xi \in H_1. \end{aligned}$$

Thus,

$$\begin{aligned} L_y(u) &= \int_0^1 (DL_y(tu))(u)\, dt \\ &= \int_0^1 \langle u, (D\mathcal{T}(tu))^*(y)\rangle_{H_1}\, dt \\ &= \langle u, \int_0^1 (D\mathcal{T}(tu))^*(y)\, dt\rangle_{H_1}, \end{aligned}$$

and the conclusion follows directly. ■

Observe that in this form above, $\mathcal{T}^*(y,u)$ is linear in y since $(D\mathcal{T}(tu))^*(\cdot)$ is the adjoint of a linear operator, i.e., the familiar Hilbert adjoint. Thus, it is also immediate that $\mathcal{T}^*(0,u) = 0$, $\forall u \in U$. Unfortunately, Example 5.1.1 demonstrates that this linearity property is still not sufficient to provide any uniqueness features.

Example 5.1.2. For any finite $T > 0$ and positive integer m, the Banach space $L_4^m[0,T]$ can be viewed as a convex open subset of $L_2^m[0,T]$ containing the zero function. With $U = L_4^m[0,T]$, the mapping

$$\begin{aligned} \mathcal{T} &: U \mapsto L_2[0,T] \\ &: u \mapsto u^T u \end{aligned}$$

is then well defined, continuously Fréchet differentiable, and satisfies the identity $\mathcal{T}(0) = 0$. One form of the adjoint operator can be immediately extracted using the definition. Specifically,

$$\begin{aligned} \langle \mathcal{T}(u), y\rangle_{L_2} &= \int_0^T u^T(\tau)u(\tau)y(\tau)\, d\tau, \\ &= \langle u, uy\rangle_{L_2^m}, \end{aligned}$$

and thus, a suitable adjoint is given by $\mathcal{T}^*(y,u) = uy$. This same adjoint form can also be computed using Theorem 5.1.3:

$$\begin{aligned} D\mathcal{T}(u) &= 2u^T \\ (D\mathcal{T}(tu))^*(y) &= 2tuy \\ \mathcal{T}^*(y,u) &= \int_0^1 (D\mathcal{T}(tu))^*(y)\, dt = uy. \end{aligned}$$

△

Example 5.1.3. Consider a Hammerstein integral operator defined on a set $U \subset L_2^m[0,\infty)$:

$$\begin{aligned} \mathcal{S} &: U \mapsto L_2^p[0,\infty) \\ &: u \mapsto \int_0^\infty K(\tau,s) f(u(s))\, ds, \end{aligned}$$

where K is a suitable continuous kernel function, and each component function of f is C^1 with $f(0) = 0$. Then applying Theorem 5.1.3 it follows that

$$
\begin{aligned}
(D\mathcal{S}(u))(\xi) &= \int_0^\infty K(\tau,s)\frac{df}{du}(u(s))\xi(s)\,ds\\
(D\mathcal{S}(u))^*(y) &= \int_0^\infty \frac{df}{du}^T(u(s))K^T(\tau,s)y(\tau)\,d\tau\\
\mathcal{S}^*(y,u) &= \int_0^\infty \underbrace{\left[\int_0^1 \frac{df}{du}(tu(s))\,dt\right]^T}_{F(u(s))} K^T(\tau,s)y(\tau)\,d\tau\\
&= F^T(u)S_{\mathrm{L}}^*(y),
\end{aligned}
$$

where the matrix-valued function $F(\cdot)$ satisfies the identity $f(x) = F(x)x$ on a convex neighborhood of 0, and S_{L}^* denotes the usual adjoint operator for the linear integral operator with kernel K. Some specific examples include the familiar linear time-invariant case

$$
\mathcal{S}_{\mathrm{LTI}}(u) = \int_0^\infty e^{A(\tau-s)}u(s)\,ds,
$$

where A is Hurwitz and $f(x) = x$, and thus a suitable adjoint is given by

$$
\mathcal{S}_{\mathrm{LTI}}^*(y,u) = \int_0^\infty e^{A^T(\tau-s)}y(\tau)\,d\tau.
$$

Also consider the SISO FM (Frequency Modulated) system

$$
\mathcal{S}_{\mathrm{FM}}(u) = \frac{1}{\pi}\int_0^\infty e^{A(\tau-s)}sin(\pi\,u(s))\,ds,
$$

where a suitable adjoint is given by

$$
\begin{aligned}
\mathcal{S}_{\mathrm{FM}}^*(y,u) &= sinc(u)\int_0^\infty e^{A^T(\tau-s)}y(\tau)\,d\tau\\
&= sinc(u)S_{\mathrm{LTI}}^*(y).
\end{aligned}
$$

△

In order to address the product-reversal property (ii), one must first define the sense in which operators can be composed when adjoint operators are present. The situation is more complicated than the familiar case since the domain of an adjoint operator is not simply the codomain of the original operator. For example, consider the Hilbert spaces H_i, $i = 1,2,3$, the operators

$$
\begin{aligned}
\mathcal{T}&\colon H_1 \mapsto H_2 & \mathcal{S}&\colon H_2 \mapsto H_3\\
&: u \mapsto w & &: w \mapsto y
\end{aligned}
$$

and the corresponding adjoints

$$
\begin{aligned}
\mathcal{T}^*&\colon H_2 \times H_1 \mapsto H_1 & \mathcal{S}^*&\colon H_3 \times H_2 \mapsto H_2\\
&: (w,u) \mapsto \bar{u} & &: (y,w) \mapsto \bar{w}.
\end{aligned}
$$

Clearly the composition and its adjoint

$$\begin{aligned} \mathcal{ST} &: H_1 \mapsto H_3 & (\mathcal{ST})^* &: H_3 \times H_1 \mapsto H_1 \\ &: u \mapsto y & &: (y,u) \mapsto \bar{u}. \end{aligned}$$

are well defined, but no *direct* composition like $\mathcal{T}^*\mathcal{T}$ or $\mathcal{T}^*\mathcal{S}^*$ is possible as in the classic setting. Still some *formal* compositions can be defined which have great utility in a variety of situations.

Definition 5.1.2. *Let H_i, $i = 1, 2, 3$, be a collection of Hilbert spaces. Assume $\mathcal{T} : H_1 \mapsto H_2$ and $\mathcal{S} : H_2 \mapsto H_3$ are two operators with well-defined adjoint operators. Define the following operator products:*

$$\begin{aligned} (\mathcal{S}^*\mathcal{T})_1 &: H_1 \times H_2 \mapsto H_2 \quad \textit{[when } H_2 = H_3\textit{]} \\ &: (u, w) \mapsto \mathcal{S}^*(\mathcal{T}(u), w) \end{aligned}$$

$$\begin{aligned} (\mathcal{S}^*\mathcal{T})_2 &: H_3 \times H_1 \mapsto H_1 \\ &: (y, u) \mapsto \mathcal{S}^*(y, \mathcal{T}(u)). \end{aligned}$$

△

The main application of this definition is in regards to the product-reversal property.

Theorem 5.1.4. *(product-reversal) Let H_i, $i = 1, 2, 3$, be a collection of Hilbert spaces. Assume $\mathcal{T} : H_1 \mapsto H_2$ and $\mathcal{S} : H_2 \mapsto H_3$ are two operators with well-defined adjoint operators. Then the following identity holds:*

$$(\mathcal{ST})^* \cong (\mathcal{T}^*(\mathcal{S}^*\mathcal{T})_2)_1.$$

Proof. The claim follows straightforwardly from the defining property (5.2). Observe that for any $(y, u) \in H_3 \times H_1$:

$$\begin{aligned} \langle u, (\mathcal{ST})^*(y,u)\rangle_{H_1} &= \langle \mathcal{ST}(u), y\rangle_{H_3} \\ &= \langle \mathcal{T}(u), \mathcal{S}^*(y, \mathcal{T}(u))\rangle_{H_2} \\ &= \langle u, \mathcal{T}^*(\mathcal{S}^*(y, \mathcal{T}(u)), u)\rangle_{H_1} \\ &= \langle u, (\mathcal{T}^*(\mathcal{S}^*\mathcal{T})_2)_1(y,u)\rangle_{H_1}. \end{aligned}$$

■

In order to compute adjoints of general adjoint operators for the double adjoint property (iii), the concept of a partial adjoint operator is needed. The idea is based on a direct generalization of identity (5.2).

Definition 5.1.3. *For any set of Hilbert spaces H_i, $i = 1, \ldots, m+1$ and an operator*

$$\begin{aligned} \mathcal{U} &: H_1 \times H_2 \times \ldots \times H_m \mapsto H_{m+1} \\ &: (u_1, u_2, \ldots, u_m) \mapsto y, \end{aligned} \tag{5.4}$$

a jth partial adjoint of $\mathcal{U}$ is any mapping of the form

$$\mathcal{U}^{*_j} : H_{m+1} \times H_1 \times H_2 \ldots \times H_m \mapsto H_j,$$

where

$$\langle \mathcal{U}(u_1, u_2, \ldots, u_m), y \rangle_{H_{m+1}} = \langle u_j, \mathcal{U}^{*_j}(y, u_1, u_2, \ldots, u_m) \rangle_{H_j},$$
$$\forall u_i \in H_i, \quad y \in H_{m+1}, \quad \text{for} \quad i = 1, \ldots, m. \qquad \triangle$$

These definitions produce the following double adjoint identities.

Theorem 5.1.5. *(double adjoints) Let H_1 and H_2 be two Hilbert spaces and $\mathcal{T} : H_1 \mapsto H_2$ be an operator with a well defined adjoint. Then it follows that*

$$(\mathcal{T}^*)^{*_1}(\bar{u}, y, u)|_{\bar{u}=u} \cong \mathcal{T}(u)$$
$$(\mathcal{T}^*)^{*_2}(\bar{u}, y, u)|_{\bar{u}=u} \cong \mathcal{T}^*(y, u)$$

for all $u \in H_1, y \in H_2$, assuming all the partial adjoints exist.

Proof. With respect to the first identity, observe that the first partial adjoint of $\mathcal{T}^*(y, u)$ fulfills

$$\begin{aligned} \langle y, (\mathcal{T}^*)^{*_1}(\bar{u}, y, u) \rangle|_{\bar{u}=u} &= \langle \mathcal{T}^*(y, u), \bar{u} \rangle|_{\bar{u}=u} \\ &= \langle y, \mathcal{T}(u) \rangle. \end{aligned}$$

For the second partial adjoint of $\mathcal{T}^*(y, u)$,

$$\begin{aligned} \langle u, (\mathcal{T}^*)^{*_2}(\bar{u}, y, u) \rangle|_{\bar{u}=u} &= \langle \mathcal{T}^*(y, u), \bar{u} \rangle|_{\bar{u}=u} \\ &= \langle u, \mathcal{T}^*(y, u) \rangle. \end{aligned}$$

■

One application of this theorem is in regards to testing for self-adjointness.

Definition 5.1.4. *Let H be a Hilbert space and $\mathcal{S} : H \mapsto H$ be a mapping with a well defined adjoint operator $\mathcal{S}^* : H \times H \mapsto H$. $\mathcal{S}$ is* self-*adjoint if*

$$\mathcal{S}^*(\bar{u}, u)|_{\bar{u}=u} \cong \mathcal{S}(u), \quad \forall u \in H.$$

Observe that an operator like $\mathcal{T}^*\mathcal{T}(u) := (\mathcal{T}^*\mathcal{T})_1(\bar{u}, u)|_{\bar{u}=u} = \mathcal{T}^*(\mathcal{T}(u), u)$ is always self-adjoint since one may write in terms of the first partial adjoint

$$\begin{aligned} \langle \mathcal{T}^*(\mathcal{T}(u), u), \bar{u} \rangle_H &= \langle \mathcal{T}(u), (\mathcal{T}^*)^{*_1}(\bar{u}, \mathcal{T}(u), u) \rangle_H \\ &= \langle u, \mathcal{T}^*((\mathcal{T}^*)^{*_1}(\bar{u}, \mathcal{T}(u), u), u) \end{aligned}$$

or in terms of the second partial adjoint

$$\langle \mathcal{T}^*(\mathcal{T}(u), u), \bar{u} \rangle_H = \langle u, (\mathcal{T}^*)^{*_2}(u, T(u), \bar{u}) \rangle_H.$$

By definition it then follows that

$$(\mathcal{T}^*\mathcal{T})^*(\bar{u}, u) \cong \mathcal{T}^*((\mathcal{T}^*)^{*_1}(\bar{u}, \mathcal{T}(u), u), u)$$
$$(\mathcal{T}^*\mathcal{T})^*(\bar{u}, u) \cong (\mathcal{T}^*)^{*_2}(u, T(u), \bar{u})$$

In either case, the identities in Theorem 5.1.5 yield the required property:

$$(\mathcal{T}^*\mathcal{T})^*(\bar{u}, u)|_{\bar{u}=u} \cong (\mathcal{T}^*\mathcal{T})(u).$$

■

Example 5.1.4. Consider the operator and its suitable choice of nonlinear Hilbert adjoint given in Example 5.1.2, where now m=1. Then

$$\mathcal{T}^*(y,u)|_{y=u} = uy|_{y=u} = u^2 = \mathcal{T}(u).$$

So $\mathcal{T}$ is self-adjoint.

Example 5.1.5. Reconsider Example 5.1.3 where m=p=1. Even in this SISO case, the corresponding Hammerstein operator is rarely self-adjoint since:

$$\begin{aligned}\mathcal{S}^*(y,u)|_{y=u} &= F(u)S_{\mathrm{L}}^*(u)\\ &\neq \mathcal{S}(u).\end{aligned}$$

△

The final property under consideration is the "C^*-identity" (iv). Unlike the linear case, at present only an inequality is known to relate the two norms in question.

Theorem 5.1.6. *(C^*-inequality) Let H_1 and H_2 be Hilbert spaces. Assume $\mathcal{T} : H_1 \mapsto H_2$ is a bounded operator with a well-defined adjoint operator. Then the following inequality holds:*

$$\|\mathcal{T}\|^2 \leq \|\mathcal{T}^*\mathcal{T}\|.$$

Proof. For any fixed $u \in H_1$ and employing the Schwarz inequality,

$$\begin{aligned}\|\mathcal{T}(u)\|_{H_2}^2 &= \langle \mathcal{T}(u), \mathcal{T}(u)\rangle_{H_2}\\ &= \langle \mathcal{T}^*(\mathcal{T}(u),u),u\rangle_{H_1}\\ &= \langle \mathcal{T}^*\mathcal{T}(u),u\rangle_{H_1}\\ &\leq \|\mathcal{T}^*\mathcal{T}(u)\|_{H_1}\|u\|_{H_1}\end{aligned}$$

Dividing both sides by $\|u\|_{H_1}^2$ and taking the supremum over all $u \neq 0$ gives the final result. ■

The above theorems show that one almost has a complete nonlinear extension of a linear adjoint map defined on a C^* algebra, except for the equality in property (iv).

The section is concluded by considering how the Fréchet derivative interactswith nonlinear Hilbert adjoints. This is important because of its relationship to the eigen-structure of the Hankel operator and its application to the spectral analysis in the next chapters. Given an operator $\mathcal{U}$ of the form (5.4), its Fréchet derivative with respect to u_j at $(u_1, u_2, \ldots, u_m)$ is denoted by $D_j\mathcal{U}(u_1, u_2, \ldots, u_m)$. The situation is greatly simplified by the fact that $D_j\mathcal{U}(u_1, u_2, \ldots, u_m)$ is a linear operator defined on H_j.

Theorem 5.1.7. *Let H_1 and H_2 be two Hilbert spaces and $\mathcal{T} : H_1 \mapsto H_2$ be an operator with a well defined Hilbert adjoint. Assuming both $\mathcal{T}$ and $\mathcal{T}^*$ are Fréchet differentiable, then the following identities hold:*

1. $(D_1\mathcal{T}^*(y,u))^*(u) = \mathcal{T}(u)$
2. $(D(\mathcal{T}^*\mathcal{T}(u)))^*(u) = 2\,(D\mathcal{T}(u))^*(\mathcal{T}(u)) - \mathcal{T}^*\mathcal{T}(u)$.

Proof.

1. For any $u \in H_1$ and $\xi, y \in H_2$ observe that

$$\begin{aligned}
&D_y\langle \mathcal{T}^*(y,u),u\rangle_{H_1}(\xi) = D_y\langle y,\mathcal{T}(u)\rangle_{H_2}(\xi)\\
&\langle D_1\mathcal{T}^*(y,u)(\xi),u\rangle_{H_1} = \langle \xi,\mathcal{T}(u)\rangle_{H_2}\\
&\langle \xi,(D_1\mathcal{T}^*(y,u))^*(u)\rangle_{H_2} = \langle \xi,\mathcal{T}(u)\rangle_{H_2}.
\end{aligned}$$

2. Similarly, for any $u, \xi \in H_1$ and $y \in H_2$

$$\begin{aligned}
&D_u\langle \mathcal{T}^*(y,u),u\rangle_{H_1}(\xi) = D_u\langle y,T(u)\rangle_{H_2}(\xi)\\
&\langle D_2\mathcal{T}^*(y,u)(\xi),u\rangle_{H_1} + \langle \mathcal{T}^*(y,u),\xi\rangle_{H_1} = \langle y, D\mathcal{T}(u)(\xi)\rangle_{H_2}\\
&\langle \xi,(D_2\mathcal{T}^*(y,u))^*(u)\rangle_{H_1} = \langle \xi,(D\mathcal{T}(u))^*(y)\rangle_{H_2} - \langle \xi,\mathcal{T}^*(y,u)\rangle_{H_1}.
\end{aligned}$$

3. First observe that for any $u, \xi \in H_1$

$$\begin{aligned}
\langle u, D\left(\mathcal{T}^*(\mathcal{T}(u),u)\right)(\xi)\rangle_{H_1} = \langle u, &D_1\left(\mathcal{T}^*(\mathcal{T}(u),u)\right)(D\mathcal{T}(u)(\xi))\\
&+D_2\left(\mathcal{T}^*(\mathcal{T}(u),u)\right)(\xi)\rangle_{H_1}.
\end{aligned}$$

Now, using the previous two identities it follows that

$$\begin{aligned}
\left(D\left(\mathcal{T}^*\mathcal{T}(u)\right)\right)^*(u) &= \left(D_1\left(\mathcal{T}^*(\mathcal{T}(u),u)\right)(D\mathcal{T}(u))\right)^*(u)\\
&\quad + \left(D_2\left(\mathcal{T}^*(\mathcal{T}(u),u)\right)\right)^*(u)\\
&= (D\mathcal{T}(u))^*\left(D_1\left(\mathcal{T}^*(\mathcal{T}(u),u)\right)\right)^*(u)\\
&\quad + \left(D_2\left(\mathcal{T}^*(\mathcal{T}(u),u)\right)\right)^*(u)\\
&= (D\mathcal{T}(u))^*\left(\mathcal{T}(u)\right) - \mathcal{T}^*(\mathcal{T}(u),u)\\
&\quad + (D\mathcal{T}(u))^*\left(\mathcal{T}(u)\right).
\end{aligned}$$

■

5.2 Variational and adjoint systems

This section is devoted to the state-space characterization of variational and adjoint systems based on [15], which were intensively utilized in e.g., [73, 78, 5] as a preparation for the main results in the following sections. In order to handle variational and adjoint systems of Hankel operators in this paper, we generalize the results in [15] using [20, 21] so that they can treat initial and final states as input and output of the system.

