HAMILTONIAN REALIZATIONS OF NONLINEAR ADJOINT OPERATORS

Kenji Fujimoto [*,**] **Jacquelien M.A. Scherpen** [*]
W. Steven Gray [***]

[*] *Faculty of Information Technology and Systems*
Department of Electrical Engineering
Delft University of Technology
P.O.Box 5031, 2600 GA Delft, The Netherlands
[**] *Department of Systems Science*
Graduate School of Informatics, Kyoto University
Uji, Kyoto 611-0011 Japan
[***] *Department of Electrical and Computer Engineering*
Old Dominion University
Norfolk, Virginia 23529-0246, U.S.A.
`fujimoto@i.kyoto-u.ac.jp`
`J.M.A.Scherpen@its.tudelft.nl, gray@ece.odu.edu`

In: Proc. IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control, Princeton, USA, 2000, pp. 39–44

Abstract: This paper addresses state-space realizations for nonlinear adjoint operators. In particular the relationship among nonlinear Hilbert adjoint operators, Hamiltonian extensions and port-controlled Hamiltonian systems are clarified. The characterization of controllability, observability and Hankel operators, and controllability and observability functions will be derived based on it. Furthermore a duality between the controllability and observability functions will be proven. The state-space realizations of such operators provide new insights to nonlinear control systems theory.

Keywords: Nonlinear control systems, Adjoint operators, Hamiltonian control systems, Hamiltonian extensions, Legendre transformations

1. INTRODUCTION

Adjoint operators play an important role in the linear control systems theory. They provide duality between inputs and outputs of linear systems. The properties with respect to input, e.g. controllability and stabilizability issues, of linear systems directly reduce to the dual results with respect to output, observability and detectability issues. Let us consider a linear operator (transfer function) $\Sigma(s) : E \to F$ with Hilbert spaces E and F. Then its adjoint operator $\Sigma'(s) : F' \to E'$ is isomorphic to $\Sigma^T(-s) : F \to E$. Thus the adjoint can be easily described by a state-space realization if the operator $\Sigma(s)$ has a finite dimensional state-space realization. The objective of this paper is to provide the nonlinear extension of such adjoint operators.

Nonlinear adjoint operators can be found in the mathematics literature, e.g. (Batt, 1970), and they are expected to play a similar role in the nonlinear control systems theory. So called nonlinear Hilbert adjoint operators are introduced in (Gray and Scherpen, 1998; Scherpen and Gray, 1999) as a special class of nonlinear adjoint operators. The definition of them is as follows: The *nonlinear Hilbert adjoint* of $\Sigma : E \to F$ with Hilbert spaces E and F is an operator $\Sigma^* : F \times E \to E$ such that

$$\langle \Sigma(u), y \rangle_F = \langle u, \Sigma^*(y, u) \rangle_E \tag{1}$$

holds for $\forall u \in E$, $\forall y \in F$. The existence of such operators in input-output sense was shown in (Gray and Scherpen, 1999) but their state-space realizations are not available so far.

On the other hand, Hamiltonian extensions (Crouch and van der Schaft, 1987) are used to characterize state-space adjoints of nonlinear control systems. In (Scherpen and van der Schaft, 1994; Ball and van der Schaft, 1996) Hamiltonian extensions are used extensively as the realizations of adjoint operators to characterize norm preserving properties. Indeed in the linear case the Hamiltonian extension of a given system is the Hilbert adjoint system. However, in the nonlinear case, this is not such a straightforward issue.

In this paper we give the state-space realization of nonlinear Hilbert adjoint operators. Firstly we show some relationship between the nonlinear Hilbert adjoint operators and Hamiltonian extensions but this does not give a complete characterization of adjoint operators. Secondly we derive a more complete characterization of the state-space realizations of nonlinear Hilbert adjoint operators which have the form of port-controlled Hamiltonian systems (Maschke and van der Schaft, 1992). The state-space realizations of such operators provide a characterization of the observability functions, controllability functions and Hankel operators which supplies a set of similarity invariants related to input-state and state-output behaviours of nonlinear systems (Scherpen, 1993; Gray and Scherpen, 1998; Scherpen and Gray, 1999). Hence it is expected that the adjoint operators in this paper will extend the existing results on the realization theory for nonlinear control systems a bit further. Furthermore we show a duality between the observability and controllability functions.

