On the Nonuniqueness of Balanced Nonlinear Realizations

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Abstract

The notion of balanced realizations for nonlinear state space model reduction problems was first introduced by Scherpen in 1993. Analogous to the linear case, the so called *singular value functions* of a system describe the relative importance of each state component from an input-output point of view. In this paper it is shown that the procedure for nonlinear balancing has some interesting ambiguities that do not occur in the linear case. Specifically, it appears that the singular value functions as currently defined are dependent on a particular factorization of the observability function. It is shown by example that in a fixed coordinate frame this factorization is not unique, and thus other distinct definitions for the singular value functions and balanced realizations are possible.

1. Introduction

The notion of balanced realizations for nonlinear state space model reduction problems was first introduced by Scherpen in [7]-[8]. Analogous to the Gramians matrices used in the linear case, controllability and observability (energy) functions are used to determine how important each state component is in influencing the input-output map of the system. These functions are then transformed, through a change of coordinates, into a simultaneous diagonal form in order to identify the so called *singular value functions* of the system. In the linear case, these functions are equivalent to the square of the (constant) Hankel singular values of the system. State truncation is finally accomplished by examining the singular value functions in a neighborhood of 0 and deleting states that correspond to the smallest singular value functions in a local sense.

The procedure for nonlinear balancing, however, has some interesting ambiguities that do not occur in the linear case. Specifically, it appears that the singular value functions defined in [7]-[8] are dependent on a particular factorization of the observability function which follows from the Fundamental Theorem of Calculus. It will be easily shown by example that in a fixed coordinate frame this factorization is not unique, and thus other distinct definitions for the singular value functions are possible. Of course, this is of great concern in model reduction applications since decisions about state deletion should only depend on the coordinate frame of the state space and on intrinsic qualities of input-output map. So in this paper we examine this issue in detail and explain the precise nature of the nonuniqueness problem. Furthermore, given a fixed factorization, we also present some results on the nonuniqueness of singular value functions via *norm preserving* coordinate transformations. It is possible that both of these nonuniqueness problems are somehow related, but that topic is not pursued here.

The paper is organized as follows. In Section 2, the background for the problem is provided by reviewing some standard definitions in connection with nonlinear balanced realizations. Then a simple example is provided to illustrate the nonuniqueness phenomena considered in this paper. In Section 3, we first consider the nonuniqueness of the factorization of the observability function via so call *null matrix functions*. This idea leads to some results about the relationship between singular value functions coming from different factorizations. We conclude with a discussion of the role of norm preserving coordinate transformations in determining the singular value functions.

The mathematical notation used throughout is fairly standard. Vector norms are represented by $\|x\| = \sqrt{x^T x}$ for $x \in \mathbb{R}^n$. $L_2(a,b)$ represents the set of Lebesgue measurable functions, possibly vector-valued, with finite L_2 norm $\|x\|_{L_2} = \sqrt{\int_a^b \|x(t)\|^2\, dt}$. If $L : \mathbb{R}^n \mapsto \mathbb{R}$ is a differentiable function, then its partial derivative $\frac{\partial L}{\partial x}$ will be the row vector of partial derivatives $\frac{\partial L}{\partial x_i}$ where $i = 1, \ldots, n$.

2. The Nature of the Problem

In this section, the background for the problem is first outlined by reviewing some standard definitions in connection with nonlinear balanced realizations. All of this material has been adapted from [7]-[8]. Then a simple example is provided to illustrate the nonuniqueness phenomena considered in this paper.

Let $\mathcal{M}$ be an n-dimensional smooth manifold, and let

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

be a system defined in terms of local coordinates on $\mathcal{M}$. It is assumed that f, g, and h are smooth vector fields on $\mathcal{M}$ and that $f(0) = 0$ and $h(0) = 0$. The corresponding controllability and observability functions (or energy functions, collectively) for such a system are defined below.

Definition 2.1 *The* **controllability** *and* **observability functions** *for the system (f,g,h) are defined, respectively, as*

$$L_c(x) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,\ x(0)=x}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2\, dt$$

and

$$L_o(x) = \frac{1}{2}\int_0^\infty \|y(t)\|^2\, dt,$$

when $x(0) = x$, and $u(t) = 0$ for $0 \le t < \infty$.