Consider an operator $S : \mathcal{U} \to L_2^r[t^0, t^1]$ defined on a (possibly infinite) time interval $[t^0, t^1] \subset \mathbb{R}$ described by the state-space realization

$$u \mapsto y = S(u) : \begin{cases} \dot{x} = f(x,u,t),\; x(t^0) = x^0 \\ y = h(x,u,t) \end{cases} \tag{5.5}$$

with $x(t) \in \mathcal{X}^0 \subset \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^r$, and $\mathcal{U} \subset L_2^m[t^0, t^1]$ an open neighborhood of 0. Here we assume $(x, u) = (0, 0)$ is an equilibrium, i.e. $f(0, 0, t) = 0$ and $h(0, 0, t) = 0$ hold for $\forall t \in \mathbb{R}$. and that all signals and functions are sufficiently smooth. Including the initial and terminal states results in a representation of the dynamical system that can also be regarded as a mapping $\mathbb{R}^n \times L_2^m \to \mathbb{R}^n \times L_2^r$ defined by

$$(x^0, u) \mapsto (x^1, y) = \hat{S}(x^0, u) : \begin{cases} \dot{x} &= f(x, u, t),\ x(t^0) = x^0 \\ y &= h(x, u, t) \\ x^1 &= x(t^1) \end{cases} \tag{5.6}$$

The variational system of $\hat{S}$ is defined by $(x^0, u, x_v^0, u_v) \mapsto (x_v^1, y_v) =$

$$\hat{S}_v(x^0, u, x_v^0, u_v) : \begin{cases} \dot{x} &= f(x, u, t), \qquad x(t^0) = x^0 \\ \dot{x}_v &= \frac{\partial f}{\partial x} x_v + \frac{\partial f}{\partial u} u_v,\ x_v(t^0) = x_v^0 \\ y_v &= \frac{\partial h}{\partial x} x_v + \frac{\partial h}{\partial u} u_v \\ x_v^1 &= x_v(t^1) \end{cases} . \tag{5.7}$$

The input-state-output set (u_v, x_v, y_v) are the so called variational input, state and output respectively and they represent the variation along the trajectory (u, x, y) of the original system $\hat{S}$.

The Hamiltonian extension $\hat{S}_a$ of $\hat{S}$ is given by a Hamiltonian control system of the form $(x^0, u, p^1, u_a) \mapsto (p^0, y_a) =$

$$\hat{S}_a(x^0, u, p^1, u_a) : \begin{cases} \dot{x} = & \frac{\partial H}{\partial p}^T = f(x, u, t), \qquad x(t^0) = x^0 \\ \dot{p} = & -\frac{\partial H}{\partial x}^T = -\frac{\partial f}{\partial x}^T p - \frac{\partial h}{\partial x}^T u_a,\ p(t^1) = p^1 \\ y_a = & \frac{\partial H}{\partial u}^T = \frac{\partial f}{\partial u}^T p + \frac{\partial h}{\partial u}^T u_a \\ p^0 = & p(t^0) \end{cases} \tag{5.8}$$

with the Hamiltonian

$$H(x, p, u, u_a) := p^T f(x, u, t) + u_a^T h(x, u, t). \tag{5.9}$$

The structure already reveals a form that corresponds to the linear adjoint notion. In the sequel this issue is studied in more detail.

For understanding the meaning of the Hamiltonian extension and its relation with the adjoint of a system, we first consider the concept of Gâteaux differentiation for dynamical systems. In our framework, it is important to take an input-output point of view and to include the initial and final states. Furthermore, Gâteaux differentiation plays an important role in the analysis of the properties of Hankel operators as may be seen in Chapter 7. To this end, we state the definition of Gâteaux differentiation.

Definition 5.2.1. *(Gâteaux differential) Suppose X and Y are Banach spaces, $U \subset X$ is open, and $S : U \to Y$. Then S is said to be Gâteaux differentiable at $x \in U$ if, for all $\zeta \in X$ the following limit exists:*

$$dS(x)(\zeta) = \lim_{\varepsilon \to 0} \frac{S(x+\varepsilon\zeta) - S(x)}{\varepsilon} = \frac{d}{d\varepsilon}\, S(x+\varepsilon\zeta)|_{\varepsilon=0}\,. \tag{5.10}$$

We write $dS(x)(\zeta)$ for the Gâteaux differential of S at x in the "direction" ζ.

Then we can prove the following property for the variational system (5.7) which is a generalized version of the results stated in [15].

Theorem 5.2.1. *Consider an operator $\hat{S}$ with the state-space realization (5.6). Suppose that the trajectory of the state $x_v(t)$ of $\hat{S}_v(u, u_v)$ in (5.7) is uniquely determined for $\forall x^0 \in \mathcal{X}^0 \subset \mathbb{R}^n$, $\forall u \in \mathcal{U} \subset L_2^m[t^0, t^1]$, $\forall x_v^0 \in \mathbb{R}^n$ and $\forall u_v \in L_2^m[t^0, t^1]$ where $\mathcal{X}^0$ and $\mathcal{U}$ are open neighborhoods of 0 in $\mathbb{R}^n$ and $L_2^m[t^0, t^1]$ respectively. Then*

$$\hat{S} : \mathcal{X}^0 \times \mathcal{U} \to \mathbb{R}^n \times L_2^r[t^0, t^1] \text{ is Gâteaux differentiable}$$
$$\Leftrightarrow$$
$$\hat{S}_v \text{ is a mapping of } \mathcal{X}^0 \times \mathcal{U} \times \mathbb{R}^n \times L_2^m[t^0, t^1] \to \mathbb{R}^n \times L_2^r[t^0, t^1]$$

Furthermore the Gâteaux differential of $\hat{S}$ is given by

$$d\hat{S}(x^0, u)(x_v^0, u_v) = \hat{S}_v((x^0, u), (x_v^0, u_v)). \tag{5.11}$$

Proof. This proof is based on the results in [15]. Let $(u(t,\varepsilon), x(t,\varepsilon), y(t,\varepsilon))$, $t \in [t^0, t^1]$ denote a family of input-state-output trajectories of $\hat{S}$ parameterized by ε. Then we have

$$\begin{aligned}
\frac{\partial y}{\partial \varepsilon}(t,0) &= \frac{\partial h}{\partial x}(x(t,0), u(t,0), t)\frac{\partial x}{\partial \varepsilon}(t,0) + \frac{\partial h}{\partial u}(x(t,0), u(t,0), t)\frac{\partial u}{\partial \varepsilon}(t,0)\\
\frac{d}{dt}\frac{\partial x}{\partial \varepsilon}(t,0) &= \frac{\partial}{\partial \varepsilon}\frac{dx(t,0)}{dt}\\
&= \frac{\partial f}{\partial x}(x(t,0), u(t,0), t)\frac{\partial x}{\partial \varepsilon}(t,0) + \frac{\partial f}{\partial u}(x(t,0), u(t,0), t)\frac{\partial u}{\partial \varepsilon}(t,0)
\end{aligned}$$

and moreover

$$\begin{aligned}
\frac{dx(t,0)}{dt} &= f(x(t,0), u(t,0), t)\\
y(t,0) &= h(x(t,0), u(t,0), t).
\end{aligned}$$

Therefore the trajectories $(\partial u/\partial\varepsilon(t,0), \partial x/\partial\varepsilon(t,0), \partial y/\partial\varepsilon(t,0))$ coincide with the input-state-output trajectories of the variational system $\hat{S}_v$. Now let $u(t,\varepsilon) = u(t) + \varepsilon v(t)$. Then we obtain

$$\begin{aligned}
d\hat{S}((x^0, u), (x_v^0, v)) &= \lim_{\varepsilon \to 0} \frac{\hat{S}(x^0 + \varepsilon x_v^0, u + \varepsilon v) - \hat{S}(x^0, u)}{\varepsilon}\\
&= \lim_{\varepsilon \to 0} \frac{(x(t^1,\varepsilon) - x(t^1,0), y(t,\varepsilon) - y(t,0))}{\varepsilon}\\
&= (\frac{\partial x}{\partial \varepsilon}(t^1,0), \frac{\partial y}{\partial \varepsilon}(t,0)).
\end{aligned}$$

Due to the assumption that the state trajectory $x_v(t)$ of the variational system $\hat{S}_v$ is uniquely determined, we can conclude that the existence of $d\hat{S}$ is equivalent to that $\hat{S}_v$ is an operator on L_2 spaces. As a result, $\hat{S}_v$ coincides with the Gâteaux differential $d\hat{S}$. This proves the theorem. ∎

Perhaps more well-known than the Gâteaux differential is the Fréchet derivative, which is especially useful for analysis of nonlinear static functions. A Fréchet derivative is a special class of Gâteaux differentiation and it coincides with Gâteaux differential if it is continuous and linear in the second argument "direction". See e.g. [8].

Next we give the formal justification of calling the Hamiltonian extension $\hat{S}_a$ the adjoint form of the variational system $\hat{S}_v$, as is done in [15]. The most general form of the Hamiltonian extension, i.e., including arbitrary initial conditions, can be seen as the differential version of the next section.

Theorem 5.2.2. *Consider an operator $\hat{S}$ with the state-space realization (5.6). Suppose that the assumptions in Theorem 5.2.1 hold and that $\hat{S}_a$ satisfies*

$$\begin{aligned} u \in \mathcal{U} \subset L_2^m[t^0, t^1],\ u_a \in L_2^r[t^0, t^1],\ x(t^0) \in \mathcal{X}^0 \subset \mathbb{R}^n,\ p(t^1) \in \mathbb{R}^n \\ \Rightarrow\ p(t^0) \in \mathbb{R}^n. \end{aligned} \tag{5.12}$$

Then there holds

$$(\hat{S}_v(x^0, u))^*(p^1, u_a) = \hat{S}_a(x^0, u)(p^1, u_a) \tag{5.13}$$

with the inner product on $\mathbb{R}^n \times L_2[t^0, t^1]$.

Proof. Let the Hamiltonian function be given by $H_v = p^T x_v$, and denote $x_v^0 := x_v(t^0)$, $x_v^1 := x_v(t^1)$, $p^0 := p(t^0)$ and $p^1 := p(t^1)$ for simplicity. Then we have

$$\begin{aligned} \frac{dH_v}{dt} &= p^T \dot{x}_v + x_v^T \dot{p} = p^T \left(\frac{\partial f}{\partial x} x_v + \frac{\partial f}{\partial u} u_v \right) - x_v^T \left(\frac{\partial f}{\partial x}^T p + \frac{\partial h}{\partial x}^T u_a \right) \\ &= p^T \frac{\partial f}{\partial u} u_v - x_v^T \frac{\partial h}{\partial x}^T u_a \\ &= \left(p^T \frac{\partial f}{\partial u} + u_a^T \frac{\partial h}{\partial u} \right) u_v - \left(x_v^T \frac{\partial h}{\partial x}^T + u_v^T \frac{\partial h}{\partial u}^T \right) u_a \\ &= y_a^T u_v - y_v^T u_a. \end{aligned}$$

This reduces to

$$\begin{aligned} \langle y_a, u_v \rangle_{L_2^m} - \langle y_v, u_a \rangle_{L_2^r} &= \int_{t^0}^{t^1} \left(y_a^T u_v - y_v^T u_a \right) dt = \int_{t^0}^{t^1} \frac{dH_v}{dt} dt \\ &= H_v|_{t=t^1} - H_v|_{t=t^0} \\ &= \langle x_v^1, p^1 \rangle_{\mathbb{R}^n} - \langle x_v^0, p^0 \rangle_{\mathbb{R}^n}. \end{aligned}$$

Therefore

$$\langle (x_v^1, y_v), (p^1, u_a) \rangle_{\mathbb{R}^n \times L_2^r} = \langle (x_v^0, u_v), (p^0, y_a) \rangle_{\mathbb{R}^n \times L_2^m} \tag{5.14}$$

holds with the inner product on $\mathbb{R}^n \times L_2$. Now, substituting $(x_v^1, y_v) = \hat{S}_v((x^0, u), (x_v^0, u_v))$ and $(p^0, y_a) = \hat{S}_a((x^0, u), (p^1, u_a))$ implies (5.13). This completes the proof. ■

We now prove some properties of this system which are related to the nonlinear Hilbert adjoint operator.

Proposition 5.2.1. *Consider the Hamiltonian extension (5.8) of S. Suppose f and h are time-invariant, i.e. $f = f(x, u)$ and $h = h(x, u)$. Define scalar valued functions H_1, H_2 and H_3 as*

$$H_1 = H - \frac{\partial H}{\partial u_a} u_a, \quad H_2 = H - \frac{\partial H}{\partial u} u, \quad H_3 = H - \frac{\partial H}{\partial u} u - \frac{\partial H}{\partial u_a} u_a.$$

Then the following relations hold:

$$\begin{aligned} \frac{dH}{dt} &= y_a^T \dot{u} + y^T \dot{u}_a, & \frac{dH_1}{dt} &= y_a^T \dot{u} - \dot{y}^T u_a \\ \frac{dH_2}{dt} &= -\dot{y}_a^T u + y^T \dot{u}_a, & \frac{dH_3}{dt} &= -\dot{y}_a^T u - \dot{y}^T u_a. \end{aligned} \tag{5.15}$$

Proof. The first equation in (5.15) follows from

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial p} \dot{p} + \frac{\partial H}{\partial u} \dot{u} + \frac{\partial H}{\partial u_a} \dot{u}_a \\ &= \frac{\partial H}{\partial x} \frac{\partial H}{\partial p}^T - \frac{\partial H}{\partial p} \frac{\partial H}{\partial x}^T + y_a^T \dot{u} + y^T \dot{u}_a = y_a^T \dot{u} + y^T \dot{u}_a. \end{aligned}$$

Then the time derivative of the other functions are obtained straightforwardly. ■

This proposition shows that the Hamiltonian extension has a close relationship to nonlinear Hilbert adjoint operators. For example, the property for H_1 in (5.15) implies the mapping $u_a \mapsto y_a$ is the nonlinear Hilbert adjoint of the variational mapping $\dot{u} \mapsto \dot{y}$, while the property for H_2 shows that the original mapping $u \mapsto y$ coincides with the adjoint of the variational map $\dot{u}_a \mapsto \dot{y}_a$. This relation can be utilized to derive a state-space realization of a nonlinear Hilbert adjoint operator of an input-affine nonlinear system under some additional assumptions. The property for H_3 in (5.15) is similar to a basic property of *physical* Hamiltonian control systems, namely the so called *energy balancing* [15]. By studying those properties it becomes clear that the Hamiltonian extension of S is not a state-space realization of the nonlinear Hilbert adjoint of S. However, the above proposition does provide a tool to obtain such state-space realization via the Hamiltonian extensions of S as can be seen in the next corollary.

Corollary 5.2.1. *Consider the input-output system S with state-space realization (5.5) defined for $t \in [t^0, t^1] \subset \mathbb{R}$. Suppose f and h are time-invariant and input-affine, i.e. $f \equiv g_0(x) + g(x)u$ and $h \equiv k_0(x) + k(x)u$ for some smooth functions g_0, g, k_0 and k. Suppose moreover that*

$$\begin{aligned} & u \in L_2^m[t^0, t^1],\ u_b \in L_2^r[t^0, t^1] \\ & \quad \Rightarrow\ \|x(t^1)\| < \infty,\ \|p_1(t^0)\| < \infty,\ \|p_2(t^0)\| < \infty \end{aligned} \tag{5.16}$$

holds for the state-space system $(u_b, u) \mapsto y_b = S^*(u_b, u)$:

$$\begin{cases} \dot{x} = g_0(x) + g(x)u, & x(t^0) = 0 \\ \dot{p}_1 = -\frac{\partial g_0}{\partial x}^T p_1 - \frac{\partial k_0}{\partial x}^T p_2, & p_1(t^1) = 0 \\ \dot{p}_2 = u_b, & p_2(t^1) = 0 \\ y_b = \left(\frac{\partial (g^T p_1)}{\partial x} + \frac{\partial (k^T p_2)}{\partial x}\right) g_0(x) & \\ \qquad -g^T(x)\left(\frac{\partial g_0}{\partial x}^T p_1 + \frac{\partial k_0}{\partial x}^T p_2\right) + k^T(x)u_b. & \end{cases} \tag{5.17}$$

Then a state-space realization of the nonlinear Hilbert adjoint $S^* : L_2^m[t^0, t^1] \times L_2^r[t^0, t^1] \to L_2^m[t^0, t^1]$ is given by (5.17).

Proof. System (5.17) is derived by defining $u_b := \dot{u}_a$, $y_b := \dot{y}_a$ and $p_2 := u_a$ within the Hamiltonian extension (5.8). Note that from the input-affine form of f and h, we have

$$H_2(x, p_1, p_2) = p_1^T g_0(x) + p_2^T k_0(x).$$

The initial condition of (5.17) and $g_0(0) = 0$ and $k_0(0) = 0$ imply that

$$H_2(x(t^0), p_1(t^0), p_2(t^0)) = H_2(x(t^1), p_1(t^1), p_2(t^1)) = 0.$$

Then it follows from (5.15) in Proposition 5.2.1 that

$$\begin{aligned} 0 = [H_2]_{t=t^0}^{t=t^1} &= \int_{t^0}^{t^1} \frac{dH_2}{dt} dt = \langle y, \dot{u}_a \rangle_{L_2^r[t^0,t^1]} - \langle u, \dot{y}_a \rangle_{L_2^m[t^0,t^1]} \\ &= \langle y, u_b \rangle_{L_2^r[t^0,t^1]} - \langle u, y_b \rangle_{L_2^m[t^0,t^1]}. \end{aligned}$$

Substituting $y = S(u)$ and $y_b = S^*(u_b, u)$ yields the defining equation of a nonlinear Hilbert adjoint operator

$$\langle S(u), u_b \rangle_{L_2^r[t^0,t^1]} = \langle u, S^*(u_b, u) \rangle_{L_2^m[t^0,t^1]},$$

and this completes the proof. ■

Observe that state-space realization (5.17) has $(2n + r)$ states, and corresponds in the linear case to $(s\, S(s)(1/s))^* = s\, S^*(s)(1/s)$. Intuitively this follows from the original definition of the Hamiltonian extension as the adjoint of the variational system. Of course this input-output mapping coincides with $S^*(s)$ for a linear system, but not generally for nonlinear systems. Furthermore, observe that this realization requires the restrictive assumption (5.16) because otherwise $t \in (-\infty, \infty)$ in (5.16), i.e. $t^0 \to -\infty$

and $t^1 \to \infty$, implies the anti-stability of a non minimum phase operator $S^*(s)(1/s)$. Hence, in general assumption (5.16) holds only for a finite time interval $t \in [t^0, t^1]$.