2. LINEAR SYSTEMS AS A PARADIGM

This section gives some examples of linear adjoint operators which play an important role in the linear systems theory, see e.g. (Zhou *et al.*, 1996). We present them here in a way that clarifies the line of thinking in the nonlinear case. Consider a causal linear input-output system $\Sigma : L_2^m[0,\infty) \to L_2^r[0,\infty)$ with a state-space realization

$$u \mapsto y = \Sigma(u) : \begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases} \quad (2)$$

where $x(0) = 0$. The Laplace transformation gives its transfer function matrix

$$\Sigma(s) := C(sI - A)^{-1}B. \quad (3)$$

Its adjoint operator is isomorphic to $\Sigma^* : L_2^r[0,\infty) \to L_2^m[0,\infty)$ given by

$$\Sigma^*(s) := \Sigma^T(-s) = B^T(-sI - A^T)^{-1}C^T \quad (4)$$

with a state-space realization

$$u_a \mapsto y_a = \Sigma^*(u_a) : \begin{cases} \dot{x} &= -A^T x - C^T u_a \\ y_a &= B^T x \end{cases} \quad (5)$$

where $x(\infty) = 0$. Here u_a and y_a have the same dimensions as y and u respectively. It satisfies the definition for Hilbert adjoint operators, namely,

$$\langle \Sigma(u), u_a \rangle_{L_2^r} = \langle u, \Sigma^*(u_a) \rangle_{L_2^m}. \quad (6)$$

Since u_a has the same dimension as y we can calculate the magnitude of operators as

$$\|\Sigma(u)\|_{L_2^r}^2 = \langle \Sigma(u), \Sigma(u) \rangle_{L_2^r} = \langle u, \Sigma^* \circ \Sigma(u) \rangle_{L_2^m}$$

by substituting $u_a = \Sigma(u)$. This relation can be utilized to derive the singular values of the corresponding input-output map.

The above representation is for general linear systems on L_2. We can relate it to the observability and controllability operators. They are given by $\mathcal{O} : \mathbb{R}^n \to L_2^r[0,\infty)$ and $\mathcal{C} : L_2^m[0,\infty) \to \mathbb{R}^n$

$$x^0 \mapsto y = \mathcal{O}(x^0) := Ce^{At}x^0 \quad (7)$$

$$u \mapsto x^0 = \mathcal{C}(u) := \int_0^\infty e^{A\tau} Bu(\tau) d\tau. \quad (8)$$

Note that these operators $\mathcal{O}$ and $\mathcal{C}$ are also operators on Hilbert spaces, hence their adjoint operators are given by $\mathcal{O}^* : L_2^m[0,\infty) \to \mathbb{R}^n$ and $\mathcal{C}^* : \mathbb{R}^n \to L_2^r[0,\infty)$

$$u_a \mapsto x^0 = \mathcal{O}^*(u_a) := \int_0^\infty e^{A^T\tau} C^T u_a(\tau) d\tau \quad (9)$$

$$x^0 \mapsto y_a = \mathcal{C}^*(x^0) := B^T e^{A^T t} x^0. \quad (10)$$

It can be easily checked that they satisfy

$$\langle \mathcal{O}(x^0), u_a \rangle_{L_2^r} = \langle x^0, \mathcal{O}^*(u_a) \rangle_{\mathbb{R}^n} \quad (11)$$

$$\langle \mathcal{C}(u), x^0 \rangle_{\mathbb{R}^n} = \langle u, \mathcal{C}^*(x^0) \rangle_{L_2^m}. \quad (12)$$

These adjoint operators can be used to calculate the observability and controllability Gramians:

$$\begin{aligned} \|\mathcal{O}(x^0)\|_{L_2^r}^2 &= \langle x^0, \mathcal{O}^* \circ \mathcal{O}(x^0) \rangle_{\mathbb{R}^n} \\ &= \langle x^0, \textstyle\int_0^\infty CA^\tau A^{T\tau} C^T d\tau \; x^0 \rangle_{\mathbb{R}^n} \\ &= \langle x^0, Q \; x^0 \rangle_{\mathbb{R}^n} \quad (13) \\ \|\mathcal{C}^*(x^0)\|_{L_2^m}^2 &= \langle x^0, \mathcal{C}^{**} \circ \mathcal{C}^*(x^0) \rangle_{\mathbb{R}^n} \\ &= \langle x^0, \textstyle\int_0^\infty B^T A^{T\tau} A^\tau B d\tau \; x^0 \rangle_{\mathbb{R}^n} \\ &= \langle x^0, P \; x^0 \rangle_{\mathbb{R}^n} \quad (14) \end{aligned}$$

These imply $Q = \mathcal{O}^* \circ \mathcal{O}$ and $P = \mathcal{C}^{**} \circ \mathcal{C}^*$.

3. STATE-SPACE REALIZATION OF NONLINEAR ADJOINT OPERATORS

This section is devoted to the state-space characterization of nonlinear Hilbert adjoint operators. We will show some relationship between nonlinear Hilbert adjoint operators and Hamiltonian extensions in section 3.1 and give the state-space realization of adjoint operators based on port-controlled Hamiltonian systems in section 3.2.