In order for a balanced realization to exist, the following properties of the system are assumed throughout the paper:

1. f is asymptotically stable on some neighborhood Y of 0.
2. The system (f,g,h) is zero-state observable on Y.
3. L_c and L_o exist and are smooth on Y.
4. $\frac{\partial^2 L_c}{\partial x^2}(0) > 0$ and $\frac{\partial^2 L_o}{\partial x^2}(0) > 0$.

The next collection of results form the core of the standard nonlinear balancing procedure.

Lemma 2.1 [5] *Let L be a smooth real-valued function on a convex neighborhood $V \subset \mathbb{R}^n$ of 0 with L(0)=0. Then L exhibits the factorization*

$$L(x) = a(x)x,$$

where a is the smooth vector field on V with component functions

$$a_i(x) = \int_0^1 \frac{\partial L}{\partial x_i}(tx_1, \ldots, tx_n)\, dt.$$

Observe that $a(0) = \frac{\partial L}{\partial x}(0)$, and that *any* factorization of the form $L(x) = \tilde{a}(x)x$ necessarily has the property that $\tilde{a}(0) = \frac{\partial L}{\partial x}(0)$. The following lemma comes from applying Morse's Lemma to L_c [5], and the above lemma twice to L_o.

Lemma 2.2 *For a system (f,g,h) with corresponding energy functions (L_c, L_o), there exists a coordinate transformation $x = \phi(\bar{x})$, $\phi(0) = 0$, defined on a neighborhood V of 0 which converts the system into an* **input-normal realization**, *where*

$$\begin{aligned} \bar{L}_c(\bar{x}) &:= L_c(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T\bar{x} \\ \bar{L}_o(\bar{x}) &:= L_o(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T M(\bar{x})\bar{x} \end{aligned}$$

with M an $n \times n$ symmetric matrix-valued function having smooth component functions on $\bar{V} := \phi^{-1}(V)$ and $M(0) = \frac{\partial^2 L_o}{\partial x^2}(0)$.

Analogous to the above observation, *any* factorization of the form $\bar{L}_o(\bar{x}) = \frac{1}{2}\bar{x}^T M'(\bar{x})\bar{x}$ necessarily has the property that $M'(0) = \frac{\partial^2 L}{\partial x^2}(0)$. In order to diagonalize M, the following technical lemma is needed.

Lemma 2.3 [3] *If there exists a neighborhood $\bar{V}$ of 0, where the number of distinct eigenvalues of M is constant everywhere $\bar{V}$, then the eigenvalues and orthonormalized eigenvectors (λ_i, p_i), $i = 1, \ldots, n$ of M are smooth functions of $\bar{x} \in \bar{V}$.*

Theorem 2.1 *For a system (f,g,h) satisfying the condition in Lemma 2.3, there exists a coordinate transformation $x = \psi(z)$, $\psi(0) = 0$, defined on a neighborhood U of 0 which converts the system into a* **input-normal/output-diagonal realization**, *where*

$$\begin{aligned} \tilde{L}_c(z) &:= L_c(\psi(z)) = \frac{1}{2}z^T z, \\ \tilde{L}_o(z) &:= L_o(\psi(z)) = \frac{1}{2}z^T diag(\tau_1(z), \ldots, \tau_n(z))z \end{aligned}$$

with $\tau_1(z) \ge \ldots \ge \tau_n(z)$ being smooth functions on $W := \psi^{-1}(U)$.

The set of functions τ_i, $i = 1, \ldots, n$ are called the *singular value functions* of (f, g, h). The final step of this balancing procedure is given below.

Theorem 2.2 *For the system in Theorem 2.1, there exists a coordinate transformation $\bar{z} = \eta(z)$, $\eta(0) = 0$, defined on the neighborhood W of 0 which converts the system into a* **balanced realization**, *where*

$$\begin{aligned}
\check{L}_c(\bar{z}) &:= \tilde{L}_c(\eta(\bar{z})) \\
&= \frac{1}{2}\bar{z}^T diag(\sigma(\bar{z}_1)^{-1}, \ldots, \sigma(\bar{z}_n)^{-1})\bar{z} \\
\check{L}_o(\bar{z}) &:= \tilde{L}_o(\eta(\bar{z})) \\
&= \frac{1}{2}\bar{z}^T diag(\sigma_1(\bar{z}_1)^{-1}\tau_1(\eta^{-1}(\bar{z})), \ldots, \\
&\qquad \sigma_n(\bar{z}_n)^{-1}\tau_n(\eta^{-1}(\bar{z})))\bar{z},
\end{aligned}$$

with $\sigma(\bar{z}_i) := \tau_i(0, \ldots, 0, \eta_i^{-1}(\bar{z}_i), 0, \ldots, 0)^{\frac{1}{2}}$ for $i = 1, \ldots, n$.

