Balancing For Nonlinear Systems

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Abstract

We present a method of balancing for nonlinear systems which is an extension of balancing for linear systems in the sense that it is based on the input and output energy of a system. We deal with the input and output energy function of a stable nonlinear systems and propose a method to use these functions to get a balanced form for a stable nonlinear system. It is a local result, but gives 'broader' results then we obtain by just linearizing the system.

Keywords: balancing, nonlinear systems, energy functions, Hamilton-Jacobi equations, Hankel singular values.

1 Introduction

Balancing for linear systems is a well known subject on which there has been a lot of research in the last decade. It started with a paper of Moore [6] in 1981, where balancing is introduced with the aim of using it as a tool for model reduction. If a linear system is in balanced form the Hankel singular values are a measure for the importance of state components. This means that the influence of the corresponding state component on the output and input energy is measured by a Hankel singular value. If a Hankel singular value is small the influence of the state component on the output and input energy is respectively low and high and thus this state component may be deleted in order to obtain a reduced-order model. A Hankel norm error bound of model reduction based on balancing is given by Glover in [1].

Here we give a set up of balancing for nonlinear systems. The intuitive idea behind model reduction for linear systems can be extended to nonlinear systems. Again, as in the linear case, the importance of state components can be measured in terms of the input and output energy. Instead of the Hankel singular values we define for nonlinear systems singular value functions, measuring again the importance of a state component.

In section 2 we give a very brief review on balancing for linear systems. Section 3 contains properties of the input and output energy functions for linear systems. These properties are instrumental in the set up for balancing of nonlinear systems. In section 4 we go into balancing for nonlinear systems and define the singular value functions. We propose a procedure to bring a nonlinear system in balanced form. Finally in section 5 we give some conclusions.

Throughout this paper we will use a fairly standard notation. We denote by $x^T x$ or $\| x \|^2$ the squared norm of a vector $x \in \mathbb{R}^n$. We say that $u : (-\infty, 0) \to \mathbb{R}^m$ is in $L_2(-\infty, 0)$ if $\int_{-\infty}^0 \| u(t) \|^2 \, dt < \infty$. By $\frac{\partial L}{\partial x}(x)$ we denote the row-vector of partial derivatives of a differentiable function $L : \mathbb{R}^n \to \mathbb{R}$. Furthermore we denote by $x(t_2) = \varphi(t_2, t_1, x_1, u)$ the solution on time t_2 of the system $\dot{x} = f(x) + g(x)u$ with initial condition $x(t_1) = x_1$ and input $u : [t_1, t_2] \to \mathbb{R}^m$.

2 Review of balancing for linear systems

Consider a linear system:

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{1}$$

where $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. We assume throughout (1) is stable, controllable and observable.

Definition 2.1 The controllability and observability function of a linear system are defined as respectively

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2} \int_{-\infty}^{0} \| u(t) \|^2 \, dt \tag{2}$$

$$L_o(x_0) = \frac{1}{2} \int_0^{\infty} \| y(t) \|^2 \, dt, \;\; x(0) = x_0, \;\; u \equiv 0 \tag{3}$$

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The value of the controllability function at x_0 is the minimum amount of control energy required to reach the state x_0 and the value of the observability function at x_0 is the amount of output energy generated by the state x_0. The following results are well known (cf. [6]):

Theorem 2.2 *For system (1) we have* $L_c(x_0) = \frac{1}{2}x_0^T W^{-1} x_0$ *where* $W = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$ *is the controllability gramian and* $L_o(x_0) = \frac{1}{2}x_0^T M x_0$ *where* $M = \int_0^\infty e^{A^Tt}C^TCe^{At}dt$ *is the observability gramian. Furthermore* W *and* M *are the unique positive definite solutions of the following Lyapunov equations:*

$$AW + WA^T = -BB^T \tag{4}$$

$$A^TM + MA = -C^TC \tag{5}$$

Theorem 2.3 *There exists a state space transformation* $x = S\bar{x}$ *for system (1) such that the transformed system*

