$\mathcal{H}_\infty$ Balancing for Nonlinear Systems

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Abstract

In previously obtained balancing methods for nonlinear systems a past and a future energy function are used to bring the nonlinear system in balanced form. By considering a different pair of past and future energy functions that are related to the $\mathcal{H}_\infty$ control problem for nonlinear systems we define $\mathcal{H}_\infty$ balancing. Furthermore, we investigate the monotonicity of the Hamilton-Jacobi-Bellman equations that appear in this balancing method. The method is used as a tool for model reduction.

Keywords: $\mathcal{H}_\infty$ balancing, nonlinear systems, Hamilton-Jacobi equations, model reduction.

1 Introduction

Recently, several authors have studied the $\mathcal{H}_\infty$ control problem for nonlinear systems (see e.g. Ball *et al.* [1], Isidori and Astolfi [9], Başar and Bernhard [2], Van der Schaft [21, 22, 23]). The developments in both $\mathcal{H}_\infty$ control, and balancing for nonlinear systems motivate the study of a balancing technique that is related to the $\mathcal{H}_\infty$ control problem. Furthermore, the nature of the $\mathcal{H}_\infty$ control problem compared to the nonlinear version of the LQG problem motivates to extend the HJB (Hamilton-Jacobi-Bellman) balancing technique that is introduced by Scherpen and Van der Schaft [28] to an $\mathcal{H}_\infty$ balancing technique.

$\mathcal{H}_\infty$ balancing for linear systems is introduced by Mustafa and Glover [16, 17, 18]. Indeed, for linear systems we may interpret $\mathcal{H}_\infty$ balancing as an extension of LQG balancing, since we consider the solutions to the $\mathcal{H}_\infty$ Control and Filter Algebraic Riccati equations instead of the solutions to the corresponding Riccati equations in the LQG case. For $\gamma \to \infty$ (γ being the disturbance attenuation level), $\mathcal{H}_\infty$ balancing converges to LQG balancing.

We may also interpret $\mathcal{H}_\infty$ balancing for nonlinear systems as an extension of the linear formulation of $\mathcal{H}_\infty$ balancing, since the same past and future energy function may be used to define nonlinear $\mathcal{H}_\infty$ balancing.

In the linear case several papers have treated the monotonicity of the maximal solutions of algebraic Riccati equations (e.g. Wimmer [35]). These results are used to obtain the relation between $\mathcal{H}_\infty$ balancing and LQG balancing, and between $\mathcal{H}_\infty$ balancing and balancing of stable linear systems (e.g. Mustafa [16]). In this paper we give some results on the monotonicity (with regard to γ) of the stabilizing solutions of the Hamilton-Jacobi equations that correspond to $\mathcal{H}_\infty$ balancing for nonlinear systems. We obtain a relation between those solutions and the solutions to the Hamilton-Jacobi equations of HJB (Hamilton-Jacobi-Bellman) balancing and balancing of stable nonlinear systems.

In Section 2 we briefly treat some energy functions and their properties as they are important for the use of two different balancing techniques. $\mathcal{H}_\infty$ balancing for linear systems is reviewed in Section 3. Further, we review the nonlinear $\mathcal{H}_\infty$ control problem in Section 4. In Section 5 we state the main results. Here we treat the past and future energy function that are brought in $\mathcal{H}_\infty$ balanced form. They are the stabilizing solutions to Hamilton-Jacobi equations, and we treat some properties of those solutions. In Section 6 we treat model reduction based on $\mathcal{H}_\infty$ balancing. Finally, in Section 7 we give conclusions.

Throughout this paper we use a fairly standard notation. We denote by $x^T x$ or $\| x \|^2$ the squared norm of a vector $x \in \mathbb{R}^n$. We say that $u : (-\infty, 0) \to \mathbb{R}^m$ is in $L_2(-\infty, 0)$ if $\int_{-\infty}^0 \| u(t) \|^2 \, dt < \infty$. By $\frac{\partial L}{\partial x}(x)$ we denote the row-vector of partial derivatives of a differentiable function $L : \mathbb{R}^n \to \mathbb{R}$. Furthermore we denote by $x(t_2) = \varphi(t_2, t_1, x_1, u)$ the solution on time t_2 of the system $\dot{x} = f(x) + g(x)u$ with initial condition $x(t_1) = x_1$ and input $u : [t_1, t_2] \to \mathbb{R}^m$.