5.2.1 The Hankel, observability and controllability operator and their differentials

This section introduces state-space realizations for the adjoints of operators related to the Hankel operator by using the previous results. These realizations are then used in the next section for singular value analysis of nonlinear dynamical operators. We only consider time invariant, input-affine, sufficiently smooth nonlinear systems without direct feed-through in the form of

$$S : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{5.18}$$

defined on the time interval $t \in (-\infty, \infty)$. It is assumed throughout that S is asymptotically stable and L_2-stable in the sense that $u \in L_2^m(-\infty, 0]$ implies that $S(u)$ restricted to $[0, \infty)$ is in $L_2^r[0, \infty)$. It is also assumed throughout that the system is asymptotically reachable from 0, i.e., the time-reversal system is asymptotically stabilizable, i.e., $\forall x_0 \in \mathbb{R}^n$, starting in $x(-\infty) = 0$ there exists a $u \in L_2(-\infty, 0)$ such that $x(0) = x^0$, see e.g. [75].

A Hankel operator of a nonlinear system can be seen as a straightforward extension of the concept of a Hankel operator of a linear system. In this light, it is natural to study the related observability and controllability operators for a nonlinear system. State-space characterization of nonlinear observability and controllability operators can be given as intuitively clear extensions from the linear case. The observability and controllability operators are mappings of $\mathbb{R}^n \to L_2^r[0, \infty)$ and $L_2^m[0, \infty) \to \mathbb{R}^n$, and their state space realizations are given by

$$x^0 \mapsto y = \mathcal{O}_0(x^0) : \begin{cases} \dot{x} = f(x) \; x(0) = x^0 \\ y = h(x) \end{cases} \tag{5.19}$$

$$u \mapsto x^1 = \mathcal{C}(u) \; : \begin{cases} \dot{x} = f(x) + g(x)\mathcal{F}_-(u) \; x(-\infty) = 0 \\ x^1 = x(0) \end{cases} . \tag{5.20}$$

Here $\mathcal{F}_-$ is the time flipping operator. Furthermore the Hankel operator $\mathcal{H}_0 : L_2^m[0, \infty) \to L_2^r[0, \infty)$ of S is given by

$$\mathcal{H}_0 := S \circ \mathcal{F}_-. \tag{5.21}$$

In finite time, these operators [34, 75] can be given via Chen-Fliess functional expansions [39], but we do not need these representations here. Clearly there holds $\mathcal{H}_0 = \mathcal{O}_0 \circ \mathcal{C}$. which relation has been studied in [34]. The state-space realizations of the differentiations $d\mathcal{O}_0$, $d\mathcal{C}$ and $d\mathcal{H}_0$ are given by the following lemma which will be utilized in the succeeding sections.

Lemma 5.2.1. *Consider the operator S with the state-space realization (5.18). Suppose that*

$$u, u_v \in L_2^m(-\infty, 0],\ x(-\infty) = x_v(-\infty) = 0 \ \Rightarrow \quad y|_{[0,\infty)}, y_v|_{[0,\infty)} \in L_2^r[0,\infty),\ x(0), x_v(0) \in \mathbb{R}^n \tag{5.22}$$

holds with $\hat{S}$ and $\hat{S}_v$ where $y|_{[0,\infty)}$ denotes the signal y restricted to the time domain $[0,\infty)$. Then the state-space realizations of $d\mathcal{O}_0 : \mathbb{R}^n \times \mathbb{R}^n \to L_2^r[0,\infty)$, $d\mathcal{C} : L_2^m[0,\infty) \times L_2^m[0,\infty) \to \mathbb{R}^n$ and $d\mathcal{H}_0 : L_2^m[0,\infty) \times L_2^m[0,\infty) \to L_2^r[0,\infty)$ are given by

$$y_v = d\mathcal{O}_0(x^0)(x_v^0) : \begin{cases} \dot{x} &= f(x) \quad x(0) = x^0 \\ \dot{x}_v &= \frac{\partial f}{\partial x} x_v \ x_v(0) = x_v^0 \\ y_v &= \frac{\partial h}{\partial x} x_v \end{cases} \tag{5.23}$$

$$x_v^0 = d\mathcal{C}(u)(u_v) : \begin{cases} \dot{x} &= f(x) + g(x)\mathcal{F}_-(u) \qquad x(-\infty) = 0 \\ \dot{x}_v &= \frac{\partial (f + g\mathcal{F}_-(u))}{\partial x} x_v + g(x)\mathcal{F}_-(u_v) \ x_v(-\infty) = 0 \\ x_v^0 &= x_v(0) \end{cases} \tag{5.24}$$

$$y_v = d\mathcal{H}_0(u)(u_v) : \begin{cases} \dot{x} &= f(x) + g(x)\,\mathcal{F}_-(u) \qquad x(-\infty) = 0 \\ \dot{x}_v &= \frac{\partial (f + g\mathcal{F}_-(u))}{\partial x} x_v + g(x)\mathcal{F}_-(u_v) \ x_v(-\infty) = 0 \\ y_v &= \frac{\partial h}{\partial x} x_v \end{cases} . \tag{5.25}$$

Proof: The proof directly follows from Theorem 5.2.1 and the definitions of $\mathcal{O}_0$, $\mathcal{C}$ and $\mathcal{H}_0$. ∎

Corollary 5.2.2. *Consider the operator S with the state-space realization (5.18). Suppose that the assumptions in Lemma 5.2.1 hold. Then*

$$\mathcal{O}_{dS} = d\mathcal{O}_0 \tag{5.26}$$

$$\mathcal{C}_{dS} = (\mathcal{C}, d\mathcal{C}) \tag{5.27}$$

$$\mathcal{H}_{dS} = d\mathcal{H}_0. \tag{5.28}$$

Note that the right hand side of (5.27) is slightly different from those in (5.26) and (5.28). This is due to the dimension of the output signal. The adjoints of the operators of Lemma 5.2.1 can be obtained by applying Theorem 5.2.2, which results in the following lemma.

Lemma 5.2.2. *Consider the operator S with the state-space realization (5.18). Suppose that the assumptions in Lemma 5.2.1 hold and that*

$$u \in L_2^m(-\infty, 0],\ u_a \in L_2^r[0,\infty),\ x(-\infty) = p(\infty) = 0 \quad \Rightarrow \ y_a|_{[0,\infty)} \in L_2^m[0,\infty),\ p(0) \in \mathbb{R}^n \tag{5.29}$$

with $\hat{S}_a$. Then state-space realizations of
$(d\mathcal{O}_0(x^0))^* : L_2^r[0,\infty)(\times\mathbb{R}^n) \to \mathbb{R}^n$*,*
$(d\mathcal{C}(u))^* : \mathbb{R}^n(\times L_2^m[0,\infty)) \to L_2^m[0,\infty)$ *and*
$(d\mathcal{H}_0(u))^* : L_2^r[0,\infty)(\times L_2^m[0,\infty)) \to L_2^m[0,\infty)$ *are given by*

$$p^0 = (d\mathcal{O}_0(x^0))^*(u_a) : \begin{cases} \dot{x} &= f(x) \qquad\qquad\qquad\qquad x(0) = x^0 \\ \dot{p} &= -\frac{\partial f}{\partial x}^T(x)\, p - \frac{\partial h}{\partial x}^T(x)\, u_a \quad p(\infty) = 0 \\ p^0 &= p(0) \end{cases} \tag{5.30}$$

$$y_a = (d\mathcal{C}(u))^*(p^1) : \begin{cases} \dot{x} &= f(x) + g(x)\mathcal{F}_-(u) \quad x(-\infty) = 0 \\ \dot{p} &= -\frac{\partial(f+g\mathcal{F}_-(u))}{\partial x}^T(x)\, p \quad p(0) = p^1 \\ y_a &= \mathcal{F}_+(g^T(x)\, p) \end{cases} \tag{5.31}$$

$$y_a = (d\mathcal{H}_0(u))^*(u_a) : \begin{cases} \dot{x} &= f(x) + g(x)\,\mathcal{F}_-(u) \qquad x(-\infty) = 0 \\ \dot{p} &= -\frac{\partial(f+g\mathcal{F}_-(u))}{\partial x}^T p - \frac{\partial h}{\partial x}^T u_a \quad p(\infty) = 0 \\ y_a &= \mathcal{F}_+(g^T(x)\, p) \end{cases} . \tag{5.32}$$

Proof: To begin with, substituting $t^0 = 0$, $t^1 = \infty$, $p^1 = p(\infty) = 0$, $u_v = 0$ for the equation (5.14) in the proof of Theorem 5.2.2 in Appendix yields

$$\langle y_v, u_a\rangle_{L_2^r} = \langle (x_v^1, y_v), (0, u_a)\rangle_{\mathbb{R}^n\times L_2^r} = \langle (x_v^0, 0), (p^0, y_a)\rangle_{\mathbb{R}^n\times L_2^m} = \langle x_v^0, p^0\rangle_{\mathbb{R}^n}.$$

Substituting moreover $y_v = d\mathcal{O}_0(x^0)(x_v^0)$ and $p^0 = (d\mathcal{O}_0(x^0))^*(u_a)$ as in (5.30) yields

$$\langle d\mathcal{O}_0(x^0)(x_v^0), u_a\rangle_{L_2^r} = \langle x_v^0, (d\mathcal{O}_0(x^0))^*(u_a)\rangle_{\mathbb{R}^n}.$$

This proves the first part.

The second part can be proved in a similar way as in the first part. Substituting $t^0 = -\infty$, $t^1 = 0$, $x_v^0 = x_v(-\infty) = 0$, $u_a = 0$ for the equation (5.14) yields

$$\begin{aligned} \langle x_v^1, p^1\rangle_{\mathbb{R}^n} &= \langle (x_v^1, y_v), (p^1, 0)\rangle_{\mathbb{R}^n\times L_2^r} = \langle (0, \mathcal{F}_-(u_v)), (p^0, y_a)\rangle_{\mathbb{R}^n\times L_2^m} \\ &= \langle \mathcal{F}_-(u_v), y_a\rangle_{L_2^m} = \langle u_v, \mathcal{F}_+(y_a)\rangle_{L_2^m}. \end{aligned}$$

Substituting moreover $x_v^1 = d\mathcal{C}(u)(u_v)$ and $y_a = (d\mathcal{C}(u))^*(p^1)$ as in (5.31) yields

$$\langle d\mathcal{C}(u)(u_v), p^1\rangle_{\mathbb{R}^n} = \langle u_v, (d\mathcal{C}(u))^*(p^1)\rangle_{L_2^m}.$$

This proves the second part.

In order to prove the last part we use the relation (5.25) for arbitrary signals $u_v \in \mathcal{U}_v \subset L_2^m[0,\infty)$ and $u_a \in L_2^r[0,\infty)$. Then

$$\begin{aligned}\langle u_a, d\mathcal{H}_0(u)(u_v)\rangle_{L_2^r} &= \langle u_a, d\mathcal{O}_0(\mathcal{C}(u))(d\mathcal{C}(u)(u_v))\rangle_{L_2^r}\\ &= \langle (d\mathcal{O}_0(\mathcal{C}(u)))^*(u_a), d\mathcal{C}(u)(u_v)\rangle_{L_2^r}\\ &= \langle (d\mathcal{C}(u))^* \circ (d\mathcal{O}_0(\mathcal{C}(u)))^*(u_a), u_v\rangle_{L_2^r}\\ &\equiv \langle (d\mathcal{H}_0(u))^*(u_a), u_v\rangle_{L_2^r}.\end{aligned}$$

Therefore we obtain

$$(d\mathcal{H}_0(u))^*(u_a) = (d\mathcal{C}(u))^* \circ (d\mathcal{O}_0(\mathcal{C}(u)))^*(u_a). \tag{5.33}$$

We can check the state-space realization of the right hand side of the above equation using (5.30) and (5.31) coincide with the left hand side given by (5.32). This completes the proof. ∎

Lemma 5.2.2 can be seen as the differential version of results from the next section. It is readily checked that for linear systems the above characterizations yield the well-known state-space characterizations of these operators.

5.3 Port-controlled Hamiltonian state space realizations of the nonlinear Hilbert adjoint

We now proceed by studying general state-space realizations of nonlinear Hilbert adjoints of input-output systems. For that, we use the the concept of (time-varying) port-controlled Hamiltonian systems, and we include the initial and terminal condition in the system definition. First, consider a possibly time-varying continuous input-output system $S : L_2^m[t^0, t^1] \to L_2^r[t^0, t^1]$ defined on a (possibly infinite) time interval $t \in [t^0, t^1] \subset \mathbb{R}$. Its state-space realization is given by (5.5). We also consider the operator version of the input-output system S in (5.5) as (5.6). Note that $\mathbb{R}^n \times L_2^m[t^0, t^1]$ is a Hilbert space with the inner product

$$\langle (x, u), (\bar{x}, \bar{u})\rangle_{\mathbb{R}^n \times L_2^m[t^0,t^1]} := \langle x, \bar{x}\rangle_{\mathbb{R}^n} + \langle u, \bar{u}\rangle_{L_2^m[t^0,t^1]}. \tag{5.34}$$

By a specialized version of the Fundamental Theorem of Integral Calculus it is well known that f and h can be factorized. These factorizations can be used in combination with the concept of a port-controlled Hamiltonian system in order to obtain a $2n$ dimensional state-space realization of a nonlinear Hilbert adjoint operator of $\hat{S}$, as is done in the following theorem.

Proposition 5.3.1. *Consider the operator* $\hat{S} : \mathbb{R}^n \times L_2^m[t^0, t^1] \to \mathbb{R}^n \times L_2^r[t^0, t^1]$ *with a state space realization as defined in (5.6) and with* $t \in [t^0, t^1] \subset \mathbb{R}$. *Furthermore, consider the corresponding port-controlled Hamiltonian system* $H_{\hat{S}} : \mathbb{R}^{2n} \times L_2^{m+r}[t^0, t^1] \to \mathbb{R}^{2n} \times L_2^{m+r}[t^0, t^1]$ *given by*

$$
(x^0, p^1, \hat{u}) \mapsto (x^1, p^0, \hat{y}) =
$$

$$
H_{\hat{S}}(x^0, p^1, \hat{u}) : \begin{cases} \dot{\hat{x}} &= \hat{J}(\hat{x}, \hat{u}, t) \frac{\partial \hat{H}}{\partial \hat{x}}^T + \hat{g}(\hat{x}, \hat{u}, t)\, \hat{u} \\ \hat{y} &= \hat{g}^T(\hat{x}, \hat{u}, t) \frac{\partial \hat{H}}{\partial \hat{x}}^T + \hat{D}(\hat{x}, \hat{u}, t)\, \hat{u} \\ \hat{x}(t^0) = (x^0, p^0) \\ \hat{x}(t^1) = (x^1, p^1) \end{cases} \tag{5.35}
$$

with $\hat{x}(t) := (x(t), p(t)) \in \mathbb{R}^n \times \mathbb{R}^n$*,* $\hat{u}(t) := (u(t), u_a(t)) \in \mathbb{R}^m \times \mathbb{R}^r$*,* $\hat{y}(t) := (y_a(t), -y(t)) \in \mathbb{R}^m \times \mathbb{R}^r$ *and*

$$
\hat{H}(\hat{x}) := p^T x \tag{5.36}
$$

$$
\hat{J}(\hat{x}, \hat{u}, t) := \begin{pmatrix} 0 & A(x,u,t) \\ -A^T(x,u,t) & 0 \end{pmatrix} \tag{5.37}
$$

$$
\hat{g}(\hat{x}, \hat{u}, t) := \begin{pmatrix} B(x,u,t) & 0 \\ 0 & -C^T(x,u,t) \end{pmatrix} \tag{5.38}
$$

$$
\hat{D}(\hat{x}, \hat{u}, t) := \begin{pmatrix} 0 & D^T(x,u,t) \\ -D(x,u,t) & 0 \end{pmatrix}. \tag{5.39}
$$

Here $A(x,u,t) \in \mathbb{R}^{n \times n}$*,* $B(x,u,t) \in \mathbb{R}^{n \times m}$*,* $C(x,u,t) \in \mathbb{R}^{r \times n}$ *and* $D(x,u,t) \in \mathbb{R}^{r \times m}$ *are appropriate matrices such that*

$$
f(x,u,t) = A(x,u,t)\, x + B(x,u,t)\, u \tag{5.40}
$$

$$
h(x,u,t) = C(x,u,t)\, x + D(x,u,t)\, u \tag{5.41}
$$

hold. Suppose that

$$
\begin{aligned} \|x^0\| < \infty,\ \|p^1\| < \infty,\ u \in L_2^m[t^0, t^1],\ u_a \in L_2^r[t^0, t^1] \\ \Rightarrow\ \|x^1\| < \infty,\ \|p^0\| < \infty. \end{aligned} \tag{5.42}
$$

Then the mapping $(x^0, p^1, u, u_a) \mapsto (p^0, y_a)$ *corresponding to the state-space realization (5.35) is a state-space realization of a nonlinear Hilbert adjoint operator* $\hat{S}^* : \mathbb{R}^{2n} \times L_2^m[t^0, t^1] \times L_2^r[t^0, t^1] \to \mathbb{R}^n \times L_2^r[t^0, t^1]$ *of* $\hat{S}$*.*

Remark 5.3.1. The port-controlled Hamiltonian system (5.35) reduces down to

$$
H_{\hat{S}} : \begin{cases} \dot{x} = f(x,u,t), & x(t^0) = x^0 \\ \dot{p} = -A^T(x,u,t)\, p - C^T(x,u,t)\, u_a, & p(t^1) = p^1 \\ y_a = B^T(x,u,t)\, p + D^T(x,u,t)\, u_a \\ y = h(x,u,t) \\ x^1 = x(t^1) \\ p^0 = p(t^0) \end{cases} \tag{5.43}
$$

by a direct calculation. The proposition states that the state-space realization corresponding to the mapping $(x^0, p^1, u, u_a) \mapsto (p^0, y_a)$ is a state-space realization of a nonlinear Hilbert adjoint operator $\hat{S}^*$, where the mapping

$(x^0, u) \mapsto (x^1, y)$ coincides with the original operator $\hat{S}$. Assumption (5.42) can be regarded as a stability requirement for the x-subsystem and an anti-stability requirement for the p-subsystem. As in the linear case, (5.42) is always satisfied locally when the time interval $[t^0, t^1]$ is sufficiently small or when the system S in (5.5) with $u = 0$ is locally exponentially stable. Furthermore, the decomposition in (5.40) and (5.41) is always possible on a convex neighborhood of 0, since $f(0, 0, t) \equiv 0$ and $h(0, 0, t) \equiv 0$. Moreover, the matrix valued functions A, B, C and D satisfy

$$\begin{aligned} A(0,0,t) &= \tfrac{\partial f}{\partial x}(0,0,t),\ B(0,0,t) = \tfrac{\partial f}{\partial u}(0,0,t) \\ C(0,0,t) &= \tfrac{\partial h}{\partial x}(0,0,t),\ D(0,0,t) = \tfrac{\partial h}{\partial u}(0,0,t). \end{aligned} \tag{5.44}$$

An explicit procedure to calculate these matrices is given in Lemma 6.1.1. Notice that the decomposition in (5.40) and (5.41) is not unique, which implies that the state-space realization of the nonlinear Hilbert adjoint as given by (5.35) is not unique. △