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \qquad y = \bar{C}\bar{x} \tag{6}$$

is in balanced form, i.e.:

$$\bar{W} = \bar{M} = \Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix} \tag{7}$$

with $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n$ *and* $\bar{W} = \bar{M} = S^{-1}WS^{-T} = S^TMS$ *are the controllability and observability gramian of the transformed system (6). Here the* σ_i*'s, i=1,..,n, are the Hankel singular values, i.e. the singular values of the Hankel operator of the system (see [1]).*

For system (6) we have as controllability and observability function respectively $\bar{L}_c(\bar{x}_0) = \frac{1}{2}\bar{x}_0^T\Sigma^{-1}\bar{x}_0$ and $\bar{L}_c(\bar{x}_0) = \frac{1}{2}\bar{x}_0^T\Sigma\bar{x}_0$. For small σ_i the amount of control energy required to reach the state $\hat{x} = (0\ldots0\ \bar{x}_i\ 0\ldots0)$ is large while the output energy generated by this state $\hat{x}$ is small. Hence if $\sigma_k >> \sigma_{k+1}$, the state components $\bar{x}_{k+1}$ to $\bar{x}_n$ are not important from this energy point of view and can be removed to reduce the number of state components of the model.

3 The controllability and observability function of nonlinear systems

Consider a smooth, i.e., C^∞, nonlinear system of the form

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x) \tag{8}$$

where $u = (u_1 \ldots u_m) \in \mathbb{R}^m$, $y = (y_1 \ldots y_p) \in \mathbb{R}^p$ and $x = (x_1 \ldots x_n)$ are local coordinates for a smooth state space manifold denoted by M. Throughout we assume that the system has an equilibrium. Without loss of generality we take this equilibrium in 0, i.e. $f(0) = 0$ and we also take $h(0) = 0$. The controllability and observability function, respectively L_c and L_o, of system (8) are defined in the same way as in section 2 for linear systems. Again the value of the controllability function at x_0 is the minimum amount of control energy required to reach x_0 and the value of the observability function at x_0 is the amount of output energy generated by x_0.

Definition 3.1 The controllability and observability function of a nonlinear system are defined as respectively

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^0 \| u(t) \|^2\, dt \tag{9}$$

$$L_o(x_0) = \frac{1}{2}\int_0^\infty \| y(t) \|^2\, dt,\ x(0) = x_0,\ u \equiv 0 \tag{10}$$

These functions do not necessarily exist. In particular, L_o can be infinite if the system is unstable and if x_0 can not be reached from 0, then by convention $L_c(x_0)$ will be infinite. In this section we assume throughout that L_c and L_o are *finite*. Also, for the rest of this paper we assume L_c and L_o are *smooth* functions of x.

Theorem 3.2 *If 0 is an asymptotically stable equilibrium of* $f(x)$ *on a neighborhood* W *of 0, then for all* $x \in W$, $L_o(x)$ *is the unique smooth solution of*

$$\frac{\partial L_o}{\partial x}(x)f(x) + \frac{1}{2}h^T(x)h(x) = 0, \qquad L_o(0) = 0 \tag{11}$$

under the assumption that (11) has a smooth solution on W*. Furthermore for all* $x \in W$, $L_c(x)$ *is the unique smooth solution of*

$$\frac{\partial L_c}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial L_c}{\partial x}(x)g(x)g^T(x)\frac{\partial^T L_c}{\partial x}(x) = 0, \quad L_c(0) = 0 \tag{12}$$

under the assumption that (12) has a smooth solution $\bar{L}_c$ *on* W *and that 0 is an asymptotically stable equilibrium of* $-(f(x) + g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ *on* W.

Proof Assume (11) has on W as smooth solution $\bar{L}_o(x)$. Then $\frac{1}{2}h^T(x)h(x) = -\frac{\partial \bar{L}_o}{\partial x}(x)f(x)$. L_o is defined as in definition 3.1 and thus:

$L_o(x_0) = \frac{1}{2}\int_0^\infty h^T(x(t))h(x(t))dt =$
$-\int_0^\infty \frac{\partial \bar{L}_o}{\partial x}(x(t))f(x(t))dt = -\int_0^\infty \frac{\partial}{\partial t}\bar{L}_o(x(t))dt =$
$-\bar{L}_o(x(\infty)) + \bar{L}_o(x(0)) = \bar{L}_o(x_0), \qquad \forall x_0 \in W$

since $x(0) = x_0$ and $x(\infty) = 0$ by the asymptotic stability of $f(x)$. Hence part 1 is proven. For part 2 we assume (12) has on W a smooth solution $\bar{L}_c(x)$. Then $\frac{\partial \bar{L}_c}{\partial x}(x)f(x) = -\frac{1}{2}\frac{\partial \bar{L}_c}{\partial x}(x)g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x)$. As in definition 3.1 we consider the inputs u such that $x(0) = x_0 \in W$ and $x(-\infty) = 0$, then

$\frac{d}{dt}\bar{L}_c(x) = \frac{\partial \bar{L}_c}{\partial x}(x)\dot{x} = \frac{\partial \bar{L}_c}{\partial x}(x)f(x) + \frac{\partial \bar{L}_c}{\partial x}(x)g(x)u =$
$\frac{1}{2}u^Tu - \frac{1}{2}u^Tu + \frac{\partial \bar{L}_c}{\partial x}(x)g(x)u$
$-\frac{1}{2}\frac{\partial \bar{L}_c}{\partial x}(x)g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x) =$
$\frac{1}{2} \| u \|^2 - \frac{1}{2} \| u - g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x) \|^2,$

and thus

$$\bar{L}_c(x_0) = \int_{-\infty}^0 \frac{d}{dt}\bar{L}_c(x(t))dt = \frac{1}{2}\int_{-\infty}^0 \| u(t) \|^2 \, dt$$
$$-\frac{1}{2}\int_{-\infty}^0 \| u(t) - g^T(x(t))\frac{\partial^T \bar{L}_c}{\partial x}(x(t)) \|^2 \, dt$$
$$\leq \frac{1}{2}\int_{-\infty}^0 \| u(t) \|^2 \, dt, \qquad \forall x_0 \in W$$

Hence $\bar{L}_c(x_0)$ is a lower bound for $\frac{1}{2}\int_{-\infty}^0 \| u(t) \|^2 \, dt$. It is clear that for $u = g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x)$, $\bar{L}_c$ is equal to its lower bound. By asymptotic stability of $-(f(x) + g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ on W this latter input is such that $x(-\infty) = 0$. Therefore for all $x_0 \in W$

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^0 \| u(t) \|^2 \, dt$$
$$= \bar{L}_c(x_0) \qquad \blacksquare$$

Remark 3.3 Equation (11) is a nonlinear Lyapunov type of equation and equation (12) is a Hamilton-Jacobi equation associated with an optimal control problem.

Remark 3.4 If L_o is a solution of (11), we can conclude from the negative semi-definiteness of $\frac{\partial L_o}{\partial x}(x)f(x)$ that L_o is decreasing along f. Since f is asymptotically stable, 0 is a minimum for L_o and hence L_o is *non-negative*. If L_c is a solution of (12), we can conclude from the negative semi-definiteness of $\frac{\partial L_c}{\partial x}(x)(-(f(x)+g(x)g(x)^T\frac{\partial^T L_c}{\partial x}(x)))$, that L_c is decreasing along $-(f(x) + g(x)g(x)^T\frac{\partial^T L_c}{\partial x}(x))$. Again, since $-(f(x) + g(x)g(x)^T\frac{\partial^T L_c}{\partial x}(x))$ is asymptotically stable, 0 is a minimum for L_c and hence L_c is *non-negative*.

Remark 3.5 If $A = \frac{\partial f}{\partial x}(0)$ is asymptotically stable then locally about 0 (11) and (12) have smooth solutions, see [8].

Remark 3.6 If we replace the condition that (12) has a smooth solution $\bar{L}_c$ on W and 0 is an asymptotically stable equilibrium of $-(f(x)+g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ by the condition that (12) has a smooth solution $\bar{L}_c$ that is positive definite, then we get the same result, see theorem 3.7.