2 Some energy functions for nonlinear systems

In this section we review some results of Scherpen [24] and Scherpen and Van der Schaft [28]. Consider a smooth, i.e., C^∞, nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x), \end{aligned} \tag{1}$$

where $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$, $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$, and $x = (x_1, \ldots, x_n)$ are local coordinates for a smooth state space manifold denoted by M. Furthermore, $f, g_1, \ldots, g_m$ are smooth vector fields on M, where $g = (g_1, \ldots, g_m)$, and $h = (h_1, \ldots, h_p)^T$ is the smooth output map of the system. Throughout we assume that the system has an equilibrium. Without loss of generality we take this equilibrium in 0, i.e. $f(0) = 0$. Also, we take $h(0) = 0$.

Definition 2.1 The *controllability* and *observability function* of a nonlinear system (1) are

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2} \int_{-\infty}^{0} \| u(t) \|^2 \, dt, \tag{2}$$

and

$$L_o(x_0) = \frac{1}{2} \int_0^{\infty} \| y(t) \|^2 \, dt, \quad x(0) = x_0, \quad u(t) \equiv 0, \quad 0 \le t < \infty, \tag{3}$$

respectively. □

The value of the controllability function at x_0 is the minimum amount of control energy required to reach the state x_0, and the value of the observability function at x_0 is the amount of output energy generated by x_0. These functions do not necessarily exist, i.e., are not necessarily finite. The following theorem gives some conditions under which L_o and L_c exist.

Theorem 2.2 (see [24]) *Assume that $f(x)$ is asymptotically stable on a neighborhood W of 0 and that*

$$\frac{\partial \check{L}_o}{\partial x}(x) f(x) + \frac{1}{2} h^T(x) h(x) = 0, \qquad \check{L}_o(0) = 0. \tag{4}$$

has a smooth solution $\check{L}_o$ for all $x \in W$. Then L_o exists and is the unique smooth solution of (4) for all $x \in W$.

Furthermore, assume that

$$\frac{\partial \check{L}_c}{\partial x}(x) f(x) + \frac{1}{2} \frac{\partial \check{L}_c}{\partial x}(x) g(x) g^T(x) \frac{\partial^T \check{L}_c}{\partial x}(x) = 0, \qquad \check{L}_c(0) = 0. \tag{5}$$

has a smooth solution $\check{L}_c$ *for all* $x \in W$ *such that*

$$-(f(x) + g(x)g^T(x)\frac{\partial^T \check{L}_c}{\partial x}(x)) \tag{6}$$

is asymptotically stable on W. *Then* $L_c(x)$ *exists and is the unique smooth solution of (5), such that (6) is asymptotically stable, for all* $x \in W$. □

Remark 2.3 The reverse implications of this theorem are also true. If L_o and L_c exist and are smooth functions of x, then they are solutions to the Hamilton-Jacobi equations (4) and (5), respectively. □

Remark 2.4 Equation (4) is a nonlinear Lyapunov type of equation and equation (5) is a Hamilton-Jacobi equation associated with an optimal control problem. □

Remark 2.5 Consider the conditions in Theorem 2.2 that (5) has a smooth solution $\check{L}_c$ on W and that $-(f(x) + g(x)g^T(x)\frac{\partial^T \check{L}_c}{\partial x}(x))$ is asymptotically stable on W. We may replace them by the conditions that (5) has a smooth solution $\check{L}_c$ on W which is positive definite, and that $f(x)$ is asymptotically stable on W, see Scherpen [24]. □

Definition 2.6 The system (1) is *reachable from* x_0 if for any $\bar{x} \in M$ there exists a $\bar{t} \geq 0$, and input u such that $\bar{x} = \varphi(\bar{t}, 0, x_0, u)$.

The system (1) is *zero-state observable* if $u(t) \equiv 0, y(t) \equiv 0$ implies $x(t) \equiv 0$, i.e., for all $x \in M$, $h(\varphi(t, 0, x, 0)) = 0$, for all $t \geq 0 \Rightarrow \varphi(t, 0, x, 0) = 0$, for all $t \geq 0$. □

The asymptotic stability of f together with the zero-state observability of the system gives a sufficient condition for L_o being positive definite.

Theorem 2.7 (see [24]) *Assume* $f(x)$ *is asymptotically stable on a neighborhood* W *of 0. If the system (1) is zero-state observable and* L_o *exists and is smooth on* W, *then* $L_o(x_0) > 0$, $\forall x_0 \in W$, $x_0 \neq 0$.