3.1 *Adjoint operators and Hamiltonian extensions*

This subsection shows the relationship between nonlinear Hilbert adjoint operators and Hamiltonian extensions. Let us consider an input-output system $\Sigma : L_2^m(\Omega) \to L_2^r(\Omega)$ defined on a (possibly infinite) time interval $\Omega = [t^0, t^1] \subseteq \mathbb{R}$ which has a state-space realization

$$u \mapsto y = \Sigma(u) : \begin{cases} \dot{x} = f(x,u) \; x(t^0) = 0 \\ y = h(x,u) \end{cases} \quad (15)$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$. Here we assume the origin is an equilibrium, i.e. $f(0,0) = 0$ and $h(0,0) = 0$ hold and that all signals and functions are sufficiently smooth.

The Hamiltonian extension of Σ is given by a Hamiltonian control system (Crouch and van der Schaft, 1987)

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p}^T = f(x,u) \\ \dot{p} = -\frac{\partial H}{\partial x}^T = -\left(\frac{\partial f}{\partial x}^T p + \frac{\partial h}{\partial x}^T u_a\right) \\ y_a = \frac{\partial H}{\partial u}^T = \frac{\partial f}{\partial u}^T p + \frac{\partial h}{\partial u}^T u_a \\ y = \frac{\partial H}{\partial u_a}^T = h(x,u) \end{cases} \quad (16)$$

with initial conditions $x^1 := x(t^1) = 0$, $p^0 := p(t^0) = 0$ and the Hamiltonian

$$H(x,p,u,u_a) := p^T f(x,u) + u_a^T h(x,u). \quad (17)$$

We now prove some properties of this system which are related to nonlinear Hilbert adjoint operators.

Proposition 1 *Consider the Hamiltonian extension (16) of Σ as in (15). Define scalar functions H_1, H_2 and H_3 as*

$$H_1(x,p,u) := H - \frac{\partial H}{\partial u_a} u_a \quad (18)$$
$$H_2(x,p,u,u_a) := H - \frac{\partial H}{\partial u} u \quad (19)$$
$$H_3(x,p,u,u_a) := H - \frac{\partial H}{\partial u} u - \frac{\partial H}{\partial u_a} u_a. \quad (20)$$

Then the following relations hold.

$$\frac{dH}{dt} = y_a^T \dot{u} + y^T \dot{u}_a \quad (21)$$
$$\frac{dH_1}{dt} = y_a^T \dot{u} - \dot{y}^T u_a \quad (22)$$
$$\frac{dH_2}{dt} = -\dot{y}_a^T u + y^T \dot{u}_a \quad (23)$$
$$\frac{dH_3}{dt} = -\dot{y}_a^T u - \dot{y}^T u_a \quad (24)$$

Proof. Equation (21) follows from

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial u}\dot{u} + \frac{\partial H}{\partial u_a}\dot{u}_a \quad (25)$$
$$= \frac{\partial H}{\partial x}\frac{\partial H}{\partial p}^T - \frac{\partial H}{\partial p}\frac{\partial H}{\partial x}^T + y_a^T \dot{u} + y^T \dot{u}_a \quad (26)$$
$$= y_a^T \dot{u} + y^T \dot{u}_a. \quad (27)$$

Hence the time derivative of the other functions are obtained by

$$\frac{dH_1}{dt} = \frac{d}{dt}\left(H - y^T u_a\right) = y_a^T \dot{u} - \dot{y}^T u_a \quad (28)$$
$$\frac{dH_2}{dt} = \frac{d}{dt}\left(H - y_a^T u\right) = -\dot{y}_a^T u + y^T \dot{u}_a \quad (29)$$
$$\frac{dH_3}{dt} = \frac{d}{dt}\left(H - y_a^T u - y^T u_a\right) = -\dot{y}_a^T u - \dot{y}^T u_a. \quad (30)$$

This proves the proposition. □

This proposition shows that the Hamiltonian extension has a close relationship to nonlinear Hilbert adjoint operators. Roughly speaking, e.g. (21) shows that the mapping $(\dot{u}_a, \dot{u}) \mapsto (-y_a)$ is a nonlinear Hilbert adjoint of the map $\check{\Sigma} : \dot{u} \mapsto y$ on $L_2[t^0, t^1]$ provided $H|_{t=t^0} = H|_{t=t^1}$ holds, because

$$\langle \check{\Sigma}(\dot{u}), \dot{u}_a \rangle_{L_2^r} := \langle y, \dot{u}_a \rangle_{L_2^r} = \langle \dot{u}, -y_a \rangle_{L_2^m} =: \langle \dot{u}, \check{\Sigma}^*(\dot{u}_a, \dot{u}) \rangle_{L_2^m}$$

holds. However we cannot obtain the nonlinear Hilbert adjoint Σ^* itself directly from Hamiltonian extensions.