Theorem 3.7 *Assume 0 is an asymptotically stable equilibrium of f on W and (12) has a smooth solution $\bar{L}_c$ on W. Then $\bar{L}_c(x_0) > 0$ for $x_0 \in W$, $x_0 \neq 0$, if and only if 0 is an asymptotically stable equilibrium of $-(f(x) + g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ on W.*

Proof Assume $\bar{L}_c(x_0) > 0$ for $x_0 \in W$, $x_0 \neq 0$. We know that on W

$$\frac{\partial \bar{L}_c}{\partial x}(x)(-(f(x) + g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x)) =$$
$$-\frac{1}{2}\frac{\partial \bar{L}_c}{\partial x}(x)g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x) \leq 0$$

This negative semi-definiteness implies that $\bar{L}_c$ is a Lyapunov function for $-(f(x) + g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ which therefore is stable on W. To prove asymptotic stability we need to find the maximal invariant set of $-(f(x)+g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ in $V := \{x | \frac{\partial \bar{L}_c}{\partial x}(x)g(x) = 0\}$. This is the same as finding the maximal invariant set of $-f(x)$ in V. By (12) we know that $V = \{x | \frac{\partial \bar{L}_c}{\partial x}(x)f(x) = 0\} = \{x | \frac{d}{dt}\bar{L}_c(x(t)) = 0\}$. Since 0 is an asymptotically stable equilibrium of f and $\bar{L}_c$ positive definite on W we conclude from this that the maximal invariant set in V is $\{0\}$ and LaSalle's invariance principle thus implies that $-(f(x) + g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ is asymptotically stable on W.

For the if part of the theorem we use theorem 3.2. This states that $\bar{L}_c = L_c$ on W, where L_c is the controllability function of system (8). Furthermore we know from the proof of theorem 3.2 that the minimum is taken for $u = g^T(x)\frac{\partial^T L_c}{\partial x}(x)$. Hence

$$L_c(x_0) = \frac{1}{2}\int_{-\infty}^0 \frac{\partial L_c}{\partial x}(x(t))g(x(t))g^T(x(t))\frac{\partial^T L_c}{\partial x}(x(t))dt$$

Let now $x_0 \neq 0$. If $\frac{\partial L_c}{\partial x}(x(t))g(x(t)) = 0$ for $-\infty \leq t \leq 0$ then $u(t) = 0$, for all t, $-\infty \leq t \leq 0$. However, since f is asymptotically stable, we cannot have $x(-\infty) = 0$ and $x(0) = x_0 \neq 0$. Hence we have a contradiction and thus there exists a t, $-\infty \leq t \leq 0$, such that $\frac{\partial L_c}{\partial x}(x(t))g(x(t))g^T(x(t))\frac{\partial^T L_c}{\partial x}(x(t)) > 0$. This implies that $L_c(x_0) > 0, \forall x_0 \in W, x_0 \neq 0$. $\blacksquare$

For the following definition see e.g. [8].

Definition 3.8 The system (8) is reachable from x_0 if for any $\bar{x} \in M$ there exists a $\bar{t} \geq 0$ and input u such that $\bar{x} = \varphi(\bar{t}, 0, x_0, u)$.
The system (8) is zero-state observable if any trajectory such that $u(t) \equiv 0, y(t) \equiv 0$ implies $x(t) \equiv 0$, i.e., for all $x \in M$, $h(\varphi(t, 0, x, 0)) = 0, t \geq 0 \Rightarrow \varphi(t, 0, x, 0) = 0, t \geq 0$.

The following theorems are related to some results in [3] and [8].

Theorem 3.9 *If the system (8) is zero-state observable and (11) has a smooth positive definite solution $\tilde{L}_o$, then the system $\dot{x} = f(x)$ is locally asymptotically stable. If $\tilde{L}_o$ is proper (i.e. for each $c > 0$ the set $\{x \in M | 0 \leq \tilde{L}_o(x) \leq c\}$ is compact), then $\dot{x} = f(x)$ is globally asymptotically stable.*

Proof $\frac{\partial \tilde{L}_o}{\partial x}(x)f(x) = -\frac{1}{2}h^T(x)h(x) \leq 0$ and by the zero-state observability $\frac{\partial \tilde{L}_o}{\partial x}(x)f(x) = 0 \Rightarrow x = 0$ under the dynamics of the system. Global asymptotic stability follows by LaSalle's invariance principle. $\blacksquare$