The observability and controllability function of an asymptotically stable system may be used to bring the system in balanced form, see Scherpen [24]. For unstable nonlinear systems different methods have been developed in Scherpen and Van der Schaft [28]. One of these methods, the HJB (Hamilton-Jacobi-Bellman) balancing method, is based on a different pair of energy functions as defined in the following definition.

Definition 2.8 The *past* and *future energy function* of a nonlinear system are defined as

$$K^-(x_0) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} (\| y(t) \|^2 + \| u(t) \|^2) dt, \tag{7}$$

and

$$K^+(x_0) = \min_{\substack{u \in L_2(0,\infty) \\ x(\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{0}^{\infty} (\| y(t) \|^2 + \| u(t) \|^2) dt \tag{8}$$

respectively. □

We assume that the system (1) is zero-state observable, and that $K^+(x_0)$ and $K^-(x_0)$ exist (in particular, are finite) and are smooth functions for every x_0 in some neighborhood Y of 0. From optimal control theory we know that K^+ and K^- are smooth non-negative solutions to some Hamilton-Jacobi-Bellman equations (e.g. Lee and Markus [12]). Furthermore, the minimizing input for K^+ is $u = -g(x)^T \frac{\partial^T K^+}{\partial x}(x)$, and for K^- it is $u = g(x)^T \frac{\partial^T K^-}{\partial x}(x)$. Summarizing we get the following theorem:

Theorem 2.9 *K^+ is on Y the smooth non-negative solution of the Hamilton-Jacobi-Bellman equation*

$$\frac{\partial K^+}{\partial x}(x) f(x) - \frac{1}{2}\frac{\partial K^+}{\partial x}(x) g(x) g(x)^T \frac{\partial^T K^+}{\partial x}(x) + \frac{1}{2} h^T(x) h(x) = 0 \tag{9}$$

with $K^+(0) = 0$, *such that*

$$f(x) - g(x)g(x)^T \frac{\partial^T K^+}{\partial x}(x) \tag{10}$$

is asymptotically stable.

Furthermore, K^- *is on* Y *the smooth non-negative solution of the Hamilton-Jacobi-Bellman equation*

$$\frac{\partial K^-}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial K^-}{\partial x}(x)g(x)g^T(x)\frac{\partial^T K^-}{\partial x}(x) - \frac{1}{2}h(x)^T h(x) = 0 \tag{11}$$

with $K^-(0) = 0$, *such that*

$$-\left(f(x) + g(x)g(x)^T \frac{\partial^T K^-}{\partial x}(x)\right) \tag{12}$$

is asymptotically stable. □

Remark 2.10 If we assume there exists a smooth solution K of (9), such that (10) is asymptotically stable, then K^+ as in (8) exists, and equals K. A similar statement holds with regard to (11) and K^- (e.g. Lee and Markus [12]). □

Like in Remark 2.5 the asymptotic stability of (10) and (12) is equivalent to the positive definiteness of the future and past energy function, respectively.

Theorem 2.11 *Assume (9) has a smooth proper solution* K *on* Y*. Then* $K(x_0) > 0$ *for* $x_0 \in Y$, $x_0 \neq 0$, *if and only if* $f(x) - g(x)g(x)^T \frac{\partial^T K}{\partial x}(x)$ *is asymptotically stable on* Y*. Similarly, assume (11) has a smooth proper solution* $\bar{K}$ *on* Y*, then* $\bar{K}(x_0) > 0$ *for* $x_0 \in Y$, $x_0 \neq 0$, *if and only if* $-(f(x) + g(x)g(x)^T \frac{\partial^T \bar{K}}{\partial x}(x))$ *is asymptotically stable on* Y*.* □

The past and future energy function of unstable nonlinear systems may be used to bring the system in HJB balanced form (see Scherpen and Van der Schaft [28]).

3 Review on $\mathcal{H}_\infty$ balancing for linear systems

$\mathcal{H}_\infty$ balancing for linear systems has been introduced by Mustafa and Glover [16, 17, 18]. A first motivation for $\mathcal{H}_\infty$ balancing is similar to the main motivation for LQG balancing, i.e., model reduction of the system and a corresponding $\mathcal{H}_\infty$ compensator. The set of invariants defined in [16, 17, 18] are related to invariants defined by Weiland [33]. First we review the formulation of Mustafa and Glover [16, 17, 18].