Note that along coordinate axes it is easily verified for $i = 1, \ldots, n$ that:

$$\begin{aligned}
\check{L}_c(0, \ldots, 0, \bar{z}_i, 0, \ldots, 0) &= \tfrac{1}{2}\bar{z}_i^2\sigma(\bar{z}_i)^{-1} \\
\check{L}_o(0, \ldots, 0, \bar{z}_i, 0, \ldots, 0) &= \tfrac{1}{2}\bar{z}_i^2\sigma(\bar{z}_i).
\end{aligned} \tag{1}$$

We now introduce an example to illustrate the nonuniqueness features of the above balancing procedure.

Example 2.1 Consider a second order system with energy functions

$$\begin{aligned}
L_c(x) &= \frac{1}{2}\left(x_1^2 + x_2^2\right) \\
L_o(x) &= \frac{1}{2}\left(\frac{3}{2}x_1^2 + x_1x_2 + \frac{3}{2}x_2^2\right)
\end{aligned}$$

for all $x \in \mathcal{M} = \mathbb{R}^2$. Applying Lemma 2.1 directly, the corresponding input-normal form has energy functions:

$$\begin{aligned}
&L_c(x) = \tfrac{1}{2}\, x^T x \\
&L_o(x) = \tfrac{1}{2}x^T M(x)x = \tfrac{1}{2}\, x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} x.
\end{aligned}$$

Since M is constant in this representation, the singular value functions appear to be the constant functions: $\tau_1(z) = 2$, $\tau_2(z) = 1$ in the diagonalized coordinate frame $x = \psi(z)$. The situation is, however, more complex than it first appears. While the factorization in Lemma 2.1 certainly yields a valid input-normal form realization, it is easily seen that this form is *not* unique. For example, consider the smooth symmetric matrix function

$$A(x) = c_1(x)\begin{bmatrix} -2x_2 & x_1 \\ x_1 & 0 \end{bmatrix} + c_2(x)\begin{bmatrix} 0 & x_2 \\ x_2 & -2x_1 \end{bmatrix},$$

where $c_1, c_2 \in C^\infty(\mathbb{R}^2)$, the ring of smooth real-valued functions defined on $\mathbb{R}^2$. Since $x^T A(x)x = 0$ everywhere on $\mathbb{R}^2$ and $A(0) = 0$, another input-normal form in the *same* coordinate system is:

$$\begin{aligned}
L_c(x) &= \tfrac{1}{2}\, x^T x \\
L_o(x) &= \tfrac{1}{2}x^T(M(x) + A(x))x \\
&:= \tfrac{1}{2}x^T M'(x)x \\
&= \tfrac{1}{2}\, x^T \begin{bmatrix} \frac{3}{2} - 2c_1(x)x_2 & \frac{1}{2} + c_1(x)x_1 + c_2(x)x_2 \\ \frac{1}{2} + c_1(x)x_1 + c_2(x)x_2 & \frac{3}{2} - 2c_2(x)x_1 \end{bmatrix} x.
\end{aligned} \tag{2}$$

For most choices of c_1, c_2, the condition in Lemma 2.3 is satisfied, and thus M' is smoothly diagonalizable. Consider, for example, the case: $c_1(x) = x_1$ and $c_2(x) = x_2$. Then it follows that the eigenvalues of M' are $\lambda_1'(x) = 2 + (x_1 - x_2)^2$ and $\lambda_2'(x) = 1 - (x_1 + x_2)^2$, which are distinct everywhere on $\mathbb{R}^2$. The diagonalizing transformation

$$x = \psi'(z') = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} z'$$

yields the corresponding input-normal/output-diagonal form:

$$\begin{aligned}
\tilde{L}_c'(z') &:= L_c(\psi'(z')) = \frac{1}{2}(z')^T z', \\
\tilde{L}_o'(z') &:= L_o(\psi'(z')) \\
&= \frac{1}{2}(z')^T diag(\tau_1'(z'), \tau_2'(z'))z' \\
&= \frac{1}{2}(z')^T diag(2 + 2(z_2')^2, 1 - 2(z_1')^2)z'.
\end{aligned}$$