The equation (22) describes an intrinsic property of Hamiltonian extensions. The mapping $u_a \mapsto y_a$ is the nonlinear Hilbert adjoint of the variational mapping $\dot{u} \mapsto \dot{y}$ of the original mapping $u \mapsto y$ and this corresponds to the original definition of Hamiltonian extensions (Crouch and van der Schaft, 1987).

Furthermore the property (23) shows that the mapping of the original system $u \mapsto y$ corresponds to the adjoint of the variational map $\dot{u}_a \mapsto \dot{y}_a$.

Property (24) corresponds to the so called *energy balancing* property of *physical* Hamiltonian control systems (Crouch and van der Schaft, 1987).

Also the relation (23) can be utilized to obtain a state-space realization of the nonlinear Hilbert adjoint of input-affine nonlinear systems. Details about this formulation are omitted for the reason of space. This state-space realization has a $(2n + m)$-dimensional state and it corresponds to

$$\left(s\ \Sigma(s)\ \tfrac{1}{s}\right)^* = s\ \Sigma^T(-s)\ \tfrac{1}{s} \quad (31)$$

in the linear case.

3.2 *Adjoint operators and port-controlled Hamiltonian systems*

Hamiltonian extensions have some relationships with nonlinear adjoint operators but their complete state-space characterization is not obtained. This section will give a more general state-space formulation based on port-controlled Hamiltonian systems (Maschke and van der Schaft, 1992).

Let us consider a possibly time-varying system $\Sigma : L_2^m(\Omega) \to L_2^r(\Omega)$ defined on a time interval $\Omega = [t^0, t^1] \subseteq \mathbb{R}$ which has a state-space realization

$$\Sigma : \begin{cases} \dot{x} \ = f(x,u,t) \ x(t^0) = x^0 \\ y \ = h(x,u,t) \end{cases} . \qquad (32)$$

Here we assume the origin is an equilibrium, i.e. $f(0,0,t) = 0$ and $h(0,0,t) = 0$ hold for $\forall t \in \mathbb{R}$.

This system can be regarded as an operator $\hat{\Sigma} : \mathbb{R}^n \times L_2^m(\Omega) \to \mathbb{R}^n \times L_2^r(\Omega)$ with

$$(x^0, u) \mapsto (x^1, y) = \hat{\Sigma}(x^0, u) : \\ \begin{cases} \dot{x} \ = f(x,u,t) \ x(t^0) = x^0 \\ y \ = h(x,u,t) \\ x^1 = x(t^1) \end{cases} . \qquad (33)$$

Note that $\mathbb{R}^n \times L_2^m(\Omega)$ is a Hilbert space with the inner product $\langle (x^0, u), (x^1, \check{u}) \rangle_{\mathbb{R}^n \times L_2^m(\Omega)} := \langle x^0, x^1 \rangle_{\mathbb{R}^n} + \langle u, \check{u} \rangle_{L_2^m(\Omega)}$. The following proposition gives a state-space realization of the nonlinear Hilbert adjoint of $\hat{\Sigma}$.

Proposition 2 *Consider an operator $\hat{\Sigma} : \mathbb{R}^n \times L_2^m(\Omega) \to \mathbb{R}^n \times L_2^r(\Omega)$ as in (33) with $\Omega = [t^0, t^1] \subseteq \mathbb{R}$ and the corresponding port-controlled Hamiltonian system $H_{\hat{\Sigma}} : \mathbb{R}^{2n} \times L_2^{m+r}(\Omega) \to \mathbb{R}^{2n} \times L_2^{m+r}(\Omega)$ given by*

$$(x^0, p^1, \hat{u}) \mapsto (x^1, p^0, \hat{y}) = H_{\hat{\Sigma}}(x^0, p^1, \hat{u}) : \\ \begin{cases} \dot{\hat{x}} &= J(\hat{x}, \hat{u}, t) \frac{\partial H}{\partial \hat{x}}^T + g(\hat{x}, \hat{u}, t) \ \hat{u} \\ \hat{y} &= g^T(\hat{x}, \hat{u}, t) \frac{\partial H}{\partial \hat{x}}^T + \hat{D}(\hat{x}, \hat{u}, t) \ \hat{u} \\ \hat{x}(t^0) &= (x^0, p^0) \\ \hat{x}(t^1) &= (x^1, p^1) \end{cases} \qquad (34)$$

with $\hat{x} := (x, p) \in \mathbb{R}^n \times \mathbb{R}^n$, $\hat{u} := (u, u_a) \in \mathbb{R}^m \times \mathbb{R}^r$, $\hat{y} := (y_a, -y) \in \mathbb{R}^m \times \mathbb{R}^r$ and