Proof of Proposition 5.3.1. Since (5.35) is time-varying port-controlled Hamiltonian system we have that

$$\begin{aligned} \frac{d\hat{H}}{dt} &= \frac{\partial \hat{H}}{\partial \hat{x}}\left(\hat{J}\frac{\partial \hat{H}}{\partial \hat{x}}^T + \hat{g}\hat{u}\right) = \frac{\partial \hat{H}}{\partial \hat{x}}\hat{g}\hat{u} = \left(\frac{\partial \hat{H}}{\partial \hat{x}}\hat{g} + \hat{u}^T\hat{D}^T\right)\hat{u} \\ &= \hat{y}^T\hat{u} = y_a^T u - y^T u_a. \end{aligned}$$

Hence,

$$\begin{aligned} \langle y_a, u\rangle_{L_2^m[t^0,t^1]} - \langle y, u_a\rangle_{L_2^r[t^0,t^1]} &= \int_{t^0}^{t^1}(y_a^T u - y^T u_a)dt = \int_{t^0}^{t^1}\frac{d\hat{H}}{dt}dt \\ &= \hat{H}(\hat{x}(t^1)) - \hat{H}(\hat{x}(t^0)) \\ &= \langle x^1, p^1\rangle_{\mathbb{R}^n} - \langle x^0, p^0\rangle_{\mathbb{R}^n}, \end{aligned}$$

i.e.,

$$\langle (x^1, y), (p^1, u_a)\rangle_{\mathbb{R}^n\times L_2^r[t^0,t^1]} = \langle (x^0, u), (p^0, y_a)\rangle_{\mathbb{R}^n\times L_2^m[t^0,t^1]} \tag{5.45}$$

holds. Substituting $(x^1, y) = \hat{S}(x^0, u)$ and $(p^0, y_a) = \hat{S}^*((p^1, u_a), (x^0, u))$ yields the definition of a nonlinear Hilbert adjoint operator

$$\begin{aligned} &\langle \hat{S}(x^0, u), (p^1, u_a)\rangle_{\mathbb{R}^n\times L_2^r[t^0,t^1]} = \\ &\langle (x^0, u), \hat{S}^*((p^1, u_a), (x^0, u))\rangle_{\mathbb{R}^n\times L_2^m[t^0,t^1]}. \end{aligned}$$

This proves the proposition. ■

It should be noted that the characterization given in the above proposition yields a coordinate dependent state space characterization of a nonlinear Hilbert adjoint in the sense that if we apply a coordinate transformation, the port-controlled Hamiltonian structure is lost. This is due to the fact that the Hamiltonian given in (5.36) is intrinsically coordinate dependent. On the other hand, it provides very natural state-space realizations of adjoint operators because it requires rather mild assumptions. The Hamiltonian extension given in the previous subsection is coordinate free, i.e., the Hamiltonian structure is maintained due to the fact that a coordinate transformation is canonical in this case. Nevertheless, it requires more restrictive assumptions, and coordinate dependency is not necessarily an important issue for nonlinear Hilbert adjoint realizations. Of course, in the linear case the port-controlled Hamiltonian adjoint characterization results into the familiar adjoint system.

The following corollary yields a state space realization for the nonlinear Hilbert adjoint when the initial conditions are set to zero.

Corollary 5.3.1. *Consider the system S in (5.5) with the initial condition $x^0 = 0$ and let $S : L_2^m[t^0, t^1] \to L_2^r[t^0, t^1]$ denote the mapping $u \mapsto y$. Suppose the assumption (5.42) holds. Then a state-space realization of the nonlinear Hilbert adjoint $S^* : L_2^m[t^0, t^1] \times L_2^r[t^0, t^1] \to L_2^m[t^0, t^1]$ of S is given by*

$$(u_a, u) \mapsto y_a = S^*(u_a, u) : \begin{cases} \dot{x} &= f(x, u, t) \\ \dot{p} &= -A^T(x, u, t)\, p - C^T(x, u, t)\, u_a \\ y_a &= B^T(x, u, t)\, p + D^T(x, u, t)\, u_a \\ x(t^0) &= 0 \\ p(t^1) &= 0. \end{cases} \tag{5.46}$$

Proof. Substituting $x^0 = 0$ and $p^1 = 0$ in (5.45) reduces to

$$\begin{aligned} \langle y, u_a \rangle_{L_2^r[t^0,t^1]} &= \langle (x^1, y), (0, u_a) \rangle_{\mathbb{R}^n \times L_2^r[t^0,t^1]} \\ &= \langle (0, u), (p^0, y_a) \rangle_{\mathbb{R}^n \times L_2^m[t^0,t^1]} \\ &= \langle u, y_a \rangle_{L_2^m[t^0,t^1]}. \end{aligned}$$

Letting $y = S(u)$ and $y_a = S^*(u_a, u)$ yields the definition of a nonlinear Hilbert adjoint operator

$$\langle S(u), u_a \rangle_{L_2^r[t^0,t^1]} = \langle u, S^*(u_a, u) \rangle_{L_2^m[t^0,t^1]}.$$

This proves the corollary. ■

5.3.1 Observability, controllability and Hankel operators

In this section we apply the results of the previous subsection to obtain state-space realizations for nonlinear Hilbert adjoints of some useful operators in nonlinear control theory, i.e., the nonlinear Hankel, observability and controllability operators. These operators can be seen as natural extensions from the linear version of these operators. Only the time-invariant, input-affine systems are considered without direct feedthrough, that is

$$S : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{5.47}$$

defined for $t \in (-\infty, \infty)$. Assume for the remainder of the paper that $f(x)$ is asymptotically stable. State-space realizations which describe the observability and controllability operators defined in Definition 3.3.2 for $t^0 = -\infty$ and $t^1 = \infty$ are given by

$$x^0 \mapsto y = \mathcal{O}_0(x^0) : \begin{cases} \dot{x} = f(x),\, x(0) = x^0 \\ y = h(x) \end{cases} \tag{5.48}$$

$$u \mapsto x^1 = \mathcal{C}(u) \ : \begin{cases} \dot{x} = f(x) + g(x)\mathcal{F}_-(u),\, x(-\infty) = 0 \\ x^1 = x(0). \end{cases} . \tag{5.49}$$

where $\mathcal{F}_- : L_2^m[0,\infty) \to L_2^m(-\infty, 0]$ denotes the zero input for positive time flipping operator as defined in (3.12) with $u_+ = 0$, i.e.,

$$\mathcal{F}_-(u)(t) := \begin{cases} u(-t),\, t \le 0 \\ \quad 0, \quad t > 0 \end{cases} . \tag{5.50}$$

Furthermore the zero input Hankel operator $\mathcal{H}_0 : L_2^m[0,\infty) \to L_2^r[0,\infty)$ is related to $\mathcal{O}_0$ and $\mathcal{C}$ as given in Proposition 3.3.1, and is mapping past inputs to future outputs. State-space realizations for nonlinear Hilbert adjoint operators of $\mathcal{O}$, $\mathcal{C}$ and $\mathcal{H}_0$ are given in the following proposition.

Proposition 5.3.2. *Consider the operator S with state space realization (5.47). Suppose that the assumption (5.42) in Proposition 5.3.1 holds for the relevant port-controlled Hamiltonian system (5.35). Then state-space realizations of $\mathcal{O}_0^* : L_2^r[0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{C}^* : \mathbb{R}^n \times L_2^m[0,\infty) \to L_2^m[0,\infty)$ and $\mathcal{H}_0^* : L_2^r[0,\infty) \times L_2^m[0,\infty) \to L_2^m[0,\infty)$ are given, respectively, by*

$$(x^0, u_a) \mapsto p^0 = \mathcal{O}_0^*(x^0, u_a) : \begin{cases} \dot{x} &= f(x) \\ \dot{p} &= -A^T(x)\, p - C^T(x)\, u_a \\ p^0 &= p(0) \\ x(0) &= x^0 \\ p(\infty) &= 0 \end{cases} \tag{5.51}$$

$$(p^1, u) \mapsto y_a = \mathcal{C}^*(p^1, u) : \begin{cases} \dot{x} &= f(x) + g(x)\mathcal{F}_-(u) \\ \dot{p} &= -A^T(x)\, p \\ y_a &= \mathcal{F}_+(g^T(x)\, p) \\ x(-\infty) &= 0 \\ p(0) &= p^1 \end{cases} \tag{5.52}$$

$$(u_a, u) \mapsto y_a = \mathcal{H}_0^*(u_a, u) : \begin{cases} \dot{x} &= f(x) + g(x)\, \mathcal{F}_-(u) \\ \dot{p} &= -A^T(x)\, p - C^T(x)\, u_a \\ y_a &= \mathcal{F}_+(g^T(x)\, p) \\ x(-\infty) &= 0 \\ p(\infty) &= 0 \end{cases} \tag{5.53}$$

with matrices $A(x) \in \mathbb{R}^{n\times n}$ *and* $C(x) \in \mathbb{R}^{r\times n}$ *such that* $f(x) \equiv A(x)x$ *and* $h(x) \equiv C(x)x$ *hold. Here* $\mathcal{F}_+ : L_2^m(-\infty, 0] \to L_2^m[0,\infty)$ *is defined by*

$$\mathcal{F}_+(u)(t) := \begin{cases} 0, & t < 0 \\ u(-t), & t \geq 0 \end{cases}. \tag{5.54}$$

Proof. Substitute $t^0 = 0$, $t^1 = \infty$, $p^1 = p(\infty) = 0$ and $u = 0$ into equation (5.45) to obtain

$$\langle y, u_a\rangle_{L_2^r[0,\infty)} = \langle (x^1, y), (0, u_a)\rangle_{\mathbb{R}^n \times L_2^r[0,\infty)} =$$

$$\langle (x^0, 0), (p^0, y_a)\rangle_{\mathbb{R}^n \times L_2^m[0,\infty)} = \langle x^0, p^0\rangle_{\mathbb{R}^n}.$$

Substituting, moreover, $y = \mathcal{O}_0(x^0)$ and $p^0 = \mathcal{O}_0^*(x^0, u_a)$ as in (5.51) yields

$$\langle \mathcal{O}_0(x^0), u_a\rangle_{L_2^r[0,\infty)} = \langle x^0, \mathcal{O}_0^*(x^0, u_a)\rangle_{\mathbb{R}^n}.$$

This proves the first part. The second part can be proved in a similar way. Substituting $t^0 = -\infty$, $t^1 = 0$, $x^0 = x(-\infty) = 0$ and $u_a = 0$ into equation (5.45) yields

$$\begin{aligned} \langle x^1, p^1\rangle_{\mathbb{R}^n} &= \langle (x^1, y), (p^1, 0)\rangle_{\mathbb{R}^n \times L_2^r(-\infty,0]} \\ &= \langle (0, \mathcal{F}_-(u)), (p^0, y_a)\rangle_{\mathbb{R}^n \times L_2^m(-\infty,0]} \\ &= \langle \mathcal{F}_-(u), y_a\rangle_{L_2^m(-\infty,0]} = \langle u, \mathcal{F}_+(y_a)\rangle_{L_2^m[0,\infty)}. \end{aligned} \tag{5.55}$$

Then, substituting $x^1 = \mathcal{C}(u)$ and $y_a = \mathcal{C}^*(p^1, u)$ in (5.52) yields

$$\langle \mathcal{C}(u), p^1\rangle_{\mathbb{R}^n} = \langle u, \mathcal{C}^*(p^1, u)\rangle_{L_2^m(-\infty,0]}.$$

This proves the second part. The last part is proved by noting that equation (5.55) implies that

$$\mathcal{F}_-^* = \mathcal{F}_+, \qquad \mathcal{F}_+^* = \mathcal{F}_-.$$

Combining this with the linear adjoint property of the time flipping operators and the definition of the zero-input Hankel operator, we obtain

$$\begin{aligned} \mathcal{H}_0^*(u_a, u) &= (S \circ \mathcal{F}_-)^*(u_a, u) \\ &= \mathcal{F}_-^* \circ S^*(u_a, \mathcal{F}_-(u)) \\ &= \mathcal{F}_+ \circ S^*(u_a, \mathcal{F}_-(u)) \end{aligned}$$

This completes the proof. ∎

Remark 5.3.2. By using Proposition 3.3.1 we obtain

$$\begin{aligned}\langle \mathcal{H}_0(u), u_a\rangle_{L_2^r[0,\infty)} &= \langle \mathcal{O}_0 \circ \mathcal{C}(u), u_a\rangle_{L_2^r[0,\infty)} \\ &= \langle u, \mathcal{C}^*(\mathcal{O}_0^*(u_a, \mathcal{C}(u)), u)\rangle_{L_2^m[0,\infty)} \\ &= \langle u, \mathcal{H}_0^*(u_a, u)\rangle_{L_2^m[0,\infty)}.\end{aligned}$$

Hence, we obtain for $\mathcal{H}_0^*$ the following equality

$$\mathcal{H}_0^*(u_a, u) = \mathcal{C}^*(\mathcal{O}_0^*(u_a, \mathcal{C}(u)), u). \tag{5.56}$$

Further, it can be seen that the trajectories of the states (x, p) in the realizations of $\mathcal{C}^*$ in (5.52), $\mathcal{O}_0^*$ in (5.51), and $\mathcal{C}$ in (5.49) and the state-space realization in the right hand side of (5.56) coincide. Thus, the combination of the state space realizations corresponding to the operators in (5.56) results in (5.53). △

5.3.2 Observability and controllability functions

In this subsection, duality relationships among the observability and controllability functions, operators and nonlinear Gramian extensions are discussed. It is assumed throughout that L_o and L_c as defined in Definition 3.1.1 exist and are smooth. Specifically, the relation as in the equations (3.14) can now be given as

$$\begin{aligned}L_o(x^0) &= \frac{1}{2}\|\mathcal{O}_0(x^0)\|_{L_2^r}^2 = \frac{1}{2}\langle x^0, \mathcal{O}_0^*(\mathcal{O}_0(x^0), x^0)\rangle_{\mathbb{R}^n} = \frac{1}{2}\langle x^0, p^0\rangle_{\mathbb{R}^n} \\ &=: \frac{1}{2}\langle x^0, \phi(x^0)\rangle_{\mathbb{R}^n}.\end{aligned} \tag{5.57}$$

Here $p^0 = p(0)$ is the initial state of the state-space realization of $\mathcal{O}_0^*$ in (5.51) with input $(x^0, u_a) = (x^0, \mathcal{O}_0(x^0))$. The function $\phi(x^0)$ can always be expressed as $\phi(x^0) = Q(x^0)\, x^0$ using a square symmetric matrix $Q(x^0)$. In the linear case this matrix equals the observability Gramian. Furthermore, note that the relation

$$L_o(x(t)) = \frac{1}{2}\langle x(t), p(t)\rangle_{\mathbb{R}^n} \tag{5.58}$$

holds along the trajectory of the state-space realization (5.51) of $\mathcal{O}_0^*(x^0, \mathcal{O}_0(x^0))$. Particularly, in the case $x(t) \in \mathbb{R}$, the function ϕ in equation (5.57) is readily computed as follows

Example 5.3.1. Consider the system (5.47) with $n = m = r = 1$, i.e., $x(t), u(t), y(t) \in \mathbb{R}$. The observability operator $\mathcal{O}_0 : \mathbb{R} \to L_2[0, \infty)$ and its Hilbert adjoint $\mathcal{O}_0^* : L_2[0, \infty) \times \mathbb{R} \to \mathbb{R}$ are given by

$$x^0 \mapsto y = \mathcal{O}_0(x^0) : \begin{cases} \dot{x} = f(x),\, x(0) = x^0 \\ y = h(x) \end{cases}$$
$$(x^0, u_a) \mapsto p^0 = \mathcal{O}_0^*(x^0, u_a) : \begin{cases} \dot{x} \;= f(x), & x(0) = x^0 \\ \dot{p} \;= -\frac{f(x)}{x}\, p - \frac{h(x)}{x}\, u_a, & p(\infty) = 0 \\ p^0 = p(0) \end{cases}$$

with $f(x)$ and $h(x)$ locally (about 0) Lipschitz continuous. Since ϕ is unique in the case $n = 1$, it follows from (5.58) and (5.57) that the states of $\mathcal{O}_0^*(x^0, \mathcal{O}_0(x^0))$ satisfy $(x(t), p(t)) = (x(t), \phi(x(t)))$ for $\forall t \in [0, \infty)$ with a scalar valued function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\phi(0) = 0$. Therefore

$$\dot{p} = -\frac{f(x)}{x}\, \phi(x) - \frac{h(x)}{x}\, h(x) \equiv \frac{d\phi(x)}{dx}\, \dot{x} = \frac{d\phi(x)}{dx}\, f(x).$$

The solution of this equation is given by

$$\phi(x) = -\frac{1}{x} \int_0^x \frac{h(\xi)^2}{f(\xi)} d\xi.$$

As explained above, the function $(\phi(x)/x)$ is the nonlinear extension of the observability Gramian. Furthermore, the function ϕ can be used to calculate L_o via (5.57). As a very simple illustrating example, take $f(x) = -x$ and $h(x) = x^2$, then it is straightforwardly computed that $\phi(x) = \frac{1}{4}x^3$, and $L_o(x) = \frac{1}{8}x^4$. △

As can be seen from equation (3.15), in the controllability case, there does not exist as nice a relation for $\mathcal{C}^*$. Instead, one must consider $\mathcal{C}^\dagger(x^1)$, the pseudo-inverse of $\mathcal{C}(u)$ yielding the input that asymptotically reaches x^1 (from 0 at $t = -\infty$) with the minimum amount of energy, i.e., equation (3.16). Analogous to equation (5.57) it follows that

$$L_c(x^1) \;=\; \frac{1}{2}\|\mathcal{C}^\dagger(x^1)\|^2_{L_2^m} = \frac{1}{2}\langle x^1, {\mathcal{C}^\dagger}^*(\mathcal{C}^\dagger(x^1), x^1)\rangle_{\mathbb{R}^n} \tag{5.59}$$
$$=: \frac{1}{2}\langle x^1, \varphi(x^1)\rangle_{\mathbb{R}^n}. \tag{5.60}$$

In the linear case $\varphi(x^1) = P^{-1}x^1$ where P is the controllability Gramian. The state-space realization of $\mathcal{C}^\dagger : \mathbb{R}^n \to L_2^m[0, \infty)$ is obtained by noting that the minimum energy input is given by $g^T(x)\frac{\partial L_c}{\partial x}^T(x)$, and by taking the time-reversion into account. Then it is described by

$$x^1 \to y_a = \mathcal{C}^\dagger(x^1) : \begin{cases} \dot{x} \;= -f(x) - g(x)g^T(x)\frac{\partial L_c}{\partial x}^T(x) \;\; x(0) = x^1 \\ y_a = g(x)\frac{\partial L_c}{\partial x}^T(x). \end{cases} \tag{5.61}$$

In the linear case $\mathcal{C}^\dagger(x^1) = \mathcal{C}^*(\varphi(x^1), u) = \mathcal{C}^*(\varphi(x^1))$ reduces to

$$L_c(x^1) = \frac{1}{2}\|\mathcal{C}^*(\varphi(x^1))\|^2_{L_2^m} = \frac{1}{2}\langle\varphi(x^1), P\,\varphi(x^1)\rangle_{\mathbb{R}^n}.$$

However in the nonlinear setting

$$\mathcal{C}^\dagger(x^1) \neq \mathcal{C}^*(\varphi(x^1), u)$$

in general. Nevertheless, we are able to obtain a duality result along the linear lines of thinking in the following proposition. It concerns a duality between the observability and controllability functions.