Theorem 3.10 *Assume 0 is an asymptotically stable equilibrium of $f(x)$ on a neighborhood W of 0. If the system (8) is zero-state observable and (11) has on W the smooth solution L_o, then $L_o(x_0) > 0$, $\forall x_0 \in W$, $x_0 \neq 0$.*

Proof By theorem 3.2 and definition 3.1 $\forall x_0 \in W$

$$L_o(x_0) = \frac{1}{2}\int_0^\infty \parallel y(t) \parallel^2 dt, \; x(0) = x_0, \; u(t) \equiv 0$$

Zero-state observability implies that for some $\tau > 0$ we have $h(\varphi(t, 0, x_0, 0)) \neq 0$ for $0 \leq t < \tau$. Hence for $x_0 \neq 0$

$$L_o(x_0) = \frac{1}{2}\int_0^\infty h^T(x(t))h(x(t))dt > 0 \qquad \blacksquare$$

4 Balancing for nonlinear systems

In the rest of this paper we consider nonlinear systems of the form (8) with controllability and observability function L_c respectively L_o as given in definition 3.1 and with the following standing assumptions:

1. 0 is an asymptotically stable equilibrium of $f(x)$ on some neighborhood Y of 0
2. $A = \frac{\partial f}{\partial x}(0)$ is asymptotically stable
3. the system is zero-state observable and reachable from 0 on Y
4. L_o exists on Y
5. (11) and (12) have smooth solutions on Y
6. 0 is an asymptotically stable equilibrium of $-(f(x)+g(x)g^T(x)\frac{\partial^T L_c}{\partial x}(x))$ on Y
7. $\frac{\partial^2 L_c}{\partial x^2}(0) > 0$ and $\frac{\partial^2 L_o}{\partial x^2}(0) > 0$

The reachability from 0 implies that L_c exists on Y, i.e. is finite. Assumption 6 can be replaced by $\det\frac{\partial^2 L_c}{\partial x^2}(0) \neq 0$ and $\det\frac{\partial^2 L_o}{\partial x^2}(0) \neq 0$, since we already know that both are non-negative definite matrices. By section 3 we know that these assumptions imply among other things that L_o is the smooth positive definite solution of (11) and L_c the smooth positive definite solution of (12). These assumptions also imply that (A, B) is controllable and that (C, A) is observable, where $B = g(0)$ and $C = \frac{\partial h}{\partial x}(0)$.

Lemma 4.1 *There exists a coordinate transformation $x = \phi(\bar{x})$, $\phi(0) = 0$, such that $L_c(x)$ in the new coordinates $\bar{x} = \phi^{-1}(x)$ is of the following form:*

$$L_c(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T\bar{x} \tag{13}$$

Furthermore we can write $L_o(x)$ in the new coordinates $\bar{x} = \phi^{-1}(x)$ in the following form:

$$L_o(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T M(\bar{x})\bar{x} \;\; \textit{with} \;\; M(0) = \frac{\partial^2 L_o}{\partial x^2}(0) \tag{14}$$

and $M(\bar{x})$ is a $n\times n$ symmetric matrix with entries which are smooth functions of $\bar{x}$.

Proof Since 0 is a minimum of the observability and controllability function we have beside $L_o(0) = 0$ and $L_c(0) = 0$ that $\frac{\partial L_o}{\partial x}(0) = 0$ and $\frac{\partial L_c}{\partial x}(0) = 0$. Therefore we can apply Morse's Lemma (see lemma 2.2 in [5]) to L_c. In this case it means that there exists local coordinates $\bar{x} = (\bar{x}_1 \ldots \bar{x}_n)$ such that $x = \phi(\bar{x})$, $\phi(0) = 0$ and such that $L_c(x)$ in the new coordinates $\bar{x} = \phi^{-1}(x)$ has the form (13). Finally (14) follows by repeated application of lemma 2.1 from [5]. $\blacksquare$

Lemma 4.2 *If there exists a neighborhood V of 0 where the number of distinct eigenvalues of $M(\bar{x})$ is constant for $\bar{x} \in V$, then on V the eigenvalues $\lambda_i(\bar{x})$, $i = 1,..,n$, are smooth functions of $\bar{x}$, as well as the associated eigenvectors.*