Thus, we see immediately that a different factorization of L_o, via the introduction of the matrix-valued function A, leads to a different set of singular value functions. Note, however, that they *are* identical along respective coordinate directions, i.e., $\tau_i'(0, \ldots, 0, z_i', 0, \ldots, 0) = \tau_i(0, \ldots, 0, z_i, 0, \ldots, 0)$ for i=1,2. Furthermore, observe that any coordinate transformation of the form $x = \nu(y) = T(y)y$ with $T(y)^T T(y) = I$ applied to the original system transforms the energy functions in (2) to yet another input-normal/output-diagonal form after applying the diagonalizing transformation $y = \hat{\psi}(\hat{z})$:

$$\begin{aligned}
\hat{L}_c(\hat{z}) &:= L_c((\nu \circ \hat{\psi})(\hat{z})) = \frac{1}{2}\hat{z}^T \hat{z}, \\
\hat{L}_o(\hat{z}) &:= L_o((\nu \circ \hat{\psi})(\hat{z})) = \frac{1}{2}\hat{z}^T diag(\hat{\tau}_1(\hat{z}), \hat{\tau}_2(\hat{z}))\hat{z},
\end{aligned}$$

where $\hat{\tau}_i(\hat{z}) = \lambda_i((\nu \circ \hat{\psi})(\hat{z}))$, $i = 1, 2$. Thus seemingly different sets of singular value functions are potentially related by an orthogonal coordinate transformation, but that is not readily apparent in this example. In the next section we consider these issues in detail.

3. Sources of Nonuniqueness

In this section we examine two sources of nonuniqueness in computing the singular value functions of a system: the addition of a null matrix function and a (nonlinear) orthogonal coordinate transformation.

Null Matrix Functions

Let V be an open neighborhood of 0, and let $C^\infty(V)$ denote the abelian ring of smooth real-valued functions defined on V. (Addition and multiplication are defined in the obvious point-wise fashion on V, see for example [4].) Let $M_n(C^\infty(V))$ denote the set of $n \times n$ matrices with components from $C^\infty(V)$. Using the usual notions of matrix addition and multiplication, $M_n(C^\infty(V))$ is an associative ring with identity [2]. The subset $S_n(C^\infty(V))$ consists of all symmetric matrices in $M_n(C^\infty(V))$. We are interested in the following subset of $S_n(C^\infty(V))$.

Definition 3.1 *The subset $\mathcal{A}(V) \subset S_n(C^\infty(V))$ is the set of matrix-valued functions, A, with the following properties:*

- **i.** $A(0) = 0$.
- **ii.** $x^T A(x)x = 0,\ x \in V$.

Any $A \in \mathcal{A}(V)$ is called a **null matrix function** on V. Some properties of $\mathcal{A}(V)$ are considered in the following lemma, and then an application of this idea is given in the subsequent lemma.

Lemma 3.1 *For any neighborhood V of 0, the following statements are true:*

- **i.** *$\mathcal{A}(V)$ is a vector space over R.*
- **ii.** *$\mathcal{A}(V)$ is a module over $C^\infty(V)$.*
- **iii.** *The matrix $A \equiv 0$ is the only constant matrix in $\mathcal{A}(V)$.*
- **iv.** *The relation $M \sim M' \Leftrightarrow M - M' \in \mathcal{A}(V)$ is an equivalence relation on $S_n(C^\infty(V))$.*

Proof: Proofs of these statements are elementary. ∎

Lemma 3.2 *On any neighborhood V of 0 and for any $M, M' \in S_n(C^\infty(V))$*

$$x^T M(x)x = x^T M'(x)x,\ x \in V \Leftrightarrow M \sim M'.$$

Proof: The proof is trivial using the fact that the equivalence on the left-hand side also implies $M(0) = M'(0)$ ∎