Proposition 5.3.3. *Consider the system S with state space realization (5.47), observability function $L_o(x)$ and controllability function $L_c(x)$. It is assumed that $f(x)$ is asymptotically stable and that $L_o(x)$ and $L_c(x)$ exist and are smooth. Consider the system*

$$\begin{cases} \dot{p} &= A^T(\phi_i(p))p + C^T(\phi_i(p))u_a \\ y_a &= g^T(\phi_i(p))p. \end{cases} \tag{5.62}$$

Let $x = \phi_c(p)$ denote the inverse mapping of $p = \frac{\partial L_c}{\partial x}^T(x)$. Suppose that (5.62) has observability function $\tilde{L}_o(p)$ and that $i = c$. Then $\tilde{L}_o(p)$ is given by the Legendre transformation

$$\tilde{L}_o(p) = -L_c(x) + p^T x. \tag{5.63}$$

Let $x = \phi_o(p)$ denote the inverse mapping of $p = \frac{\partial L_o}{\partial x}^T(x)$. Suppose that (5.62) has controllability function $\tilde{L}_c(p)$ and that $i = o$. Then $\tilde{L}_c(p)$ is given by the Legendre transformation

$$\tilde{L}_c(p) = -L_o(x) + p^T x. \tag{5.64}$$

Proof. It follows from Theorem 3.1.2 that the controllability function $L_o(x)$ of the system S is the unique anti-stabilizing solution of the Hamilton-Jacobi equation

$$\frac{\partial L_c}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial L_c}{\partial x}(x)g(x)g^T(x)\frac{\partial L_c}{\partial x}^T(x) = 0, \quad L_c(0) = 0, \tag{5.65}$$

and the observability function $\tilde{L}_o(p)$ of (5.62) for $i = c$ is the unique solution of the Lyapunov equation

$$\frac{\partial \tilde{L}_o}{\partial p}(p)A(\phi_c(p))^T p + \frac{1}{2}\underbrace{p^T g(\phi_c(p))g^T(\phi_c(p))p}_{y_a^T y_a} = 0, \quad \tilde{L}_o(0) = 0. \tag{5.66}$$

Here $x = \phi_c(p)$ is the inverse mapping of

$$p = \frac{\partial L_c}{\partial x}^T(x).$$

By applying this transformation it follows that (5.65) and (5.66) are the same. And thus, a solution for (5.66) is given by the Legendre transformation (5.63) which has the property that

$$\frac{\partial \tilde{L}_o}{\partial p}(p) = \phi_c^T(p)$$

holds. Furthermore, this is the observability function of the p-subsystem because the Lyapunov equation has a unique solution. This proves the first part. The second part can be proved in a similar way. This completes the proof. ■

Thus we can prove a certain type of duality between input and output. This is similar to the linear case. Legendre transformations define for physical Hamiltonian systems the physical dual coordinates. However, from the difference of the adjoint notion and the above duality result, it follows that such type of duality does not hold in the exact same way for the general case, although some similarity with physical systems is obtained. For example, note that the p dynamics of (5.62) can be related to the port-controlled Hamiltonian state-space realization (5.43) of nonlinear Hilbert adjoint operators by considering the reverse-time dynamics of the p subsystem of (5.43) with $x = \phi_i(p)$ and by putting $u \equiv 0$ at the appropriate places. Legendre transformations transform a convex function into another convex function, which is necessary to relate the controllability and observability of "dual" dynamics with the original dynamics. The type of duality of Proposition 5.3.3 is the so called *duality in the sense of Young* [4], which often appears in the literature of both optimal control and optimization theory.

Notes and references

The material of this chapter is mainly taken from [21], [22], [24], and [77].

The notion of an *adjoint map* or *adjoint operator* can be found in a wide variety of mathematical contexts: functional analysis [81], differential geometry [1], differential algebra [89], representation theory for Lie algebras [7] and topological vector spaces [66]. These concepts appear primarily in a linear setting, i.e., linear maps on linear spaces, and thus are closely related to one another.

In linear systems theory, the important notion of an adjoint *state-space system* is usually defined in terms of signal sets that form Hilbert spaces, either L_2 or H_2 [90].

For example, in [6] the notion of an adjoint map is defined in terms of a dual map on a topological vector space. This idea is distinct from the adjoint map that appears in [11] which employs the Gâteaux derivative of the operator when it is well defined. Other distinct definitions can be found in [2, 13]. A set of definitions that is useful for system theoretical considerations, and in particular realization theory, is given in [12, 47, 57, 79, 86], although these papers are not addressing this application.

In a nonlinear state-space context, the adjoint system has appeared in [14], but can be given an input-output interpretation using the nonlinear Hilbert adjoint operator. [21, 22, 24].

[34, 75] [66] [78]

The above definition is more general than the definition of an adjoint operator given in [11], where the identity (5.2) is only required to hold when $y = u$. To study singular value structures, $y = T(u)$ should also be admissible. The adjoint definition of [11] is too limited for this purpose. Definition 5.1.1 is slightly different from the definition that appeared in [34, 75] since here linearity in y is an additional requirement.

In [12, 47, 57, 79, 86] the characterization of the adjoint operator given in Theorem 5.1.2 (or more correctly Theorem 5.1.3) is basically used as the definition of a unique adjoint for a homogeneous operator. Equation (5.2) is simply viewed as a property of this adjoint operator. In [79] the definition is further extended to handle homogeneous operators that depend on a single parameter $\epsilon \in [0, 1]$. In [12] nonhomogeneous operators with boundary conditions are also considered. In our case, the adjoint operator in Theorem 5.1.2 is just one of many possible solutions to Definition 5.1.1. Different definitions of adjoint operators can be found in [2] and [13]. In [2] a pseudo adjoint operator is considered in the context of Lipschitz operators. The definition in [13] is introduced specifically for solving nonlinear partial differential equations by using a nonlinear semigroup generated by an accretive operator.

[81]

[51]

Since (5.35) is time-varying port-controlled Hamiltonian system (see e.g., [25, 20])

, e.g., [90]

The original definitions of these operators via Chen-Fliess functional expansions along with more detailed discussions can be found in [34, 75].

[73].

The type of duality of Proposition 5.3.3 is the so called *duality in the sense of Young* [4], which often appears in the literature of both optimal control [87] and optimization theory [84].

[71, 48]

7. Model reduction via balancing and the Hankel operator

In the theory of continuous time linear systems, the Hankel operator supplies a set of similarity invariants, the so called Hankel singular values, which can be used to quantify the importance of each state in the corresponding input-output system [42]. The Hankel singular values are most easily computed in a state space setting using the product of the Gramian matrices, though intrinsically they depend only on the given input-output mapping. The linear Hankel theory is rather complete and the relations between and interpretations in the state space and input-output settings are fully understood.

The nonlinear extension of the state space concept of balanced realizations has been introduced in Chapter 6, mainly based on studying the past input energy and the future output energy. Recently, the relation of the state space notion of balancing for nonlinear systems with the nonlinear input-output Hankel operator has been considered, see e.g. [34, 75]. In particular, singular value functions which are nonlinear state space extension of the Hankel singular values in the linear case play an important role in the nonlinear Hankel theory. It has been shown that singular value functions are closely related to Hankel operators [34, 74]. However, there are some major differences with the linear theory, i.e., studying similarity invariance of singular value functions in relation to the nonlinear Hankel operator can be done via several interpretations of the concept of similarity invariance and may result in different conclusions. In this chapter we use the input-output interpretation to study the differential eigenstructure of the nonlinear Hankel operator, and show that this results in a new characterization of Hankel singular value functions for nonlinear systems. The relation with the state space characterization of the singular value functions is also considered.

In order to study the singular value structure of nonlinear operators, we need the results of Chapter 5 on adjoint operators, and in particular the results on the adjoints of the differentials of the Hankel, controllability and observability operators. Then we apply these results to study both the singular value structure of the Hankel operator and the eigenstructure of the Gâteaux differential of the square norm of that operator. It is shown that the eigenstructure of these operators are closely related. This eigenstructure derives an alternative definition of the singular value functions, which have a stronger relationship with the Hankel norm for nonlinear systems, other than the singular value functions given in Chapter 6 have. Furthermore a new input-normal/output-diagonalization procedure for nonlinear systems is derived based on the differential eigenstructure.

7.1 The Hankel operator and state-space balancing

The nonlinear Hilbert adjoint as defined in Section 5.1 can be related to the state-space balancing as presented in Chapter 6 in a limited sense. Let $\mathcal{H}_0(\hat{u})$ be defined as in Proposition 3.3.1, $\mathcal{F}_-$ be the time flipping operator, $\mathcal{O}_0(x)$ and $\mathcal{C}(\hat{u})$ be the corresponding (zero-input) observability and controllability operators, respectively, and recall the equations (3.14), (3.15) and (3.16).

Theorem 7.1.1. *Let (f, g, h) be an analytic n dimensional input-normal/output-diagonal realization of a causal L_2-stable input-output mapping S on a neighborhood W of 0. Define on W the collection of component vectors $\tilde{z}_j = (0, \ldots, 0, z_j, 0, \ldots, 0)$ for $j = 1, 2, \ldots, n$, and the functions $\hat{\sigma}^2(z_j) = \tau(\tilde{z}_j)$. Let v_j be the minimum energy input which drives the state from $z(-\infty) = 0$ to $z(0) = \tilde{z}_j$ and define $\hat{v}_j = \mathcal{F}_-(v_j)$. Then the functions $\{\hat{\sigma}_j\}_{j=1}^n$ are singular value functions of the Hankel operator $\mathcal{H}_0$ in the following sense:*

$$\langle \hat{v}_j, (\mathcal{H}_0^* \mathcal{H}_0)(\hat{v}_j)\rangle_{L_2} = \hat{\sigma}_j^2(z_j)\langle \hat{v}_j, \hat{v}_j\rangle_{L_2}, \quad j = 1, 2, \ldots n. \tag{7.1}$$

Proof: The following equalities follow from the various assumptions above:

$$\begin{aligned}\langle \hat{v}_j, (\mathcal{H}_0^* \mathcal{H}_0)(\hat{v}_j)\rangle_{L_2} &= \langle \hat{v}_j, \mathcal{H}_0^*(\mathcal{H}_0(\hat{v}_j), \hat{v}_j)\rangle_{L_2} \\ &= \langle \mathcal{H}_0(\hat{v}_j), \mathcal{H}_0(\hat{v}_j)\rangle_{L_2} = \langle \mathcal{O}_0\,\mathcal{C}(\hat{v}_j), \mathcal{O}_0\,\mathcal{C}(\hat{v}_j)\rangle_{L_2} \\ &= \langle \mathcal{O}_0(\tilde{z}_j), \mathcal{O}_0(\tilde{z}_j)\rangle_{L_2} = 2\,L_o(\tilde{z}_j) \\ &= \tau_j(\tilde{z}_j) z_j^2 = \hat{\sigma}_j^2(z_j) \cdot 2L_c(\tilde{z}_j) \\ &= \hat{\sigma}_j^2(z_j)\,\langle \hat{v}_j, \hat{v}_j\rangle_{L_2}.\end{aligned}$$
∎

If (f, g, h) is a balanced form, as defined in Chapter 6, then $\sigma_j(z_j) = \hat{\sigma}_j(z_j)$. Clearly, equation (7.1) describes a more limited sense of a singular value than in the linear case since it does not necessarily yield a spectral decomposition analogous to (2.3)-(2.4). Moreover, in the linear case, the following relation follows from the eigenstructure given by (2.3) and (2.4)

$$\mathcal{H}_0^* \circ \mathcal{H}_0(v_i) = \sigma_i^2 v_i \tag{7.2}$$

for each eigenvector v_i. In the nonlinear case, however, such a relation does not follow from the above theorem.

7.2 Differential eigenstructure of the Hankel operator

We now proceed with establishing a much stronger singular value structure for the Hankel operator. We first focus on the nonlinear extension of (7.2). It concerns the extension to the nonlinear case of the eigenstructure of the operator $\mathcal{H}_0^* \circ \mathcal{H}_0$, that is, it concerns the solution $\lambda \in \mathbb{R}$ and $v \in L_2^m[0,\infty)\backslash\{0\}$ of a linear equation

$$\mathcal{H}_0^* \circ \mathcal{H}_0(v) = \lambda\, v. \tag{7.3}$$

All nonzero solutions of λ are given by $\lambda = \sigma_i^2$, $i \in \{1,2,\ldots,n\}$ where σ_i's are Hankel singular values.

A simplest nonlinear generalization of (7.3) with a nonlinear Hilbert adjoint $\mathcal{H}_0^*(\cdot,\cdot)$ may be $\mathcal{H}_0^*(\mathcal{H}_0(v),v) = \lambda\, v$ which is a stronger version of (7.1) in Theorem 7.1.1. Unfortunately the solution of the above equation is not found so far. However, we can also consider the eigenstructure of another operator $u \mapsto (d\mathcal{H}_0(u))^* \circ \mathcal{H}_0(u)$ characterized by

$$(d\mathcal{H}_0(v))^* \circ \mathcal{H}_0(v) = \lambda\, v \tag{7.4}$$

where $\lambda \in \mathbb{R}$ is an eigenvalue and $v \in L_2^m[0,\infty)\backslash\{0\}$ the corresponding eigenvector. This operator has a realization given by the Hamiltonian extension, as described in Chapter 5, which is based on similar ideas as given in [15, 78, 5]. Note that the operator $u \mapsto (d\mathcal{H}_0(u))^* \circ \mathcal{H}_0(u)$ is *nonlinear* so in general the eigenstructure is different from the linear case. First of all, we prove the fact that this eigenstructure has a close relationship with the Hankel norm of S defined by

$$\|S\|_H := \sup_{\substack{u \in L_2^m[0,\infty)\\ u\neq 0}} \frac{\|\mathcal{H}_0(u)\|_2}{\|u\|_2}. \tag{7.5}$$

Theorem 7.2.1. *Consider an operator S with its Hankel operator $\mathcal{H}_0$. Assume that the Hankel operator $\mathcal{H}_0$ is differentiable. Let $v \in L_2^m[0,\infty)\backslash\{0\}$ denote the input which achieves the maximization in the definition of Hankel norm in (7.5), namely*

$$\|S\|_H = \frac{\|\mathcal{H}_0(v)\|_2}{\|v\|_2}. \tag{7.6}$$

Then v satisfies (7.4) with the eigenvalue $\lambda = \|S\|_H^2$.

Proof: The differential of $\|\mathcal{H}_0(u)\|_2/\|u\|_2$ (in the direction δu) satisfies

$$\begin{aligned}
d\left(\frac{\|\mathcal{H}_0(u)\|_2}{\|u\|_2}\right)(\delta u) &= \frac{\|u\|_2 d(\|\mathcal{H}_0(u)\|_2)(\delta u) - \|\mathcal{H}_0(u)\|_2 d(\|u\|_2)(\delta u)}{\|u\|_2^2}\\
&= \frac{\|u\|_2/\|\mathcal{H}_0(u)\|_2\langle (d\mathcal{H}_0(u))^* \circ \mathcal{H}_0(u), \delta u\rangle - \|\mathcal{H}_0(u)\|_2/\|u\|_2\langle u, \delta u\rangle}{\|u\|_2^2}\\
&= \frac{\langle (d\mathcal{H}_0(u))^* \circ \mathcal{H}_0(u) - (\|\mathcal{H}_0(u)\|_2/\|u\|_2)^2 u,\ \delta u\rangle}{\|u\|_2\ \|\mathcal{H}_0(u)\|_2} \equiv 0
\end{aligned}$$

for all variations δu at $u = v$ because it is a critical point. This reduces to

$$(d\mathcal{H}(v))^* \circ \mathcal{H}(v) \equiv (\|\mathcal{H}_0(v)\|_2/\|v\|_2)^2\, v = \|S\|_H^2\, v$$

which proves the theorem. ∎

Theorem 7.2.1 uses the necessary condition for maximization that the differential should be zero at the maximum. For the maximization in the definition of the Hankel norm (7.5) it is necessary that the input v satisfies (7.4). Therefore the eigenstructure (7.4) is worth investigating and the solutions of (7.4) are useful in what follows indeed. Now we continue to study more precise properties of the differential eigenstructure (7.4) by adopting the following assumption.

Assumption A1 Suppose that the system S in (5.18) is asymptotically stable about 0, and that there exist open neighborhoods $\mathcal{X}^0 \subset \mathbb{R}^n$ of 0 and $\mathcal{U} \subset L_2^m[0,\infty)$ of 0 such that the operators

$$\mathcal{O}_0 : \mathcal{X}^0 \to L_2^r[0,\infty) \tag{7.7}$$

$$\mathcal{C} : \mathcal{U} \to \mathcal{X}^0 \tag{7.8}$$

$$\mathcal{C}^\dagger : \mathcal{X}^0 \to \mathcal{U} \tag{7.9}$$

exist and are differentiable.

Firstly, the following lemma gives the complete characterization of the eigenvectors corresponding to non-zero eigenvalues.