$$H(\hat{x}) := p^T x \qquad (35)$$

$$J(\hat{x}, \hat{u}, t) := \begin{pmatrix} 0 & A(x,u,t) \\ -A^T(x,u,t) & 0 \end{pmatrix} \qquad (36)$$

$$g(\hat{x}, \hat{u}, t) := \begin{pmatrix} B(x,u,t) & 0 \\ 0 & -C^T(x,u,t) \end{pmatrix} \qquad (37)$$

$$\hat{D}(\hat{x}, \hat{u}, t) := \begin{pmatrix} 0 & D^T(x,u,t) \\ -D(x,u,t) & 0 \end{pmatrix} . \qquad (38)$$

Here $A(x,u,t) \in \mathbb{R}^{n \times n}$, $B(x,u,t) \in \mathbb{R}^{n \times m}$, $C(x,u,t) \in \mathbb{R}^{r \times n}$ and $D(x,u,t) \in \mathbb{R}^{r \times m}$ are appropriate matrices such that

$$f(x,u,t) = A(x,u,t) \ x + B(x,u,t) \ u \qquad (39)$$

$$h(x,u,t) = C(x,u,t) \ x + D(x,u,t) \ u \qquad (40)$$

hold. Suppose that

$$|(x^0, p^1)| < \infty, \ u, u_a \in L_2(\Omega) \\ \Rightarrow \ |(x^1, p^0)| < \infty. \qquad (41)$$

Then the mapping $(x^0, p^1, u, u_a) \mapsto (p^0, y_a)$ corresponding to the state-space realization (34) is a state-space realization of the nonlinear Hilbert adjoint operator $\hat{\Sigma}^* : \mathbb{R}^{2n} \times L_2^{m+r}(\Omega) \to \mathbb{R}^n \times L_2^r(\Omega)$ of $\hat{\Sigma}$.

By direct calculation the port-controlled Hamiltonian system (34) reduces down to

$$H_{\hat{\Sigma}} : \begin{cases} \dot{x} = f(x,u,t) \\ \dot{p} = -A^T(x,u,t) \ p - C^T(x,u,t) u_a \\ y_a = B^T(x,u,t) \ p + D^T(x,u,t) u_a \\ y = h(x,u,t) \\ x^1 = x(t^1) \\ p^0 = p(t^0) \end{cases} \qquad (42)$$

with $x(t_0) = x^0$ and $p(t_1) = p^1$ as "initial" conditions. The mapping $(x^0, p^1, u, u_a) \mapsto (p^0, y_a)$ is a state-space realization of the nonlinear Hilbert adjoint operator $\hat{\Sigma}^*$. Since the assumption (41) can be regarded as a stability requirement for the x-subsystem and an anti-stability requirement for the p-subsystem, then, as in the linear case, (41) it is always satisfied locally when Ω is finite or when the given system Σ in (32) with $u = 0$ is locally exponentially stable at least.

Proof of Proposition 2. The property of (time-varying) port-controlled Hamiltonian systems (see e.g. (Fujimoto and Sugie, 1998)) proves

$$\frac{dH}{dt} = \frac{\partial H}{\partial \hat{x}} \left(J \frac{\partial H}{\partial \hat{x}}^T + g \hat{u} \right) = \frac{\partial H}{\partial \hat{x}} g \hat{u} \\ = \left(\frac{\partial H}{\partial \hat{x}} g + \hat{u}^T \hat{D}^T \right) \hat{u} = \hat{y}^T \hat{u} = y_a^T u - y^T u_a .$$

This reduces to

$$\langle y_a, u \rangle_{L_2^m(\Omega)} - \langle y, u_a \rangle_{L_2^r(\Omega)} \\ = \int_{t^0}^{t^1} (y_a^T u - y^T u_a) dt = \int_{t^0}^{t^1} \frac{dH}{dt} dt \\ = H(\hat{x}(t^1)) - H(\hat{x}(t^0)) \\ = \langle x^1, p^1 \rangle_{\mathbb{R}^n} - \langle x^0, p^0 \rangle_{\mathbb{R}^n} .$$

Hence

$$\langle (x^1, y), (p^1, u_a) \rangle_{\mathbb{R}^n \times L_2^r(\Omega)} \\ = \langle (x^0, u), (p^0, y_a) \rangle_{\mathbb{R}^n \times L_2^m(\Omega)} \qquad (43)$$

holds. Substituting $(x^1, y) = \hat{\Sigma}(x^0, u)$ and $(p^0, y_a) = \hat{\Sigma}^*((p^1, u_a), (x^0, u))$ yields the definition of nonlinear Hilbert adjoint operators

$$\langle \hat{\Sigma}(x^0, u), (p^1, u_a) \rangle_{\mathbb{R}^n \times L_2^r(\Omega)} \\ = \langle (x^0, u), \hat{\Sigma}^*((p^1, u_a), (x^0, u)) \rangle_{\mathbb{R}^n \times L_2^m(\Omega)} . \qquad (44)$$