Another interesting observation about the set $\mathcal{A}(V)$ is its relationship to an isotropy subgroup of the matrix group:

$$GL_n(C^\infty(V)) := \{T \in M_n(C^\infty(V)) : \exists S \in M_n(C^\infty(V)) \text{ with } TS = I\},$$

where I denotes the identity matrix [6]. Viewing $GL_n(C^\infty(V))$ as a transformation group on V with the usual group action

$$\begin{aligned} \psi &: GL_n(C^\infty(V)) \times V \mapsto V \\ &: (T, x) \mapsto T(x)x, \end{aligned}$$

the isotropy subgroup for any $x \in V$ is

$$I_x := \{T \in GL_n(C^\infty(V)) : T(x)x = x\}.$$

The corresponding isotropy subgroup for V is

$$I_V := \bigcap_{x \in V} I_x.$$

Now given any symmetric element $B \in I_V$, it is immediate that $I - B \in \mathcal{A}(V)$, that is,

$$x^T(I - B(x))x = x^T(x - B(x)x) = 0.$$

However, it is easy to find examples of null matrices with no corresponding element in I_V. Specifically, it is possible for $x^T A(x)x = 0$ everywhere on V without $A(x)x = 0$. Hence, the usual methods associated with matrix groups do not completely describe the nature of $\mathcal{A}(V)$.

Returning now to our main problem, we saw in the example from the previous section that the equivalence $M \sim M'$ on $S_n(C^\infty(V))$ does not imply equivalence of their respective spectrums. This is a fundamental source of nonuniqueness in the calculation of the singular value functions of a system. However, it is still possible to make some general statements relating their spectrums. This is done using the following results.

Lemma 3.3 *If $A \in \mathcal{A}(V)$ then we can write $A(x) = [a_{ij}(x)] = [\alpha_{ij}(x)x] = [\sum_{k=1}^n (\alpha_{ij}(x))_k x_k] := [\sum_{k=1}^n \alpha_{ijk}(x)x_k]$ on V where*

- **i.** $\alpha_{ijk}(0) = \frac{\partial a_{ij}}{\partial x_k}(0)$;
- **ii.** $\alpha_{ijk}(0) + \alpha_{kij}(0) + \alpha_{jki}(0) = 0$ *for all i, j, k;*
- **iii.** $\sum_{ijk}(\alpha_{ijk}(x) + \alpha_{kij}(x) + \alpha_{jki}(x))x_i x_j x_k = 0$ *on V.*

Proof:

i. This result follows from the fact that $A(0) = 0$ and applying Lemma 2.1 componentwise to A.

ii. Since $x^T A(x)x = 0$ everywhere on V then

$$\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}(x^T A(x)x)\bigg|_{x=0} = \frac{\partial a_{ij}}{\partial x_k}(0) + \frac{\partial a_{ki}}{\partial x_j}(0) + \frac{\partial a_{jk}}{\partial x_i}(0) = 0.$$

iii. Observe that:

$$\begin{aligned} x^T A(x)x &= \sum_{ij}(\alpha_{ij}(x)x)x_i x_j \\ &= \sum_{ijk}\alpha_{ijk}(x)x_i x_j x_k = 0 \end{aligned}$$

So

$$\begin{aligned} 3\sum_{ijk}\alpha_{ijk}(x)x_i x_j x_k &= 0 \\ \sum_{ijk}(\alpha_{ijk}(x) + \alpha_{kij}(x) + \alpha_{jki}(x))x_i x_j x_k &= 0. \end{aligned}$$

■

Next consider the following result from matrix perturbation theory adapted from [1] (see p. 163).

Theorem 3.1 *Let $M_0 \in \mathbb{R}^{n\times n}$ be a simple symmetric matrix with eigenvalues $\{\lambda_i\}_{i=1}^n$ and orthonormal eigenvectors $\{p_i\}_{i=1}^n$. For $\theta \in \mathbb{R}$ and symmetric matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$ define*

$$M(\theta) = M_0 + M_1\theta + M_2\theta^2.$$

For sufficiently small $|\theta|$, the matrix $M(\theta)$ is also simple, and its corresponding eigenvalues $\{\lambda_i(\theta)\}_{i=1}^n$ and orthonormal eigenvectors $\{p_i(\theta)\}_{i=1}^n$ depend analytically on θ, i.e.,

$$\begin{aligned} \lambda_i(\theta) &= \lambda_i^{(0)} + \lambda_i^{(1)}\theta + \lambda_i^{(2)}\theta^2 + \dots \\ p_i(\theta) &= p_i^{(0)} + p_i^{(1)}\theta + p_i^{(2)}\lambda^{(2)}\theta^2 + \dots \end{aligned}$$