Lemma 7.2.1. *Consider the system S in (5.18). Suppose that Assumption A1 holds. Then a pair $\lambda \in \mathbb{R}\backslash\{0\}$ and $v \in \mathcal{U}\backslash\{0\}$ is a pair of eigenvalues and eigenvectors of the mapping $u \mapsto (d\mathcal{H}_0(u))^* \circ \mathcal{H}_0(u)$ if and only if there exists $x^0 \in \mathcal{X}^0\backslash\{0\}$ such that λ and v satisfy*

$$\begin{cases} \dot{x} = \frac{\partial H(x,p)}{\partial p}^T, & x(0) = x^0,\ x(\infty) = 0 \\ \dot{p} = -\frac{\partial H(x,p)}{\partial x}^T, & p(0) = \frac{1}{\lambda}\frac{\partial L_o}{\partial x}^T(x^0) \\ v = g^T(x)p \end{cases} \tag{7.10}$$

with the Hamiltonian

$$H(x,p) = -p^T(f(x) + \frac{1}{2}g(x)g(x)^T p).$$

Proof: Necessity is proved first. Instead of considering the state-space realization of the operator $y_a = (d\mathcal{H}_0(u))^* \circ \mathcal{H}_0(u)$ directly, we use the equation (5.33) as in the proof of Lemma 5.2.2. Note that both $\mathcal{O}_0$ and $\mathcal{C}$ are differentiable because of Assumption A1. We can observe

$$y_a = (d\mathcal{H}_0(u))^* \circ \mathcal{H}_0(u) = (d\mathcal{C}(u))^* \circ (d\mathcal{O}_0(\mathcal{C}(u)))^* \circ \mathcal{O}_0 \circ \mathcal{C}(u). \tag{7.11}$$

Let

$x^0 := \mathcal{C}(u).$

Then (7.11) reduces to

$$y_a = (d\mathcal{C}(u))^* \circ (d\mathcal{O}_0(x^0))^* \circ \mathcal{O}_0(x^0). \tag{7.12}$$

Next we consider the Gâteaux differential of $L_o(x^0)$ in the direction ζ

$$\begin{aligned}\frac{\partial L_o(x^0)}{\partial x^0}\,\zeta &= dL_o(x^0)(\zeta) = \frac{1}{2}d\|\mathcal{O}_0(x^0)\|_2^2(\zeta) = \langle\, \mathcal{O}_0(x^0), d\mathcal{O}_0(x^0)(\zeta)\,\rangle_{L_2^r}\\ &= \langle\, (d\mathcal{O}_0(x^0))^* \circ \mathcal{O}_0(x^0), \zeta\,\rangle_{\mathbb{R}^n}.\end{aligned}$$

Note that Assumption A1 implies that $x^0 \in \mathcal{X}^0$ and and that $L_o(x)$ is differentiable on $\mathcal{X}^0$. This means that

$$d\mathcal{O}_0(x^0))^* \circ \mathcal{O}_0(x^0) = \frac{\partial L_o(x^0)}{\partial x^0}^T.$$

Hence from (7.12) it follows that

$$y_a = (d\mathcal{C}(u))^*(\frac{\partial L_o(x^0)}{\partial x^0}^T).$$

It follows from Lemma 5.2.2 that the state-space realization of this operator is given by

$$\begin{cases}\dot{x} \;= f(x) + g(x)\,\mathcal{F}_-(u) \qquad x(-\infty) = 0\\ \dot{p} \;= -\frac{\partial(f+g\mathcal{F}_-(u))}{\partial x}^T(x)\,p\;\, p(0) = \frac{\partial L_o}{\partial x}^T(x^0)\\ y_a = \mathcal{F}_+(g^T(x)\,p)\end{cases}. \tag{7.13}$$

If we consider the reverse-time expression of this system (with x and p now representing the reverse time state variables) given by

$$\begin{cases}\dot{x} \;= -f(x) - g(x)u \quad x(0) = x^0 \;\; (x(\infty) = 0)\\ \dot{p} \;= \frac{\partial(f+gu)}{\partial x}^T(x)\,p \qquad p(0) = \frac{\partial L_o}{\partial x}^T(x^0)\\ y_a = g^T(x)\,p\end{cases}. \tag{7.14}$$

Now we have the causal state-space expression of the operator $y_a = (d\mathcal{H}_0(u))^* \circ \mathcal{H}_0(u)$ using x^0. Suppose $u = v$ and $y_a = \lambda v$ hold, i.e. the pair λ and v is the pair of eigenvalue and eigenvector. Then we have

$$v = \frac{1}{\lambda}y_a = \frac{1}{\lambda}g^T(x)p.$$

Let $\bar{p} := (1/\lambda)p$, then

$$\begin{cases}\dot{x} = -f(x) - g(x)g^T(x)\bar{p} \qquad x(0) = x^0\\ \dot{\bar{p}} = \frac{1}{\lambda}\dot{p}\\ \quad = \frac{1}{\lambda}\frac{\partial(f+(1/2)gg^T\bar{p})}{\partial x}^T p\\ \quad = \frac{\partial(f+(1/2)gg^T\bar{p})}{\partial x}^T \bar{p} \qquad \bar{p}(0) = \frac{1}{\lambda}\frac{\partial L_o}{\partial x}^T(x^0)\\ v = g^T(x)\bar{p}\end{cases}.$$

Sufficiency follows straightforwardly from the converse arguments. This completes the proof. ■

Lemma 7.2.1 gives the characterization of all pairs of eigenvalues and eigenvectors. Next we concentrate on a special class of these pairs. They are closely related to the energy functions $L_o(x)$ and $L_c(x)$.

Lemma 7.2.2. *Consider the system S in (5.18). Suppose that Assumption A1 holds and that there exist $\lambda \in \mathbb{R}\backslash\{0\}$ and $x^0 \in \mathcal{X}^0\backslash\{0\}$ satisfying*

$$\frac{\partial L_o}{\partial x}(x^0) = \lambda \frac{\partial L_c}{\partial x}(x^0). \tag{7.15}$$

Then λ is an eigenvalue of the mapping $u \mapsto (d\mathcal{H}_0(u))^* \circ \mathcal{H}_0(u)$ corresponding to the eigenvector

$$v = \mathcal{C}^\dagger(x^0). \tag{7.16}$$

Proof: For Assumption A1, the operator $\mathcal{C}^\dagger$ exists. Its state-space realization is given as

$$\mathcal{C}^\dagger : x^0 \mapsto y_a \begin{cases} \dot{x} \;=\; -f(x) - g(x)g^T(x)\frac{\partial L_c}{\partial x}^T \; x(0) = x^0 \\ y_a \;=\; g^T(x)\frac{\partial L_c}{\partial x}^T \end{cases} .$$

If the condition (7.15) holds, then

$$p(0) = \frac{1}{\lambda}\frac{\partial L_o}{\partial x}^T(x^0) = \frac{\partial L_c}{\partial x}^T(x^0).$$

Combined with the dynamics for p in (7.10), with the Hamilton-Jacobi Bellman equation for L_c in (3.5), and with Lemma 5.2.2, this implies that

$$p(t) \equiv \frac{\partial L_c}{\partial x}^T(x(t))$$

holds along the trajectory of $(d\mathcal{H}_0(u))^* \circ \mathcal{H}_0(u)$. ■

Remark 7.2.1. Note that in the linear case the Gramians take the role of the energy functions in the previous two lemmas, resulting in well-known results for linear systems. Indeed, for linear systems equation (7.15) reduces to

$$x^{0T} QP = \lambda\, x^{0T} \tag{7.17}$$

with observability and controllability Gramians Q and P respectively. △

Although Lemma 7.2.2 gives a sufficient condition for pairs of eigenvalues and eigenvectors in terms of the energy functions, it does not say anything about the existence of pairs x_0 and λ that fulfill (7.15). We now continue to investigate the necessity of this condition. In order to proceed, we define two scalar functions

$$\rho_{\max}(c) := \sup_{\substack{u \in \mathcal{C}^{\dagger}(\mathcal{X}^0) \\ \|u\|_2 = c}} \frac{\|\mathcal{H}_0(u)\|_2}{\|u\|_2} = \sup_{\substack{u \in \mathcal{U} \\ \|u\|_2 = c}} \frac{\|\mathcal{H}_0(u)\|_2}{\|u\|_2} \tag{7.18}$$

$$\rho_{\min}(c) := \inf_{\substack{u \in \mathcal{C}^{\dagger}(\mathcal{X}^0) \\ \|u\|_2 = c}} \frac{\|\mathcal{H}_0(u)\|_2}{\|u\|_2}. \tag{7.19}$$

$\rho_{\max}$ is closely related to the Hankel norm because if $\mathcal{U} = L_2^m[0,\infty)$ then

$$\|S\|_H = \sup_{\substack{u \in L_2^m[0,\infty) \\ u \neq 0}} \frac{\|\mathcal{H}_0(u)\|_2}{\|u\|_2} = \sup_{c>0} \rho_{\max}(c), \tag{7.20}$$

that is, $\rho_{\max}(c)$ represents the Hankel norm under the fixed input magnitude $\|u\|_2 = c$. For $\rho_{\min}(c)$ we do not have such an interpretation. Furthermore, $\rho_{\max}$ and $\rho_{\min}$ equal the maximum and minimum Hankel singular values, respectively, in the linear case. Therefore these functions can be seen as an alternative nonlinear extension of Hankel singular values other than the singular value functions $\tau_i(z)$, $i = 1, \ldots, n$, in Theorem 6.1.1. Now the result on the necessity is stated.

Theorem 7.2.2. *Consider the system S in (5.18). Suppose that Assumption A1 holds. Let $v_{\max}(c)$ and $v_{\min}(c)$ denote the inputs which achieve the maximization and minimization under an arbitrary input magnitude $\|u\|_2 = c$, in the definition of $\rho_{\max}$ and $\rho_{\min}$ in (7.18) and (7.19) respectively. i.e. they satisfy*

$$c = \|v_i(c)\|_2 \tag{7.21}$$

$$\rho_i(c) = \frac{\|\mathcal{H}_0(v_i(c))\|_2}{\|v_i(c)\|_2} \tag{7.22}$$

for $i \in \{\max, \min\}$. Then $v_{\max}(c)$ and $v_{\min}(c)$ are the eigenvectors of $u \mapsto (d\mathcal{H}_0(u))^* \circ \mathcal{H}_0(u)$ with respect to the following eigenvalues $\lambda_{\max}(c)$ and $\lambda_{\min}(c)$ respectively.

$$\lambda_i(c) = \rho_i^2(c) + \frac{c}{2}\frac{d\rho_i^2(c)}{dc}, \quad i = \{\max, \min\} \tag{7.23}$$

Furthermore, both pairs $(\lambda, x^0) = (\lambda_{\max}(c), \mathcal{C}(v_{\max}(c)))$ and $(\lambda_{\min}(c), \mathcal{C}(v_{\min}(c)))$ satisfy the condition (7.15) in Lemma 7.2.2.

Proof: Let i denote the index such that $i \in \{\max, \min\}$. Firstly we define $\xi_i(c) := \mathcal{C}(v_i(c))$ and show the existence of $\lambda_i(c)$ such that

$$\frac{\partial L_o}{\partial x}(\xi_i(c)) = \lambda_i(c)\frac{\partial L_c}{\partial x}(\xi_i(c)). \tag{7.24}$$

To this effect, let the level set of $L_c(x)$ be given by

$$\mathcal{X}_{L_c=k} := \{\, x \mid L_c(x) = k \,\}.$$

Then $\xi_i(c) \in \mathcal{X}_{L_c=\frac{c^2}{2}}$ follows from the fact that $v_i(c)$ is the input which minimizes the input energy. Indeed $\xi_i(c) \in \mathcal{X}_{L_c=\frac{c^2}{2}}$ denotes the set of the states derived by the input $\|u\|_2 = c$. Consider a curve $\eta(s) \in \mathcal{X}_{L_c=\frac{c^2}{2}}$ parameterized by a scalar variable s such that $\eta(0) = \xi_i(c)$ holds. Since $\eta(s)$ is contained in the level set $\mathcal{X}_{L_c=\frac{c^2}{2}}$,

$$\frac{dL_c(\eta(s))}{ds} = \frac{\partial L_c(\eta)}{\partial \eta}\frac{d\eta(s)}{ds} = 0 \tag{7.25}$$

holds along $\eta(s)$. Next, from the definition, we can observe the following relations

$$\sup_{\substack{u\in\mathcal{C}^\dagger(\mathcal{X}^0)\\ \|u\|_2=c}} \frac{\|\mathcal{H}_0(u)\|_2}{\|u\|_2} = \sup_{x\in\mathcal{X}^0\cap\mathcal{X}_{L_c=\frac{c^2}{2}}} \sqrt{\frac{L_o(x)}{L_c(x)}}$$

$$\inf_{\substack{u\in\mathcal{C}^\dagger(\mathcal{X}^0)\\ \|u\|_2=c}} \frac{\|\mathcal{H}_0(u)\|_2}{\|u\|_2} = \sup_{x\in\mathcal{X}^0\cap\mathcal{X}_{L_c=\frac{c^2}{2}}} \sqrt{\frac{L_o(x)}{L_c(x)}}.$$

This implies that $\eta(0) = \xi_i(c)$ maximizes (minimizes) the value (L_o/L_c) in the level set $\mathcal{X}_{L_c=\frac{c^2}{2}}$. Therefore we obtain

$$\left.\frac{d\frac{L_o(\eta(s))}{L_c(\eta(s))}}{ds}\right|_{s=0} = \frac{2}{c^2}\left.\frac{dL_o(\eta(s))}{ds}\right|_{s=0} = \frac{2}{c^2}\frac{\partial L_o(\eta)}{\partial\eta}\left.\frac{d\eta(s)}{ds}\right|_{s=0} = 0. \tag{7.26}$$

The equations (7.25) and (7.26) have to hold for all curves $\eta(s) \in \mathcal{X}_{L_c=\frac{c^2}{2}}$. Namely both $(\partial L_o/\partial x)$ and $(\partial L_c/\partial x)$ are orthogonal to the tangent space of $\mathcal{X}_{L_c=\frac{c^2}{2}}$ at $x = \xi_i(c)$. Because this tangent space is $(n-1)$-dimensional, we can conclude $(\partial L_o/\partial x)$ and $(\partial L_c/\partial x)$ are linearly dependent at $x = \xi_i(c)$. Therefore there exists a scalar constant $\lambda_i(c)$ such that (7.24) holds. Remember that $v_i(c)$ can be described by $v_i(c) = \mathcal{C}^\dagger(\xi_i(c))$. Then it follows directly from Lemma 7.2.2 that $v_i(c) = \mathcal{C}^\dagger(\xi_i(c))$ is the eigenvector of $u \mapsto (d\mathcal{H}_0(u))^* \circ \mathcal{H}_0(u)$ with respect to $\lambda_i(c)$.

Secondly we prove the equation (7.23). By the above discussion, for any vector $\zeta \in \mathbb{R}^n$ which is not orthogonal to the tangent space of $\mathcal{X}_{L_c=\frac{c^2}{2}}$ at $x = \xi_i(c)$, $\lambda_i(c)$ can be expressed as

$$\lambda_i(c) = \frac{\frac{\partial L_o}{\partial x}\zeta}{\frac{\partial L_c}{\partial x}\zeta}.$$

Let ζ be the directional differential $(d\varsigma(s)/ds)$ of another curve $\varsigma(s) = x_i^s$ passing through the maximizing (minimizing) state. Namely it goes across the level set $\mathcal{X}_{L_c=\frac{c^2}{2}}$ through $x = \xi_i(c)$ and $\varsigma(c) = \xi_i(c)$ holds. Then we can obtain

$$\lambda_i(c) = \left.\frac{\frac{\partial L_o(\varsigma)}{\partial \varsigma}\frac{d\varsigma(s)}{ds}}{\frac{\partial L_c(\varsigma)}{\partial \varsigma}\frac{d\varsigma(s)}{ds}}\right|_{s=c} = \left.\frac{\frac{dL_o(\varsigma(s))}{ds}}{\frac{dL_c(\varsigma(s))}{ds}}\right|_{s=c} = \frac{\frac{d(\rho_i^2(c)\, c^2/2)}{dc}}{\frac{d(c^2/2)}{dc}}$$

$$= \frac{c\rho_i^2(c) + \frac{c^2}{2}\frac{d\rho_i^2(c)}{dc}}{c} = \rho_i^2(c) + \frac{c}{2}\frac{d\rho_i^2(c)}{dc}$$

because of the definition of $\rho_i(c)$ (7.18) and (7.19). ■

The above result only focuses on the maximum and minimum values on the level sets. For linear systems this results in the maximum and minimum Hankel singular values. In that case, by "ruling" out these directions, and further using similar arguments, the other Hankel singular values can also be obtained. By using a similar method we can extend this result in order to obtain a new type of singular value functions, the so-called *axis singular value functions* $\rho_i(s)$'s, $i \in \{1, 2, \ldots, n\}$ which are mappings from $\mathbb{R} \to [0, \infty) \subset \mathbb{R}$. These functions fully characterize the solutions of the differential eigenstructure (7.4) (and (7.15)). The result can be obtained under the following assumption.

Assumption A2 Suppose that the Jacobian linearization of the system S has nonzero distinct Hankel singular values.

Theorem 7.2.3. *Consider the system S in (5.18). Suppose that Assumptions A1 and A2 hold. Then there exists a neighborhood $U \subset \mathbb{R}$ of 0, n smooth functions $\rho_i : U \to [0, \infty)$'s, $i \in \{1, 2, \ldots, n\}$ (the so-called* axis singular value functions*) such that*

$$\min\{\rho_i(s), \rho_i(-s)\} \geq \max\{\rho_{i+1}(s), \rho_{i+1}(-s)\} \tag{7.27}$$

holds for all $s \in U$ and $\forall i \in \{1, 2, \ldots, n-1\}$ and that there exist n distinct smooth curves $\xi_i : U \to \mathbb{R}^n$ satisfying $\xi_i(0) = 0$ and

$$L_c(\xi_i(s)) = \frac{s^2}{2}, \quad L_o(\xi_i(s)) = \frac{{\rho_i}^2(s)\, s^2}{2} \tag{7.28}$$

$$\frac{\partial L_o}{\partial x}(\xi_i(s)) = \lambda_i(s)\frac{\partial L_c}{\partial x}(\xi_i(s)) \tag{7.29}$$

with

$$\lambda_i(s) := {\rho_i}^2(s) + \frac{s}{2}\frac{d{\rho_i}^2(s)}{ds}. \tag{7.30}$$

Furthermore, $\max\{\rho_1(s), \rho_1(-s)\}$ *and* $\min\{\rho_n(s), \rho_n(-s)\}$ *coincide with* $\rho_{\max}(s)$ *and* $\rho_{\min}(s)$ *respectively for all* $s \geq 0$. *In particular, if* $U = \mathbb{R}$, *then*

$$\|S\|_H = \sup_{s \in \mathbb{R}} \rho_1(s). \tag{7.31}$$

Proof: Suppose the state-space realization is in input-normal form. Consider the equation (7.15). For the smoothness of $\partial L_o/\partial x$, there exists a smooth matrix valued function $Q : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ such that

$$\frac{\partial L_o}{\partial x} = x^T Q^T(x).$$

Assumption A2 implies that there exists a neighborhood of 0 on which $Q(x)$ is decomposed as

$$\tilde{S}^{-1}(x)Q(x)\tilde{S}(x) = \mathrm{diag}(q_1(x), q_2(x), \dots, q_n(x))$$

where $q_1 \geq q_2 \geq \dots \geq q_n$ and where $\tilde{S} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is a nonsingular smooth matrix valued function with $s_i : \mathbb{R}^n \to \mathbb{R}^n$'s

$$\tilde{S}(x) = (s_1(x),\ s_2(x), \dots, s_n(x)).$$

Hence (7.15) reduces to

$$\tilde{S}(x)\ \mathrm{diag}(q_1(x), q_2(x), \dots, q_n(x))\ \tilde{S}^{-1}(x)\ x = \lambda\, x. \tag{7.32}$$

Consider the sphere $\mathcal{X}_{\|x\|=c} := \{\ x \mid \|x\| = c\}$ and mappings $\tilde{v}_i : \mathcal{X}_{\|x\|=c} \to \mathcal{X}_{\|x\|=c}$'s defined by

$$\tilde{s}_i :\ x \mapsto \frac{c}{\|s_i(x)\|}\ s_i(x). \tag{7.33}$$

Then there exists a neighborhood $U \subset \mathcal{X}^0$ of 0 on which, for sufficiently small $c > 0$, we can choose $2n$ closed sets[1] $X_i^j \subset \mathcal{X}_{\|x\|=c}$'s ($i \in \{1, 2, \dots, n\}$, $j \in \{+, -\}$) which are homeomorphic to the $n-1$ dimensional unit disc $D^{n-1} = \{x \in \mathbb{R}^{n-1} \mid \|x\| \leq 1\}$ satisfying

$$x \in X_i^j \quad \Rightarrow \quad \tilde{s}_i(x) \in X_i^j.$$

This results from the fact that $s_i(0)$ equals the eigenvector of the observability Gramian of the Jacobian linearization of the original system and $s_i(x)$ is smooth in a neighborhood of the origin. (The image $\tilde{s}_i(\mathcal{X}_{\|x\|=c})$ can be chosen small enough by choosing a sufficiently small $c > 0$.) Then it follows from Brouwer's fixed point theorem (see e.g. [52]) that the mapping $\tilde{s}_i$ in (7.33) has a fixed point, i.e. $\exists \xi_i^j \in X_i^j$ s.t.

$$\xi_i^j = \tilde{s}_i(\xi_i^j) = \frac{c}{\|s_i(\xi_i^j)\|}\ s_i(\xi_i^j).$$

Then it can be easily checked that (7.32) holds with the eigenvectors $x = \xi_i^j$'s and the eigenvalues $\lambda = q_i(\xi_i^j)$'s. Finally define

$$\xi_i(s) := \begin{cases} \xi_i^+ \in \mathcal{X}_{\|x\|=s} & (s \geq 0) \\ \xi_i^- \in \mathcal{X}_{\|x\|=-s} & (s < 0) \end{cases}.$$

Then equation (7.29) holds. Property (7.28) can be proved similar to the proof of Theorem 7.2.2. Furthermore, the ordering $\min\{\rho_i(s), \rho_i(-s)\} \geq \max\{\rho_{i+1}(s), \rho_{i+1}(-s)\}$ follows from the fact that $\rho_i(0) = \sigma_i$ holds with the Hankel singular value σ_i of the Jacobian linearization of the system. This completes the proof. ∎

[1] $2n$ is equivalent to the number of the axis intersecting the sphere $\mathcal{X}_{\|x\|=c}$.