This proves the proposition. □

This result provides a useful tool to analyze the properties of nonlinear input-output systems with state-space realizations. It should be noted that the characterization given here uses a coordinate dependent expression because we employ the inner product on $\mathbb{R}^n$ which is intrinsically coordinate dependent. However, it may be argued that it provides very natural state-space realizations of adjoint operators because it only requires rather mild assumptions indeed.

Proposition 2 directly derives the state-space characterization of the nonlinear Hilbert adjoint operators of $L_2^m(\Omega) \to L_2^r(\Omega)$ in the usual setting.

Corollary 1 *Consider the system Σ as in (32) with the initial condition $x^0 = 0$ and let $\Sigma : L_2^m(\Omega) \to L_2^r(\Omega)$ denote the mapping $u \mapsto y$. Suppose the assumption (41) holds. Then a state-space realization of the nonlinear Hilbert adjoint $\Sigma^* : L_2^{m+r}(\Omega) \to L_2^m(\Omega)$ of Σ is given by*

$$(u_a, u) \mapsto y_a = \Sigma^*(u_a, u) :$$
$$\begin{cases} \dot{x} = f(x, u, t) & x(t^0) = 0 \\ \dot{p} = -A^T(x, u, t)\ p - C^T(x, u, t)\ u_a & p(t^1) = 0 \,. \\ y_a = B^T(x, u, t)\ p + D^T(x, u, t)\ u_a \end{cases}$$

The results presented in this section are useful for obtaining state-space characterizations of adjoints of some characteristic operators appearing in nonlinear realization theory. This matter is the topic of the next section.

4. ENERGY FUNCTIONS AND OPERATORS

4.1 *Observability, controllability and Hankel operators*

This section gives the state-space realizations for nonlinear Hilbert adjoint of some energy functions and operators. We only consider a causal L_2-stable time invariant input-affine nonlinear system without direct feed-through in the form of

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \qquad (45)$$

defined on the time interval $\Omega := (-\infty, \infty)$.

Let us consider the observability and controllability operators $\mathcal{O} : \mathbb{R}^n \to L_2^r(\Omega_+)$ and $\mathcal{C} : L_2^m(\Omega_+) \to \mathbb{R}^n$ with $\Omega_+ := [0, \infty)$ of Σ defined by

$$x^0 \mapsto y = \mathcal{O}(x^0) : \begin{cases} \dot{x} = f(x) & x(0) = x^0 \\ y = h(x) \end{cases} \qquad (46)$$

$$u \mapsto x^1 = \mathcal{C}(u) : \begin{cases} \dot{x} = f(x) + g(x)\mathcal{F}_-(u) \\ x^1 = x(0) \\ 0 = x(-\infty) \end{cases} \qquad (47)$$

These operators are originally defined (Gray and Scherpen, 1998) by using Chen-Fliess expansions for analytic nonlinear systems, see e.g.(Fliess, 1974; Isidori, 1995). These are natural generalizations of the linear operators (7) and (8). Furthermore the Hankel operator (Gray and Scherpen, 1998; Scherpen and Gray, 1999) $\mathcal{H} : L_2^m(\Omega_+) \to L_2^r(\Omega_+)$ of Σ is given by

$$\mathcal{H} := \Sigma \circ \mathcal{F}_-. \qquad (48)$$

Here $\mathcal{F}_- : L_2^m(\Omega_+) \to L_2^m(\Omega_-)$ with $\Omega_- := (-\infty, 0]$ denotes the so called *flipping operator* defined by

$$\mathcal{F}_-(u)(t) = \begin{cases} u(-t) & t \in \Omega_- \\ 0 & t \in \Omega_+ \end{cases} . \qquad (49)$$

The state-space realizations of the nonlinear Hilbert adjoint operators of $\mathcal{O}, \mathcal{C}$ and $\mathcal{H}$ are given by the following proposition.