for $i = 1, 2, \dots, n$. In particular,

$$\begin{aligned} \lambda_i^{(0)} &= \lambda_i \\ \lambda_i^{(1)} &= p_i^T M_1 p_i \\ \lambda_i^{(2)} &= p_i^T M_2 p_i + \sum_{\substack{j=1 \\ i\neq j}}^{N} \frac{1}{\lambda_i - \lambda_j}|p_i^T M_1 p_j|^2 \\ p_i^{(0)} &= p_i \\ p_i^{(1)} &= \sum_{\substack{j=1 \\ i\neq j}}^{N} \frac{p_i^T M_1 p_j}{\lambda_i - \lambda_j}. \end{aligned}$$

We now present the main results of the paper.

Theorem 3.2 *Suppose $M \in S_n(C^\infty(V))$ and $M(0)$ simple. Let $\{\lambda_i, p_i\}$ denote the smoothly defined eigenvalue and orthonormal eigenvector pairs for M on a neighborhood $\bar{V} \subset V$ of 0 (c.f. Lemma 2.3). Let $A \in \mathcal{A}(V)$ and define $M' = M + A$ with corresponding eigenvalues $\{\lambda_i'\}_{i=1}^n$. In the diagonalized coordinate frame $z = \phi^{-1}(x)$ for M, the eigenvalues of M and M' are equivalent to first order along their respective coordinate directions. That is, sufficiently close to 0*

$$\begin{aligned} &\lambda_i'(\psi(0,\dots,0,z_i,0,\dots,0)) = \\ &\qquad \lambda_i(\psi(0,\dots,0,z_i,0,\dots,0)) + \mathcal{O}(z_i^2). \quad (3) \end{aligned}$$

Proof: Let $M = P\Lambda P^T$ be the spectral decomposition of M on $\bar{V}$. Then it follows directly that for any $x \in \bar{V}$

$$\begin{aligned} M'(x) &= M(x) + A(x) \\ &= P(x)\Lambda(x)P^T(x) + A(x) \\ \underbrace{P^T(x)M'(x)P(x)}_{N(x)} &= \Lambda(x) + \underbrace{P^T(x)A(x)P(x)}_{B(x)}. \end{aligned}$$

Now set $z = P^T(x)x = \psi^{-1}(x)$ or $x = \psi(z)$, then

$$\begin{aligned} N(\psi(z)) &= \Lambda(\psi(z)) + B(\psi(z)) \\ \tilde{N}(z) &= \tilde{\Lambda}(z) + \tilde{B}(z). \end{aligned} \quad (4)$$

Note that $\tilde{N}(z)$ has the same eigenvalues as $M'(\psi(z))$ and $\tilde{B}(z) \in \mathcal{A}(\psi^{-1}(\bar{V}))$, that is,

$$\begin{aligned} \tilde{B}(0) &= B(\psi(0)) = B(0) = 0 \\ z^T\tilde{B}(z)z &= x^T P(x)\cdot P^T(x)A(x)P(x)\cdot P^T(x)x \\ &= x^T A(x)x = 0. \end{aligned}$$

Now evaluate equation (4) in the i-th coordinate direction:

$$\begin{aligned} \tilde{N}(0,\dots,0,z_i,0,\dots,0) &= \tilde{\Lambda}(0,\dots,0,z_i,0,\dots,0) + \\ &\quad \tilde{B}(0,\dots,0,z_i,0,\dots,0). \end{aligned}$$

If $|z_i|$ is sufficiently small then there exists a matrix B_i such that

$$\begin{aligned} \tilde{N}(0,\dots,0,z_i,0,\dots,0) &= \tilde{\Lambda}(0,\dots,0,z_i,0,\dots,0) + \\ &\quad B_i z_i + \mathcal{O}(z_i^2). \end{aligned}$$

In light of Theorem 3.1 it follows that

$$\begin{aligned} &\lambda_i'(\psi(0,\dots,0,z_i,0,\dots,0)) = \\ &\qquad \lambda_i(\psi(0,\dots,0,z_i,0,\dots,0)) + e_i^T B_i e_i\ z_i + \mathcal{O}(z_i^2) \end{aligned}$$

with $e_i = (\underbrace{0,\dots,0,1}_{i\text{-th position}},0,\dots,0)^T$. However, from Lemma 3.3, part ii. we have that $e_i^T B_i e_i = [B_i]_{ii} = 0$. Thus the theorem is proven. ■