Notice that the scalar variable s of $\rho_i(s)$'s, $i \in \{1, 2, \ldots, n\}$ can be negative, whereas the variable c of $\rho_j(c)$'s, $j \in \{\max, \min\}$ is non-negative, since it represents the input energy level. Both $\rho_i^2(c)$ and $\rho_i^2(-c)$, $i \in \{1, 2, \ldots, n\}$ denote the ratio L_o/L_c with respect to the prescribed input energy $L_c = (1/2)c^2$. The eigenstructure given in Theorems 7.2.2 and 7.2.3 is particularly important because it is closely related to the Hankel norm and the corresponding axis singular value functions. Indeed the axis singular value functions represent the gain of the Hankel operator at the eigenvector $u = \mathcal{C}^\dagger(\xi_i(s))$ as in (7.22), i.e.

$$\rho_i(s) = \left. \frac{\|\mathcal{H}_0(u)\|}{\|u\|} \right|_{u=\mathcal{C}^\dagger(\xi_i(s))} \tag{7.34}$$

holds. By its definition, the eigenvector $u = \mathcal{C}^\dagger(\xi_i(s))$ represents the stationary point of this gain which follows from the same arguments to the proof of Theorem 7.2.1. Furthermore, it should be noted that Theorems 7.2.2 and 7.2.3 give an input-output characterization of the Hankel operator without using state variables, that is, they are coordinate free.

Remark 7.2.2. For linear systems we have that $\lambda_i(s) = \rho_i^2(s) = \sigma_i^2$, $i = 1, \ldots, n$ where the σ_i's are the Hankel singular values. △

The effectiveness of Theorem 7.2.3 is demonstrated in the following example.

Example 7.2.1. Consider the system (5.18) with $x = (x_1, x_2) \in \mathbb{R}^2$, $u = (u_1, u_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ and f, g and h as follows which fulfills the assumptions in Theorem 6.1.1

$$f(x) = \begin{pmatrix} -9x_1 + 6x_1^2x_2 + 6x_2^3 - x_1^5 - 2x_1^3x_2^2 - x_1x_2^4 \\ -9x_2 - 6x_1^3 - 6x_1x_2^2 - x_1^4x_2 - 2x_1^2x_2^3 - x_2^5 \end{pmatrix}$$

$$g(x) = \begin{pmatrix} \frac{3\sqrt{2}(9-6x_1x_2+x_1^4-x_2^4)}{9+x_1^4+2x_1^2x_2^2+x_2^4} & \frac{\sqrt{2}(-9x_1^2-27x_2^2+6x_1^3x_2+6x_1x_2^3-(x_1^2+x_2^2)^3)}{9+x_1^4+2x_1^2x_2^2+x_2^4} \\ \frac{\sqrt{2}(27x_1^2+9x_2^2+6x_1^3x_2+6x_1x_2^3+(x_1^2+x_2^2)^3)}{9+x_1^4+2x_1^2x_2^2+x_2^4} & \frac{3\sqrt{2}(9+6x_1x_2-x_1^4+x_2^4)}{9+x_1^4+2x_1^2x_2^2+x_2^4} \end{pmatrix}$$

$$h(x) = \begin{pmatrix} \frac{2\sqrt{2}(3x_1+x_1^2x_2+x_2^3)(3-x_1^4-2x_1^2x_2^2-x_2^4)}{1+x_1^4+2x_1^2x_2^2+x_2^4} \\ \frac{\sqrt{2}(3x_2-x_1^3-x_1x_2^2)(3-x_1^4-2x_1^2x_2^2-x_2^4)}{1+x_1^4+2x_1^2x_2^2+x_2^4} \end{pmatrix}.$$

This system is zero-state observable and asymptotically stable about 0. Solving the Hamilton-Jacobi equations for L_o and L_c in (3.4) and (3.5) we obtain

$$L_c(x) = \frac{1}{2}x^Tx \tag{7.35}$$

$$L_o(x) = \frac{1}{2}\frac{36x_1^2 + 9x_2^2 + 18x_1^3x_2 + 18x_1x_2^3 + x_1^6 + 6x_1^4x_2^2 + 9x_1^2x_2^4 + 4x_2^6}{1 + x_1^4 + 2x_1^2x_2^2 + x_2^4}$$

$$= \frac{1}{2} x^T \begin{pmatrix} \frac{36+18x_1x_2+x_1^4+6x_1^2x_2^2}{1+x_1^4+2x_1^2x_2^2+x_2^4} & 0 \\ 0 & \frac{9+18x_1x_2+9x_1^2x_2^2+4x_2^4}{1+x_1^4+2x_1^2x_2^2+x_2^4} \end{pmatrix} x \tag{7.36}$$

on $\mathcal{X}^0 = \mathbb{R}^n$. We see that the controllability function is already in input-normal form and that the observability function is in output-diagonal form. A pair of singular value functions as defined in Theorem 6.1.1 are

$$\tau_1(x) = \frac{36 + 18x_1x_2 + x_1^4 + 6x_1^2x_2^2}{1 + x_1^4 + 2x_1^2x_2^2 + x_2^4} \tag{7.37}$$

$$\tau_2(x) = \frac{9 + 18x_1x_2 + 9x_1^2x_2^2 + 4x_2^4}{1 + x_1^4 + 2x_1^2x_2^2 + x_2^4} \tag{7.38}$$

which are the elements of the diagonal matrix in (7.36). The neighborhood W of 0, where the number of distinct eigenvalues is constant, is

$$W = \{\, x \mid -x_1^4 + 3x_1^2x_2^2 + 4x_2^4 < 27\},$$

i.e., $\tau_1(x) > \tau_2(x)$ for $\forall x \in W$. Next, we continue with Theorem 7.2.3. In order to obtain the $\xi_i(s)$'s we have to compute the solution of the equations (7.28) and (7.29) which reduce down to

$$s^2 = 2L_c(x) = x_1^2 + x_2^2$$
$$0 = \det \begin{pmatrix} \frac{\partial L_c}{\partial x} \\ \frac{\partial L_o}{\partial x} \end{pmatrix} = \frac{-27x_1x_2 + 9x_1^4 - 9x_2^4 + 3x_1^5x_2 + 6x_1^3x_2^3 + 3x_1x_2^5}{(1 + x_1^4 + 2x_1^2x_2^2 + x_2^4)^2}.$$

Here the second equation follows from the fact that $\partial L_c/\partial x$ is parallel to $\partial L_o/\partial x$ and that x is 2-dimensional. These equations have the following two solutions which can be obtained by a standard CAD such as Maple.

$$\xi_1(s) = \begin{pmatrix} \frac{3s}{\sqrt{9+s^4}} \\ \frac{s^3}{\sqrt{9+s^4}} \end{pmatrix} \tag{7.39}$$

$$\xi_2(s) = \begin{pmatrix} \frac{s^3}{\sqrt{9+s^4}} \\ -\frac{3s}{\sqrt{9+s^4}} \end{pmatrix} \tag{7.40}$$

For the equation (7.28), the axis singular value functions $\rho_i(s)$'s can be obtained by a direct calculation:

$$\rho_1(s) = \sqrt{\frac{L_o(\xi_1(s))}{L_c(\xi_1(s))}} = 2\sqrt{\frac{9 + s^4}{1 + s^4}} \tag{7.41}$$

$$\rho_2(s) = \sqrt{\frac{L_o(\xi_2(s))}{L_c(\xi_2(s))}} = \sqrt{\frac{9 + s^4}{1 + s^4}}. \tag{7.42}$$

Notice that both functions $\xi_i(s)$'s and $\rho_i(s)$'s are defined for all $s \in \mathbb{R}$. We can easily check that the $\lambda_i(s)$'s given by (7.30) satisfy condition (7.29). Furthermore it can be observed that

$$\min\{\rho_1(s), \rho_1(-s)\} = 2\sqrt{\frac{9+s^4}{1+s^4}} > \sqrt{\frac{9+s^4}{1+s^4}} = \max\{\rho_2(s), \rho_2(-s)\} \quad (7.43)$$

holds for all $s \in \mathbb{R}$. This implies that the relation (7.27) holds on $U = \mathbb{R}$. Therefore it follows from (7.31) that

$$\|S\|_H = \sup_{s\in\mathbb{R}} \rho_1(s) = 6. \quad (7.44)$$

Thus Theorem 7.2.3 is a powerful tool in investigating the gain structure of the Hankel operator and is a natural nonlinear extension of the linear results. △

7.3 Input-normal/output-diagonal realizations

The previous section gives a new characterization for the nonlinear extension of Hankel singular values, namely the *axis singular value functions* $\rho_i(s)$, $i = 1, \ldots, n$, in Theorem 7.2.3. This theorem is so strong that we can derive a new input-normal/output-diagonal realization by employing it repetitively. The main idea is to convert the curve $\xi_i(s)$ into each axis of the input-normal/output-diagonal coordinates in order for the gain of the Hankel norm in (7.34) to have a stationary point on each axis. The following theorem states the result.

Theorem 7.3.1. *Consider the system S in (5.18). Suppose that Assumptions A1 and A2 hold. Then there exist a neighborhood $V \subset \mathbb{R}^n$ of 0 and a coordinate transformation $x = \Phi(z)$, $\Phi(0) = 0$, on V, converting the system into an input-normal/output-diagonal form, i.e. there exist n smooth functions $\tau_i : V \to \mathbb{R}$ satisfying (6.3) and (6.4), such that*

$$z_i = 0 \;\Leftrightarrow\; \frac{\partial L_c(\Phi(z))}{\partial z_i} = 0 \;\Leftrightarrow\; \frac{\partial L_o(\Phi(z))}{\partial z_i} = 0 \quad (7.45)$$

holds for all $i \in \{1, 2, \ldots, n\}$. Furthermore

$$\tau_i(0, \ldots, 0, \underbrace{z_i}_{i\text{-th}}, 0, \ldots, 0) = \rho_i^2(z_i) \quad (7.46)$$

$$\frac{\partial \tau_i}{\partial z}(0, \ldots, 0, \underbrace{z_i}_{i\text{-th}}, 0, \ldots, 0) = (0, \ldots, 0, \underbrace{\frac{d\rho_i^2(z_i)}{dz_i}}_{i\text{-th}}, 0, \ldots, 0) \quad (7.47)$$

holds for all $i \in \{1, 2, \ldots, n\}$. In particular, if $V = \mathbb{R}^n$, then

$$\|S\|_H = \sup_{z_1\in\mathbb{R}} \sqrt{\tau_1(z_1, 0, \ldots, 0)}. \quad (7.48)$$

The proof of this theorem is very long and tedious, and breaks down into several steps. The equation (7.45) is proved by induction, where the cases $n = 1$ and $n = 2$ are studied. Then the case $n = k$ is proved, which in itself breaks down into three steps. Finally the output-diagonal form in the z coordinates is proved. The proof can be found in Appendix D. It should be noted that the proof is constructive and it actually gives a procedure to obtain the new input-normal/output-diagonal realization.

In Theorem 7.3.1 the existence of an input-normal/output-diagonal form is proved so that (7.45) and the properties in Theorem 7.2.3 hold along each axis. The relationship (7.45) is a stronger version of the equation (7.15) which can be achieved by applying Theorem 7.2.3 repetitively. This property is quite important because the input-normal/output-diagonal structure is preserved under the projection to lower dimensional subspaces spanned by each axis $\{z_1, z_2, \ldots\}$ and this is likely to play an important role in model reduction of nonlinear systems. We illustrate our final result in the following example.

Example 7.3.1. Consider the state space system in the form of (5.18) as given in Example 7.2.1. Equations (7.39) and (7.40) imply that the coordinate transformation $x = \Phi(z)$ is given by

$$x = \Phi(z) = \begin{pmatrix} \frac{3}{\sqrt{9+(z_1^2+z_2^2)^2}} & \frac{(z_1^2+z_2^2)}{\sqrt{9+(z_1^2+z_2^2)^2}} \\ \frac{-(z_1^2+z_2^2)}{\sqrt{9+(z_1^2+z_2^2)^2}} & \frac{3}{\sqrt{9+(z_1^2+z_2^2)^2}} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{7.49}$$

in the form of the rotation coordinate transformation (D.3) in Appendix D, which maps the z_i-axis into ξ_i, i.e.,

$$\xi_1(s) = \Phi(s, 0)$$
$$\xi_2(s) = \Phi(0, s).$$

See Appendix D for the detailed procedure to obtain (7.49). The coordinate transformation (7.49) converts the system vector fields and output mapping into

$$f(z) = \begin{pmatrix} -9z_1 - z_1^5 - 2z_1^3 z_2^2 - z_1 z_2^4 \\ -9z_2 - z_1^4 z_2 - 2z_1^2 z_2^3 - z_2^5 \end{pmatrix}$$
$$g(z) = \begin{pmatrix} \sqrt{18 + 2z_1^4 + 4z_1^2 z_2^2 + 2z_2^4} & 0 \\ 0 & \sqrt{18 + 2z_1^4 + 4z_1^2 z_2^2 + 2z_2^4} \end{pmatrix}$$
$$h(z) = \begin{pmatrix} \frac{(6z_1 - 2z_1^5 - 4z_1^3 z_2^2 - 2z_1 z_2^4)\sqrt{18+2z_1^4+4z_1^2 z_2^2+2z_2^4}}{1+z_1^4+2z_1^2 z_2^2+z_2^4} \\ \frac{(3z_2 - z_1^4 z_2 + 2z_1^2 z_2^3 - 2z_2^5)\sqrt{18+2z_1^4+4z_1^2 z_2^2+2z_2^4}}{1+z_1^4+2z_1^2 z_2^2+z_2^4} \end{pmatrix}.$$

The observability and controllability functions in the new coordinates are given as follows

$$L_c(\Phi(z)) = \frac{1}{2} z^T z$$
$$L_o(\Phi(z)) = \frac{1}{2} z^T \begin{pmatrix} \frac{4(9+z_1^4+2z_1^2z_2^2+z_2^4)}{1+z_1^2+2z_1^2z_2^2+z_2^4} & 0 \\ 0 & \frac{9+z_1^4+2z_1^2z_2^2+z_2^4}{1+z_1^2+2z_1^2z_2^2+z_2^4} \end{pmatrix} z =: \frac{1}{2} z^T \begin{pmatrix} \tilde{\tau}_1(z) & 0 \\ 0 & \tilde{\tau}_2(z) \end{pmatrix} z$$

which of course satisfy the HJB equations (3.4) and (3.5). It can be readily checked that the above energy functions L_o and L_c satisfy the properties (7.45), (7.46) and (7.47) on the valid region $V = \mathbb{R}^n$. Hence (7.48) implies

$$\|S\|_H = \sup_{z_1 \in \mathbb{R}} \sqrt{\tilde{\tau}_1(z_1, 0)} = 6$$

which indeed equals the outcome of Example 7.2.1. △

It can be observed from the above example that the singular value functions $\tilde{\tau}_i$'s have a close relationship with the Hankel norm of the system in the new input-normal/output-diagonal realization. The gain structure of the Hankel operator is clearly exhibited in the new coordinates.

Remark 7.3.1. The balancing method in the linear case, e.g., [53, 90], provides a balance between input and output in the sense that $Q = P$ is diagonal with the observability and controllability Gramians Q and P, and thus when the minimum amount of control energy that is necessary to reach a state component is small (i.e., the state component is easily controllable), then it follows that the amount of output energy that is generated by that state component is large (i.e., the state component is easily observable). In the case of input-normal/output diagonal realizations only the output energy that is generated by a state component is considered, while the minimum input energy for all state components is weighted equally. In the nonlinear case the input-normal/output-diagonal realization given in Theorem 7.3.1 has a similar interpretation. However, for a balanced realization, an additional step is required. So far, a procedure to convert the input-normal/output-diagonal realization of Theorem 6.1.1 into a balanced form on each coordinate axis z_i was given in Chapter 6. A procedure for balancing based on the input-normal/output-diagonal realization of Theorem 7.3.1 is a topic of on-going research, while model reduction using the new procedure of this section is the topic of the next section. △

7.4 Model reduction

This section develops a procedure for model reduction method based on the results of Section 7.3. However, since our realization from Section 7.3 is an input-normal/output-diagonal realization it is not in "strict" balanced form yet. The input-normal, output-diagonal realization essentially measures the importance of states such that the control energy is equally important for all states, and the output energy (or in other words, the observability properties of the output) are different for the different state components. Hence reduction of the input-normal, output-diagonal form of Section 7.3 is only based on the output energy of the different state components. This corresponds in a certain sense with the linear cross Gramian thinking (see e.g. [35]), i.e., by noting that the Hankel norm equals

$$\|S\|_H = \sup_{\substack{x\in\mathbb{R}^n \\ x\neq 0}} \sqrt{\frac{L_o(x)}{L_c(x)}},$$

it can be seen that for the input-normal case, $L_o(x)$ contains the information that is given by the cross Gramians. The property (7.45) can be considered of importance for model reduction of nonlinear systems. Suppose the assumptions in Theorems 7.3.1 hold and the coordinate transformation $x = \Phi(z)$ gives the balanced representation in the sense of Theorem 7.3.1, i.e. (7.45) holds. Suppose moreover that

$$\min\{\rho_k(z_k), \rho_k(-z_k)\} > \max\{\rho_{k+1}(z_{k+1}), \rho_{k+1}(-z_{k+1})\} \tag{7.50}$$

holds for all $z \in V$. Then the state variables $z_1, \ldots, z_k$ are more important in terms of energy than those $z_{k+1}, \ldots, z_n$ due to the ordering of the singular value functions ρ_i's.