Proposition 3 *Consider the operator Σ as in (45). Suppose that the assumption (41) in Proposition 2 holds for the relevant port-controlled Hamiltonian system (34). Then state-space realizations of $\mathcal{O}^* : L_2^r(\Omega_+) \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{C}^* : \mathbb{R}^n \times L_2^m(\Omega_+) \to L_2^m(\Omega_+)$ and $\mathcal{H}^* : L_2^r(\Omega_+) \times L_2^m(\Omega_+) \to L_2^m(\Omega_+)$ are given by*

$$(x^0, u_a) \mapsto p^0 = \mathcal{O}^*(x^0, u_a) :$$
$$\begin{cases} \dot{x} = f(x) & x(0) = x^0 \\ \dot{p} = -A^T(x)\ p - C^T(x)\ u_a & p(\infty) = 0 \\ p^0 = p(0) \end{cases} \qquad (50)$$

$$(p^1, u) \mapsto y_a = \mathcal{C}^*(p^1, u) :$$
$$\begin{cases} \dot{x} = f(x) + g(x)\mathcal{F}_-(u) & x(-\infty) = 0 \\ \dot{p} = -A^T(x)\ p & p(0) = p^1 \\ y_a = \mathcal{F}_+(g^T(x)\ p) \end{cases} \qquad (51)$$

$$(u_a, u) \mapsto y_a = \mathcal{H}^*(u_a, u) :$$
$$\begin{cases} \dot{x} = f(x) + g(x)\ \mathcal{F}_-(u) & x(-\infty) = 0 \\ \dot{p} = -A^T(x)\ p - C^T(x)\ u_a & p(\infty) = 0 \\ y_a = \mathcal{F}_+(g^T(x)\ p) \end{cases} \qquad (52)$$

respectively with matrices $A(x) \in \mathbb{R}^{n\times n}$ and $C(x) \in \mathbb{R}^{r\times n}$ such that $f(x) \equiv A(x)x$ and $h(x) \equiv C(x)x$ hold. Here $\mathcal{F}_+ : L_2^m(\Omega_-) \to L_2^m(\Omega_+)$ denotes another flipping operator defined by

$$\mathcal{F}_+(u)(t) = \begin{cases} 0 & t \in \Omega_- \\ u(-t) & t \in \Omega_+ \end{cases} . \qquad (53)$$

The proofs of Proposition 3 and 4 are omitted for the reason of space. It is expected that the state-space characterization of $\mathcal{O}^*, \mathcal{C}^*$ and $\mathcal{H}^*$ will provide further developments in the realization theory of nonlinear control systems.

4.2 *Observability and controllability functions*

In this subsection some relationship between the observability and controllability functions, operators and Gramians are developed.

Definition The observability function $L_o(x)$ and the controllability function $L_c(x)$ of Σ as in (45) are defined by

$$L_o(x^0) := \frac{1}{2}\int_0^\infty \|y(t)\|^2 dt,\ x(0) = x^0,\ u(t) \equiv 0$$
$$L_c(x^1) := \min_{\substack{u \in L_2^m(\Omega_-) \\ x(-\infty)=0,\, x(0)=x^1}} \frac{1}{2}\int_{-\infty}^0 \|u(t)\|^2 dt$$

respectively.

These functions are closely related to observability and controllability operators and Gramians in the linear case (13) and (14). At first we present the relation between the observability function, operator and Gramian.

$$\begin{aligned} L_o(x^0) &= \tfrac{1}{2}\|\mathcal{O}(x^0)\|^2_{L_2^r} = \tfrac{1}{2}\langle x^0, \mathcal{O}^*(\mathcal{O}(x^0), x^0)\rangle_{\mathbb{R}^n} \\ &= \tfrac{1}{2}\langle x^0, p^0\rangle_{\mathbb{R}^n} =: \tfrac{1}{2}\langle x^0, \phi(x^0)\rangle_{\mathbb{R}^n} \end{aligned} \quad (54)$$

Here $p^0 = p(0)$ is the initial state of the state-space realization of $\mathcal{O}^*$ in (50) with the input $(u_a, x^0) = (\mathcal{O}(x^0), x^0)$. The function $\phi(x^0)$ can always be rewritten by $\phi(x^0) = Q(x^0)\ x^0$ using a square symmetric matrix $Q(x^0)$. This matrix coincides with the observability Gramian in the linear case. Further it should be noticed that

$$L_o(x(t)) = \tfrac{1}{2}\langle x(t), p(t)\rangle_{\mathbb{R}^n} \quad (55)$$

holds along the trajectory of the state-space realization (50) of $\mathcal{O}^*$.