Proof This follows from Theorem 5.13a in [4]. $\blacksquare$

Theorem 4.3 *Consider system (8) and assume the condition of lemma 4.2 is fulfilled. On a neighborhood U of zero there exists a coordinate transformation $x = \psi(z)$, $\psi(0) = 0$, such that $L_c(x)$ in the new coordinates $z \in W := \psi^{-1}(U)$ is of the following form:*

$$\tilde{L}_c(z) := L_c(\psi(z)) = \frac{1}{2}z^T z \tag{15}$$

while L_o is for the new coordinates of the following form:

$$\tilde{L}_o(z) := L_o(\psi(z)) = \frac{1}{2}z^T \begin{pmatrix} \tau_1(z) & & 0 \\ & \ddots & \\ 0 & & \tau_n(z) \end{pmatrix} z \tag{16}$$

where $\tau_1(z) \geq ... \geq \tau_n(z)$ are smooth functions of z, called the <u>singular value functions</u>.

Proof From lemma 4.1 we know that there exists a transformation $x = \phi(\bar{x})$, $\phi(0) = 0$ such that in the new coordinates L_c and L_o are of the form (13) respectively (14). By lemma 4.2 we know that on V the eigenvalues of $M(\bar{x})$ and the associated eigenvectors are smooth functions of $\bar{x}$. Furthermore we know that $M(0) > 0$ which means that $M(0)$ is diagonalizable. By the smoothness of the eigenvalues and eigenvectors this implies that $M(\bar{x})$ is diagonalizable on V. Indeed, since $M(\bar{x})$ is symmetric, we can write $M(\bar{x}) = T(\bar{x})\Lambda(\bar{x})T^T(\bar{x})$ where

$$\Lambda(\bar{x}) = \begin{pmatrix} \lambda_1(\bar{x}) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(\bar{x}) \end{pmatrix}$$

with $\lambda_1(\bar{x}) \leq ... \leq \lambda_n(\bar{x})$, $\lambda_i(\bar{x}), i = 1,..,n$, are the eigenvalues of $M(\bar{x})$ and $T(\bar{x})$ is the corresponding matrix of eigenvectors with $T(\bar{x})$ an orthogonal matrix, i.e. $T^T(\bar{x})T(\bar{x}) = I$, $\bar{x} \in V$. Now we can rewrite (14) as:

$$L_o(\phi(\bar{x})) = \tfrac{1}{2}\bar{x}^T T(\bar{x})\Lambda(\bar{x})T^T(\bar{x})\bar{x}, \quad \bar{x} \in V$$

Define a new coordinate transformation $z = \nu(\bar{x}) := T^T(\bar{x})\bar{x}$. In these coordinates we get:

$$L_o(\phi(\nu^{-1}(z))) = \frac{1}{2}z^T\Lambda(\nu^{-1}(z))z, \;\; z \in W := \nu(V)$$

From (13) and $\bar{x} = T(\bar{x})z$ we get:

$$L_c(\phi(\nu^{-1}(z))) = \frac{1}{2}z^T T^T(\bar{x})T(\bar{x})z = \frac{1}{2}z^T z$$

Define $\tau_i(z) := \lambda_i(\nu^{-1}(z))$, $i = 1,..,n$, the transformation $\psi := \phi \circ \nu^{-1}$ and $U := \phi^{-1}(V)$, then the theorem is proven. ∎

Remark 4.4 For a linear system the singular value functions τ_i, $i = 1,..,n$ are constants and are the squared Hankel singular values.

The form of the controllability and observability function in (15) and (16) is not yet entirely balanced. For that we need an additional coordinate transformation. We take as transformation $\bar{z}_i = \eta_i(z_i) := \tau_i(0,..,0,z_i,0,..0)^{\frac{1}{4}} z_i$, $i = 1,..,n$ and hence $\bar{z} = \eta(z) := (\eta_1(z_1) \dots \eta_n(z_n))$ on $\bar{z} \in \bar{W} := \eta(W)$. Since $\tilde{L}_o(z) > 0$ we have that $\tau_i(0,..,0,z_i,0,..,0) > 0$, $i = 1,..,n$, for $z \in W$, $z \neq 0$ and therefore η is a well defined transformation. Define $\bar{L}_c(\bar{z}) := \tilde{L}_c(\eta^{-1}(\bar{z}))$ and $\bar{L}_o(\bar{z}) := \tilde{L}_o(\eta^{-1}(\bar{z}))$. Then (15) and (16) become respectively:

$$\bar{L}_c(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} \sigma_1(\bar{z}_1)^{-1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n(\bar{z}_n)^{-1} \end{pmatrix} \bar{z} \qquad (17)$$

$$\bar{L}_o(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} \omega_1(\bar{z}) & & 0 \\ & \ddots & \\ 0 & & \omega_n(\bar{z}) \end{pmatrix} \bar{z} \qquad (18)$$

where $\sigma_i(\bar{z}_i) = \tau_i(0,..,0,\eta_i^{-1}(\bar{z}_i),0,..,0)^{\frac{1}{2}}$ and $\omega_i(\bar{z}) = \sigma_i(\bar{z}_i)^{-1}\tau_i(\eta^{-1}(\bar{z}))$ for $i = 1,..,n$. Now (17) and (18) have the property that $\bar{L}_c(0,..,0,\bar{z}_i,0,..,0) = \frac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i)^{-1}$ and $\bar{L}_o(0,..,0,\bar{z}_i,0,..,0) = \frac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i)$ for $1,..,n$. This corresponds with the linear theory, since in that case σ_i is constant and thus σ_i is a Hankel singular value, $i = 1,..,n$. We know that $\tau_1(\bar{z}) \geq \ldots \geq \tau_n(\bar{z})$ for $\bar{z} \in \bar{W}$. In energy terms we can say that if $\tau_i(\bar{z}) > \tau_{i+1}(\bar{z})$ the state variable $\bar{z}_i$ is more important then the state variable $\bar{z}_{i+1}$ on $\bar{W}$. Similar to the concept of balancing for linear systems we call the nonlinear system *balanced* if it has a controllability and observability function of the form of (17) and (18). This means that we can balance system (8) by a coordinate transformation of the form $x = \chi(\bar{z}) := \psi(\eta^{-1}(\bar{z}))$ where ψ is as in theorem 4.3.

Example 4.5 We take system (8) which fulfills the conditions of section 4 with $x = (x_1, x_2)$, $u = (u_1, u_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$ and f, g and h as follows:

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = -\begin{pmatrix} \dfrac{625x_1 + 112x_1^3 + 552x_1^2x_2 + 639x_1x_2^2 + 216x_2^3}{625} \\ \dfrac{384x_1^3 + 625x_2 + 464x_1^2x_2 + 48x_1x_2^2 - 63x_2^3}{625} \end{pmatrix}$$

$$\begin{aligned} g(x) &= \begin{pmatrix} g_{11}(x) & g_{12}(x) \\ g_{21}(x) & g_{22}(x) \end{pmatrix} \\ &= \begin{pmatrix} \frac{3\sqrt{2}}{5} & \frac{4\sqrt{2}}{25}\sqrt{25 + 7x_1^2 + 48x_1x_2 - 7x_2^2} \\ \frac{-4\sqrt{2}}{5} & \frac{3\sqrt{2}}{25}\sqrt{25 + 7x_1^2 + 48x_1x_2 - 7x_2^2} \end{pmatrix} \\ h(x) &= \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} = \begin{pmatrix} \frac{2}{5}(3x_1 - 4x_2) \\ \frac{\sqrt{2}}{25}(4x_1 + 3x_2)^2 \end{pmatrix} \end{aligned}$$

Solving the Hamilton-Jacobi equations (11) and (12) we get:

$$L_c(x) = \frac{1}{2}x^T x, \quad L_o(x) = \frac{1}{2}x^T \begin{pmatrix} m_{11}(x) & m_{12}(x) \\ m_{21}(x) & m_{22}(x) \end{pmatrix} x$$

where $m_{12}(x) = m_{21}(x)$ and

$$\begin{aligned} m_{11}(x) &= \tfrac{2}{625}(425 + 72x_1^2 - 192x_1x_2 + 128x_2^2) \\ m_{12}(x) &= \tfrac{12}{625}(-25 + 9x_1^2 - 24x_1x_2 + 16x_2^2) \\ m_{22}(x) &= \tfrac{1}{625}(1025 + 81x_1^2 - 216x_1x_2 + 144x_2^2) \end{aligned}$$