Divide the coordinate into two parts corresponding to the division (7.50) as

$$\begin{aligned} z &= (z^a, z^b) \in \mathbb{R}^n \\ z^a &:= (z_1, \ldots, z_k) \in \mathbb{R}^k \\ z^b &:= (z_{k+1}, \ldots, z_n) \in \mathbb{R}^{n-k} \\ \begin{pmatrix} f^a(z) \\ f^b(z) \end{pmatrix} &:= \left.\frac{\partial \Phi^{-1}(x)}{\partial x}\right|_{x=\Phi(z)} f(\Phi(z)) \\ \begin{pmatrix} g^a(z) \\ g^b(z) \end{pmatrix} &:= \left.\frac{\partial \Phi^{-1}(x)}{\partial x}\right|_{x=\Phi(z)} g(\Phi(z)). \end{aligned}$$

Moreover, divide the system S into two subsystems accordingly as follows:

$$S^a : \begin{cases} \dot{z}^a = f^a(z^a, 0) + g^a(z^a, 0)u^a \\ y^a = h(\Phi(z^a, 0)) \end{cases} \tag{7.51}$$

$$S^b : \begin{cases} \dot{z}^b = f^b(0, z^b) + g^b(0, z^b)u^b \\ y^b = h(\Phi(0, z^b)) \end{cases} . \tag{7.52}$$

Then we obtain the following properties.

Lemma 7.4.1. *Consider the system S in (5.18) and the divided systems (7.51) and (7.52). Suppose the assumptions in Theorem 7.3.1 hold. Then the controllability and observability functions $L_c^a(z^a)$, $L_o^a(z^a)$, $L_c^b(z^b)$ and $L_o^b(z^b)$ of the systems S^a and S^b satisfy*

$$L_c^a(z^a) = L_c(\Phi(z^a, 0)) \tag{7.53}$$
$$L_o^a(z^a) = L_o(\Phi(z^a, 0)) \tag{7.54}$$
$$L_c^b(z^b) = L_c(\Phi(0, z^b)) \tag{7.55}$$
$$L_o^b(z^b) = L_o(\Phi(0, z^b)). \tag{7.56}$$

Proof: It follows from Theorems 7.3.1 and 3.1.2 that the controllability and observability functions satisfy

$$0 = \frac{\partial L_c(\Phi(z))}{\partial z} f(\Phi(z)) + \frac{1}{2}\frac{\partial L_c(\Phi(z))}{\partial z} g(\Phi(z)) g(\Phi(z))^{\mathrm{T}} \frac{\partial L_c(\Phi(z))}{\partial z}^{\mathrm{T}} \tag{7.57}$$
$$0 = \frac{\partial L_o(\Phi(z))}{\partial z} f(\Phi(z)) + \frac{1}{2} h(\Phi(z))^{\mathrm{T}} h(\Phi(z)). \tag{7.58}$$

Then, for (7.45), we obtain

$$\frac{\partial L_c(\Phi(z))}{\partial z^b}(z^a, 0) = \frac{\partial L_o(\Phi(z))}{\partial z^b}(z^a, 0) = 0.$$

Hence, substituting $z = (z^a, 0)$ for (7.57) and (7.58) yields

$$\begin{aligned}
0 &= \left(\frac{\partial L_c(\Phi(z))}{\partial z^a},\ 0\right)\begin{pmatrix} f^a \\ f^b \end{pmatrix} + \frac{1}{2}\left(\frac{\partial L_c(\Phi(z))}{\partial z^a}, 0\right)\begin{pmatrix} g^a \\ g^b \end{pmatrix}\left({g^a}^{\mathrm{T}},\ {g^b}^{\mathrm{T}}\right)\begin{pmatrix} \frac{\partial L_c(\Phi(z))}{\partial z^a}^{\mathrm{T}} \\ 0 \end{pmatrix} \\
&= \frac{\partial L_c^a(z^a)}{\partial z^a} f^a(z^a, 0) + \frac{1}{2}\frac{\partial L_c^a(z^a)}{\partial z^a} g^a(z^a, 0)^{\mathrm{T}} g^a(z^a, 0) \frac{\partial L_c^a(z^a)}{\partial z^a}^{\mathrm{T}}, \\
0 &= \left(\frac{\partial L_o(\Phi(z))}{\partial z^a},\ 0\right)\begin{pmatrix} f^a \\ f^b \end{pmatrix} + \frac{1}{2}\, h(\Phi(z^a, 0))^{\mathrm{T}} h(\Phi(z^a, 0)) \\
&= \frac{\partial L_o^a(z^a)}{\partial z^a} f^a(z^a, 0) + \frac{1}{2}\, h(\Phi(z^a, 0))^{\mathrm{T}} h(\Phi(z^a, 0)).
\end{aligned}$$

These relations prove (7.53) and (7.54). Equations (7.55) and (7.56) can be proved in the same way and this completes the proof. ■

Lemma 7.4.1 implies the following preservation property in the model reduction procedure which is a natural generalization of the linear case results in [65, 29].

Theorem 7.4.1. *Consider the system S in (5.18) and the divided systems (7.51) and (7.52). Suppose the assumptions in Theorem 7.3.1 hold. Then the reduced systems S^a and S^b are in the balanced form with the properties (7.45), and*

$$\rho_i^a(z_i^a) = \rho_i(z_i^a) \quad i \in \{1, 2, \ldots, k\} \tag{7.59}$$
$$\rho_i^b(z_i^b) = \rho_{i+k}(z_i^b) \quad i \in \{1, 2, \ldots, n-k\} \tag{7.60}$$

hold with ρ_i^a's and ρ_i^b's the singular value functions of the systems S^a and S^b, respectively. In particular, if $V = \mathbb{R}^n$, then

$$\|S^a\|_H = \|S\|_H. \tag{7.61}$$

Proof: The fact that S^a and S^b are again in the balanced form is obvious because of

$$\begin{aligned} z_i^a = 0 &\Leftrightarrow \frac{\partial L_c^a(z^a)}{\partial z_i^a} = \frac{\partial L_c(\Phi(z^a, 0))}{\partial z_i^a} = 0 \\ &\Leftrightarrow \frac{\partial L_o^a(z^a)}{\partial z_i^a} = \frac{\partial L_o(\Phi(z^a, 0))}{\partial z_i^a} = 0 \end{aligned}$$

which is obtained from Lemma 7.4.1. Also the equations (7.59)–(7.61) follow straightforwardly from Lemma 7.4.1. This completes the proof. ■

Remark 7.4.1. The model reduction of S into S^a (and S^b) is uniquely determined (coordinate free), although the balanced coordinate $z = \Phi^{-1}(x)$ itself is not unique. On the other hand, the model reduction method based on the model reduction procedure in Chapter 6 is coordinate dependent. △

Example 7.4.1. (continued from Example 7.3.1) Consider again the state-space system which already has input-normal/output-diagonal form, obtained in Example 7.3.1. According to the above model reduction procedure, one obtains

$$S^a : \begin{cases} \dot{z}^a = f^a(z^a) + g^a(z^a)u \\ y = h^a(z^a) \end{cases}$$

with

$$\begin{aligned} f^a(z^a) &= -9z^a - (z^a)^5 \\ g^a(z^a) &= \left(\sqrt{18 + 2(z^a)^4},\, 0\right) \\ h^a(z^a) &= \begin{pmatrix} \frac{(6z^a - 2(z^a)^5)\sqrt{18+2(z^a)^4}}{1+(z^a)^4} \\ 0 \end{pmatrix}. \end{aligned}$$

The observability and controllability functions of the reduced system S^a are given as follows:

$$\begin{aligned} L_c^a(z^a) &= \frac{1}{2}(z^a)^2 \\ L_o^a(z^a) &= \frac{1}{2}\frac{4(z^a)^2(9 + (z^a)^4)}{1 + (z^a)^4} =: \frac{1}{2}(z^a)^2\, \tau^a(z^a). \end{aligned}$$

Furthermore, the square of the Hankel norm of the reduced system S^a can be computed as

$$\|S^a\|_H^2 = \sup_{z^a \in \mathbb{R}} \frac{L_o^a(z^a)}{L_c^a(z^a)} = \sup_{z^a \in \mathbb{R}} \tau^a(z^a) = \sup_{z^a \in \mathbb{R}} \frac{4(9+(z^a)^4)}{1+(z^a)^4} = 36 = \|S\|_H^2.$$

which indeed equals that of the original S. Thus the Hankel norm is preserved. △

This example exhibits the effectiveness of the new balancing and model reduction method.

7.5 A relation with spectral analysis

Spectral theory for nonlinear operators is a diverse subject with substantial roots going back to at least the late 1960's [10]. The proliferation of definitions and approaches (see, for example, [3, 19, 26, 27, 36, 40, 64]) is partly due to the fact that no single definition completely characterizes the original operator as in the linear case. In this section, we outline an additional approach to defining a nonlinear spectrum motivated by the nature of our application and the notion of the C^1-spectrum essentially introduced in [59].

Definition 7.5.1. *Let E be a Banach space and $\mathcal{S} : E \to E$ be an operator that is continuously Fréchet differentiable on E. The C^1-spectrum of $\mathcal{S}$, $\sigma^1(\mathcal{S})$, is the set of all complex numbers λ such that $\mathcal{S} - \lambda I$ is not a diffeomorphism on E.*

For a linear operator $\mathcal{S}$, this definition reduces to the usual definition of a spectrum. The following result from the analysis of the C^1-spectrum in [3] is relevant to our study:

Theorem 7.5.1. [3] *Let $\mathcal{S}$ be an operator as described in Definition 7.5.1, then*

$$\sigma^1(\mathcal{S}) = \pi(\mathcal{S}) \cup \bigcup_{u \in E} \sigma(D\mathcal{S}(u))$$

where $\pi(\mathcal{S})$ denotes the set of all λ such that $\mathcal{S} - \lambda I$ is not proper (in the sense of [3]), and $\sigma(\mathcal{A})$ denotes the usual spectrum of a bounded linear operator $\mathcal{A}$.

This theorem reveals that the C^1-spectrum of a nonlinear operator directly involves the Fréchet derivative of the operator, i.e., $\sigma(D\mathcal{S}(u))$ is an important part of the C^1-spectrum. Since our problem is to extend the singular value definitions into the nonlinear setting, and in particular for a Hankel operator, it is the spectrum of the Fréchet derivative of the operator $\mathcal{H}_0^*\mathcal{H}_0(u)$ that is relevant. The following corollary of Theorem 7.2.3 and Theorem 5.1.7 reveals some information about $\sigma\left(D(\mathcal{H}_0^*\mathcal{H}_0)\right)$.

Corollary 7.5.1. *In the context of Theorem 7.2.3, the following relation holds:*

$$\langle (D(\mathcal{H}_0^*\mathcal{H}_0)(u_i(s)))\,(u_i(s)), u_i(s)\rangle = \left(2\lambda_i(s) - (\rho_i(s))^2\right)\langle u_i(s), u_i(s)\rangle.$$

Proof: Applying Theorem 5.1.7, property 3, gives directly

$$\begin{aligned}&\langle (D(\mathcal{H}_0^*\mathcal{H}_0)(u_i(s)))\,(u_i(s)), u_i(s)\rangle = \\ &2\langle (D\mathcal{H}_0(u_i(s)))\,(\mathcal{H}_0(u_i(s)))\,, u_i(s)\rangle - \langle \mathcal{H}_0^*\mathcal{H}_0(u_i(s)), u_i(s)\rangle.\end{aligned}$$

Then using Theorem 7.2.3, the result immediately follows. △

Note that the above corollary does not directly characterize $\sigma(D(\mathcal{H}_0^*\mathcal{H}_0))$, but it yields an eigen-equation within an inner product identity. In the general nonlinear setting, it is not possible to (exactly) extract the eigen-equation from this identity, i.e., every number in $\sigma(D(\mathcal{H}_0^*\mathcal{H}_0))$ fulfills the above equation, but numbers not in $\sigma(D(\mathcal{H}_0^*\mathcal{H}_0))$ can satisfy it too. In [3], other types of spectra are discussed, mainly for operators that are not C^1. The nonlinear Hilbert adjoint operators of the form $\mathcal{T}^*(\mathcal{T}(u), u)$ considered in Chapter 5 are generally C^1. Furthermore, our original interest in the singular value structure of the Hankel operator is related to the state-space notions of the controllability and observability energy functions and the inner product relations with their respective operators (see [34, 75]). These facts, together with the observation that our nonlinear Hilbert adjoint is defined within an inner product, motivates one to include the inner product structure directly into a spectrum definition.

Definition 7.5.2. *Let E be a Hilbert space and consider an operator $\mathcal{S} : E \to E$. Then the* inner product spectrum *is defined as*

$$\sigma_{ip}(\mathcal{S}) = \{\lambda : \exists p \neq 0 \;\; with \;\; \langle (\mathcal{S} - \lambda I)\,(p), p\rangle_E = 0\}\,.$$

It follows immediately when $\mathcal{S}$ is continuously Fréchet differentiable on E that $\sigma_{ip}(\mathcal{S}) \subset \sigma^1(\mathcal{S})$. Furthermore, in the case of a linear operator $\mathcal{S}(p) = Ap$ with $A^T = A$ and $E = \mathbb{R}^n$, it is easily verified that $\sigma_{ip}(\mathcal{S}) = \mathrm{Range}(\mathcal{R}_\mathcal{S}(p))$, where $\mathcal{R}_\mathcal{S}$ is the Rayleigh quotient of $\mathcal{S}$ defined as

$$\mathcal{R}_\mathcal{S}(p) = \frac{p^T A p}{p^T p}.$$

It is known in this case that $\sigma_{ip}(A) = [\lambda_{min}(A), \lambda_{max}(A)] \subset \mathbb{R}$, where λ_{min} (λ_{max}) denotes the smallest (largest) eigenvalue of A. The obvious extension of the Rayleigh quotient for nonlinear maps is then

$$\begin{aligned}\mathcal{R}_\mathcal{S} &: E \to \mathbb{R}\\ &: p \mapsto \frac{\langle p, \mathcal{S}(p)\rangle_E}{\langle p, p\rangle_E},\end{aligned}$$

and it straightforwardly follows that $\sigma_{ip}(\mathcal{S}) = \mathrm{Range}(\mathcal{R}_\mathcal{S})$. When $\mathcal{S}$ is homogeneous, the range of this Rayleigh quotient is a subset of the numerical range defined in [88]. Furthermore, it is related to the numerical range

$W(\mathcal{S},\mathcal{T})$ as defined in [11] for positively homogeneous operators $\mathcal{S}$ and $\mathcal{T}$ of degree k on the unit sphere $S_1(0)$ in E. Specifically, if $\mathcal{T} = I$ and $\mathcal{S}$ is positively homogeneous of degree$k$, then $\mathcal{R}_{\mathcal{S}}(S_1(0)) = W(\mathcal{S}, I)$.

In the case of a compact linear operator $\mathcal{A} : E \to E$, it is known that

$$\sigma_{ip}(\mathcal{A}^*\mathcal{A}) = (0, \tau_1^2],$$

where τ_1 is the largest singular value of $\mathcal{A}$ [45]. If rank$(\mathcal{A}^*\mathcal{A}) = n < \infty$, then this result can be further refined to

$$\sigma_{ip}(\mathcal{A}^*\mathcal{A}) = [\tau_n^2, \tau_1^2],$$

where τ_n is the smallest nonzero singular value of $\mathcal{A}$. In the case of the nonlinear system S given by (5.18) with corresponding Hankel operator $\mathcal{H}_0$, it follows immediately that $(\rho_i(s))^2 \in \sigma_{ip}\left(\mathcal{H}_0^*\mathcal{H}_0\right)$ for all $i \in \{1, 2, \dots, n\}$, and $s \in \mathbb{R}$. Furthermore,

$$\|S\|_H^2 = \sup_{s\in\mathbb{R}} \left\{{\rho_1}^2(s)\right\} = \sup\left\{\sigma_{ip}\left(\mathcal{H}_0^*\mathcal{H}_0\right)\right\}$$

and

$$\inf_{s\in\mathbb{R}} \left\{{\rho_n}^2(s)\right\} = \inf\left\{\sigma_{ip}\left(\mathcal{H}_0^*\mathcal{H}_0\right)\right\},$$

where $\|S\|_H$ is the Hankel norm of S.

The following example illustrates some of the relations between the inner product spectrum and the Hankel operator theory as it appears in this chapter.

Example 7.5.1. Consider the following state-space system

$$S\ :\ \begin{cases} \dot{z}_1 = -z_1 + z_1 z_2^2 + u_1\sqrt{2} \\ \dot{z}_2 = -z_2 - z_2^3 + u_2\sqrt{2 - 2z_1^2 + 2z_2^2} \\ y_1 = 2z_1 \\ y_2 = \sqrt{2}\left(z_2\sqrt{1 + z_2^2}\right), \end{cases}$$

where $z \in W = \{z | z_1^2 < 1\}$. Then it can be shown that (see Example 6.1.1) the controllability and observability function are given by

$$\tilde{L}_c(z) = \frac{1}{2} z^T z, \qquad \tilde{L}_o(z) = \frac{1}{2} z^T \begin{pmatrix} 2 & 0 \\ 0 & 1 + z_1^2 \end{pmatrix} z.$$

In order to obtain the $\lambda_i(\cdot)$'s in (7.29), one must compute the solutions of

$$\begin{cases} 2z_1 + z_1 z_2^2 = \lambda(s) z_1 \\ z_2 + z_1^2 z_2 = \lambda(s) z_2 \\ z_1^2 + z_2^2 = s^2. \end{cases}$$

Note that $z \in W$ implies that $s < 1$, so that $z_1^2 = \frac{1}{2}(1+s^2)$ and $z_2^2 = \frac{1}{2}(s^2-1)$ have no real solution. Thus, there remains two possibilities:

$$\begin{cases} x_1(+s) = \begin{pmatrix} s \\ 0 \end{pmatrix}, \lambda_1(+s) = 2 \\ x_1(-s) = \begin{pmatrix} -s \\ 0 \end{pmatrix}, \lambda_1(-s) = 2 \end{cases}$$

$$\begin{cases} x_2(+s) = \begin{pmatrix} 0 \\ s \end{pmatrix}, \lambda_2(+s) = 1 \\ x_2(-s) = \begin{pmatrix} 0 \\ -s \end{pmatrix}, \lambda_2(-s) = 1. \end{cases}$$

In Theorem 7.2.3 it is shown that

$$s\lambda_i(s) = \frac{d}{ds}\left({\rho_i}^2(s)\frac{s^2}{2}\right),$$

and thus it follows that

$${\rho_1}^2(\pm s) = 2,\ {\rho_2}^2(\pm s) = 1 \ \Rightarrow\ \sigma_{ip}\left(\mathcal{H}_0^*\mathcal{H}_0\right) = [1, 2] \ \Rightarrow\ \|S\|_H = \sqrt{2}.$$

△

Notes and references

The material of Section 7.1 can be found in various paper, e.g., [34, 75]. The material of Section 7.2 and 7.3 is taken from [22, 24]. The material of Section 7.4 is taken from [23], and may also be found in [20]. The results of 7.5 are taken from [77].