In the controllability case, there does not hold such a relation as in the observability case. As for the equation (54) it does follow that

$$\begin{aligned} L_c(x^1) &= \tfrac{1}{2}\|\mathcal{C}^\dagger(x^1)\|^2_{L_2^m} = \tfrac{1}{2}\langle x^1, \mathcal{C}^{\dagger *}(\mathcal{C}^\dagger(x^1), x^1)\rangle_{\mathbb{R}^n} \\ &=: \tfrac{1}{2}\langle x^1, \varphi(x^1)\rangle_{\mathbb{R}^n} \end{aligned} \quad (56)$$

with $\mathcal{C}^\dagger : \mathbb{R}^n \to L_2^m(\Omega_+)$ the pseudo-inverse of $\mathcal{C}$ such that

$$\mathcal{C}^\dagger(x^1) := \arg \min_{\mathcal{C}(u)=x^1} \|u\|_{L_2^m} \quad (57)$$

holds. The state-space realization of $\mathcal{C}^\dagger$ can be easily obtained and $L_c(x)$ can be calculated in a similar way as $L_o(x)$. This relation is slightly different from the linear case (14), since here we deal with the "inverse" of the controllability Gramian. The correspondence with the linear case can however be explored in terms of duality as follows.

Proposition 4 *Consider the system Σ as in (45) and the related port-controlled Hamiltonian system $H_{\hat{\Sigma}}$ as in (42). Suppose that $\dot{x} = f(x)$ is asymptotically stable and that Σ has the observability function $L_o(x)$ and the controllability function $L_c(x)$. Consider the reverse-time system of the p-subsystem of the related port-controlled Hamiltonian system*

$$\begin{cases} \dot{p} &= A^T(\phi(p))p + C^T(\phi(p))u_a \\ y_a &= g^T(\phi(p))p \end{cases} . \quad (58)$$

Let $x = \phi(p)$ denote the inverse mapping of $p = \frac{\partial L_c}{\partial x}^T(x)$ and suppose that this system has the observability function $\tilde{L}_o(p)$. Then $\tilde{L}_o(p)$ is given by a Legendre transformation

$$\tilde{L}_o(p) = -L_c(x) + p^T x. \quad (59)$$

Let $x = \phi(p)$ denote the inverse mapping of $p = \frac{\partial L_o}{\partial x}^T(x)$ and suppose the system (58) has a controllability function $\tilde{L}_c(p)$. Then $\tilde{L}_c(p)$ is given by a Legendre transformation as well

$$\tilde{L}_c(p) = -L_o(x) + p^T x. \quad (60)$$

Legendre transformations give *the duality in the sense of Young* (Arnold, 1989) and this duality is intrinsically coordinate dependent. Thus, as in the linear case, we have duality in the sense of Young between inputs and outputs which appears to be closely related to port-controlled Hamiltonian adjoint systems.

ACKNOWLEDGMENT

The first author would like to thank Professor Arjan van der Schaft of the University of Twente for his insightful suggestions.

5. REFERENCES

Arnold, V. I. (1989). *Mathematical Methods of Classical Mechanics*. second edition ed.. Springer-Verlag. New York.

Ball, J. A. and A. J. van der Schaft (1996). j-inner-outer factorization, j-spectral factorization, and robust control for nonlinear systems. *IEEE Trans. Autom. Contr.* **AC-41**(3), 379–392.

Batt, J. (1970). Nonlinear compact mappings and their adjoints. *Math. Ann.* **189**, 5–25.

Crouch, P. E. and A. J. van der Schaft (1987). *Variational and Hamiltonian Control Systems*. Vol. 101 of *Lecture Notes on Control and Information Science*. Springer-Verlag. Berlin.

Fliess, M. (1974). Matrices de hankel. *J. Math. Pures. Appl.* pp. 197–222.

Fujimoto, K. and T. Sugie (1998). Canonical transformation and stabilization of generalized hamiltonian systems. Submitted (1999); Preliminary version is in *Proc. 4th IFAC Symp. NOLCOS '98* pp. 544–549.

Gray, W. S. and J. M. A. Scherpen (1998). Hankel operators and gramians for nonlinear systems. *Proc. 37th IEEE Conf. on Decision and Control* pp. 3349–3353.

Gray, W. S. and J. M. A. Scherpen (1999). Hankel operators, sigular value functions and Gramian generalizations for nonlinear systems. Submitted.

Isidori, A. (1995). *Nonlinear Control Systems*. third ed.. Springer-Verlag. Berlin.

Maschke, B. M. J. and A. J. van der Schaft (1992). Port-controlled hamiltonian systems: modelling origins and system-theoretic properties. *IFAC Symp. NOLCOS* pp. 282–288.

Scherpen, J. M. A. (1993). Balancing for nonlinear systems. *Systems & Control Letters* **21**, 143–153.

Scherpen, J. M. A. and A. J. van der Schaft (1994). Normalized coprime factorization and balancing for unstable nonlinear systems. *Int. J. Control* **60**(6), 1193–1222.

Scherpen, J. M. A. and W. S. Gray (1999). On singular value functions and hankel operators for nonlinear systems. *Proc. ACC'99* pp. 2360–2364.

Zhou, K., J. C. Doyle and K. Glover (1996). *Robust and Optimal Control*. Prentice-Hall, Inc.. Upper Saddle River, N.J.