We see the system already has the form of lemma 4.1. Hence the first step of the proof of theorem 4.3 is done by taking the transformation $x = \phi(\bar{x}) = \bar{x}$. The eigenvalues of $M(x)$ are:

$$\begin{aligned} \lambda_1(x) &= \tfrac{1}{25}(25 + 9x_1^2 - 24x_1x_2 + 16x_2^2) \\ &= 1 + (\tfrac{1}{5}(3x_1 - 4x_2))^2) \\ \lambda_2(x) &= 2 \end{aligned}$$

It immediately follows that the neighborhood V of zero where the number of distinct eigenvalues is constant, is $V = \{x | (3x_1 - 4x_2)^2 < 25\}$ The unitary matrix with eigenvectors is the following:

$$T(x) = T = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$

Now the second coordinate transformation of the proof of theorem 4.3 on V is $z = \nu(x) = T^T x = \frac{1}{5}(3x_1 - 4x_2, 4x_1 + 3x_2)$. Hence the coordinate transformation ψ of theorem 4.3 is $\psi(z) = \nu^{-1}(z)$ and $W = \psi^{-1}(V) = \nu(V) = \{z | z_1^2 < 1\}$. Furthermore in the new coordinates the controllability and observability function, (15) and (16), become

$$\tilde{L}_c(z) = \frac{1}{2}z^T z, \quad \tilde{L}_o(z) = \frac{1}{2}z^T \begin{pmatrix} 2 & 0 \\ 0 & 1 + z_1^2 \end{pmatrix} z, \quad z \in W$$

where the singular value functions are $\tau_1(z) = 2$ and $\tau_2(z) = 1 + z_1^2$. The system transforms into:

$$\begin{cases} \dot{z}_1 = -z_1 + z_1 z_2^2 + u_1\sqrt{2} \\ \dot{z}_2 = -z_2 - z_2^3 + u_2\sqrt{2 - 2z_1^2 + 2z_2^2} \end{cases} \qquad \begin{cases} y_1 = 2z_1 \\ y_2 = \sqrt{2}z_2 \end{cases}$$

for $z \in W$. To bring this system in balanced form we need to take the coordinate transformation $\bar{z} = \eta(z) =$

$(2^{\frac{1}{4}}z_1, z_2)$, $\bar{z} \in \bar{W} = \nu(W) = \{\bar{z} | \bar{z}_1^2 < 2^{\frac{1}{2}}\}$. Then (17) and (18) become

$$\bar{L}_c(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} 2^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} \bar{z}$$

$$\bar{L}_o(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} 2^{\frac{1}{2}} & 0 \\ 0 & 1 + 2^{-\frac{1}{2}}\bar{z}_1^2 \end{pmatrix} \bar{z}$$

and in the coordinates $\bar{z} \in \bar{W}$ the system is in balanced form $\dot{\bar{z}} = \bar{f}(\bar{z}) + \bar{g}(\bar{z})u$, $y = \bar{h}(\bar{z})$ with

$$\bar{f}(\bar{z}) = \begin{pmatrix} -\bar{z}_1 + \bar{z}_1\bar{z}_2^2 \\ -\bar{z}_2 - \bar{z}_2^3 \end{pmatrix},$$

$$\bar{g}(\bar{z}) = \begin{pmatrix} 2^{\frac{3}{4}} & 0 \\ 0 & (2 - 2^{\frac{1}{2}}\bar{z}_1^2 + 2\bar{z}_2^2)^{\frac{1}{2}} \end{pmatrix},$$

$$\bar{h}(\bar{z}) = \begin{pmatrix} 2^{\frac{3}{4}}\bar{z}_1 \\ 2^{\frac{1}{2}}\bar{z}_2 \end{pmatrix}$$

■

5 Conclusions

We introduced balancing for stable nonlinear systems. This method is an extension of balancing for linear systems, since we considered the input and output energy function of a stable nonlinear system in a similar way as we do this for stable linear systems. The properties of the input and output energy functions for a nonlinear system match with the properties of these functions for a linear system and therefore we used these functions to balance the system about the equilibrium. The method can be used as a tool for model reduction of stable nonlinear systems.

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