Remarks:

1. In the context of the singular value functions, i.e., when $L_o(x) = \frac{1}{2}x^T M(x)x$ and $L_o'(x) = \frac{1}{2}x^T M'(x)x$, the identity (3) becomes

$$\lambda_i'(\psi(0,\ldots,0,z_i,0,\ldots,0)) = \tau_i(0,\ldots,0,z_i,0,\ldots,0) + \mathcal{O}(z_i^2).$$

The lefthand side of this identity is only equivalent to the true singular value functions for M' if the diagonalizing transformation $z' = (\psi')^{-1}(x)$ for M' is identical to the diagonalizing transformation $z = \psi^{-1}(x)$ for M. This is the case in Example 2.1 from the previous section, M and M' are simultaneously diagonalized by the same coordinate transformation.

2. In general the identity (3) is not true to second order. However, if matrix $B_1 = 0$ in the proof of Theorem 3.2 then it follows from the expression for $\lambda_i^{(2)}$ in Theorem 3.1 that we have equality up to second order. This is also the case in Example 2.1.

Norm Preserving Coordinate Transformations

A smooth coordinate transformation $y = \nu(x)$ is said to be *norm preserving* on a convex neighborhood of the origin, W, if $\|y\| = \|x\|$ for all $x \in W$. Since all such maps satisfy $\nu(0) = 0$, it follows directly from Lemma 2.1 that there exists at least one factorization of the form $\nu(x) = T(x)x$ where $T \in M_n(C^\infty(W))$. Thus, it is immediate that everywhere on W

$$\|T(x)x\|^2 = x^T T^T(x)T(x)x = x^T x$$

or equivalently $T^T(x)T(x) = I + A(x)$ for some $A \in \mathcal{A}(W)$. A specific class of norm preserving transformations are the so called *orthogonal* transformations, which are characterized by having a factorization $\nu(x) = T(x)x$ where $T^T(x)T(x) = I$. In the context of energy functions, norm preserving transformations are interesting because they preserve input-normal forms, that is,

$$\bar{L}_c(y) = \frac{1}{2}y^T y = \frac{1}{2}x^T x = L_c(x).$$

In the following theorem, we see that orthogonal coordinate transformations also preserve the singular value functions in a natural sense.

Theorem 3.3 *Consider a system (f,g,h) with singular value functions, τ_i, $i = 1,\ldots,n$ derived from a specific input-normal form: $L_c(x) = \frac{1}{2}x^T x$, $L_o(x) = \frac{1}{2}x^T M(x)x$. Any orthogonal coordinate transformation, $y = \nu(x) = T(x)x$, yields the corresponding singular value functions $\bar{\tau} = \tau \circ \nu^{-1}$, $i = 1,\ldots,n$.*

Proof: After performing the indicated coordinate transformation, the new system has an input-normal form where

$$\bar{L}_o(y) = \frac{1}{2}y^T T(\nu^{-1}(y))M(\nu^{-1}(y))T^{-1}(\nu^{-1}(y))y.$$

Since this correspondings to a similarity transformation on $M(\nu^{-1}(y))$, the theorem is proven. ■

Remarks:

1. Observe that if ν is merely norm preserving but not orthogonal, then the matrix transformation above on $M(\nu^{-1}(y))$ is only a congruence transformation. Thus, the corresponding singular value functions are not preserved.

2. Given a fixed factorization $L_c(x) = \frac{1}{2}x^T M(x)x$, let k be the number of distinct singular value functions, and let j_i be the number of times the i^{th} singular value function appears. Then it follows directly from Theorem 2.2 that both systems have the same balanced form, except for a coordinate transformation of the form

$$x = diag(T_1(\bar{x}),\ldots,T_k(\bar{x}))\bar{x} \qquad (5)$$

where the blocks $T_i(\bar{x})$, $i = 1,\ldots,k$, are $j_i \times j_i$ orthogonal matrices, i.e., $T_i(\bar{x})^T T_i(\bar{x}) = I$, with entries that are smooth functions of $\bar{x}$. This result is very analogous to what happens in the linear system case.

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