The normalized $\mathcal{H}_\infty$ control problem (see also Doyle *et al.* [4]) on which the $\mathcal{H}_\infty$ balancing method is based, is the minimum entropy problem associated with the system

$$\begin{aligned} \dot{x} &= Ax + Bu + Bd \\ y &= Cx + v \\ z &= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} Cx \\ u \end{pmatrix}. \end{aligned} \tag{13}$$

Here $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $z \in \mathbb{R}^{m+p}$, and d and v are independent Gaussian white noise processes with covariance functions $I\delta(t-\tau)$. By the form of system (13) we call the $\mathcal{H}_\infty$ control problem that is formulated normalized. Let $w := (d^T, v^T)$ and assume that the system (A, B, C) is minimal. G is the transfer matrix of the system (A, B, C). It is easily shown that the closed-loop transfer matrix H from w to z is

$$H = \begin{pmatrix} (I-GK)^{-1}G & (I-GK)^{-1}GK \\ K(I-GK)^{-1}G & K(I-GK)^{-1} \end{pmatrix}$$

where K is the transfer matrix of the controller. For notational simplicity the Laplace transform variable s is suppressed. The block diagram of the closed-loop system is given in Figure 2.1. The $\mathcal{H}_\infty$ constraint is $\|H\|_\infty < \gamma$. Here $\gamma \in \mathbb{R}$ and

$$\|H\|_\infty = \sup_{\omega \in \mathbb{R}} \lambda_{\max}^{\frac{1}{2}}(H(-j\omega)^T H(j\omega)).$$

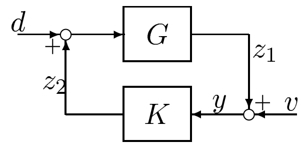

Figure 1: Block diagram of the closed-loop system.

The optimal $\mathcal{H}_\infty$ norm is $\gamma_0 := \inf\{\|H\|_\infty | \ K \text{ stabilizes } G\}$. We assume $\gamma > \gamma_0$ (the strictly suboptimal case). The entropy that has to be minimized is given by

$$I(\gamma) = -\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty} \ln|\det(I - \frac{1}{\gamma^2}H(-j\omega)^T H(j\omega))|d\omega \tag{14}$$

The normalized $\mathcal{H}_\infty$ problem is to find a controller K that stabilizes G and minimizes the entropy $I(\gamma)$, over the class of stabilized closed-loop transfer matrices H satisfying $\|H\|_\infty < \gamma$, where $\gamma > \gamma_0$. Take

$$L := (Y_\infty^{-1} - \frac{1}{\gamma^2}X_\infty)^{-1}, \tag{15}$$

where Y_∞ is the unique positive definite stabilizing solution (i.e., $\sigma(A - (1-\gamma^{-2})C^T C Y_\infty) \subset \mathbb{C}^-$) to the $\mathcal{H}_\infty$ Filter Algebraic Riccati Equation (HFARE)

$$AY_\infty + Y_\infty A^T + BB^T - (1 - \frac{1}{\gamma^2})Y_\infty C^T C Y_\infty = 0, \tag{16}$$

and where X_∞ is the unique positive definite stabilizing solution (i.e., $\sigma(A-(1-\gamma^{-2})BB^T X_\infty) \subset \mathbb{C}^-$) to the $\mathcal{H}_\infty$ Control Algebraic Riccati Equation (HCARE)

$$A^T X_\infty + X_\infty A + C^T C - (1 - \frac{1}{\gamma^2})X_\infty BB^T X_\infty = 0. \tag{17}$$

There exists a controller K that solves the normalized $\mathcal{H}_\infty$ problem if and only if $\lambda_{\max}(X_\infty Y_\infty) < \gamma^2$ (the coupling condition), and the resulting so called central controller (Doyle *et al.* [4]) takes the form

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} - (1 - \tfrac{1}{\gamma^2})BB^T X_\infty \hat{x} + LC^T(y - C\hat{x}) \\ u &= -B^T X_\infty \hat{x}. \end{aligned} \tag{18}$$

This controller does not exactly have the form that is given by Mustafa and Glover in [17], but in [7] they prove that the controller (18) also solves the problem.

Theorem 3.1 ([17]) *The eigenvalues of $X_\infty Y_\infty$ are similarity invariants and there exists a state space representation where*

$$N := X_\infty = Y_\infty = \begin{pmatrix} \vartheta_1 & & 0 \\ & \ddots & \\ 0 & & \vartheta_n \end{pmatrix} \tag{19}$$

with $\vartheta_1 \geq \vartheta_2 \geq \ldots \geq \vartheta_n > 0$. This state space representation is called a $\mathcal{H}_\infty$ balanced representation *or $\mathcal{H}_\infty$* balanced form *and the ϑ_i's are called the $\mathcal{H}_\infty$ characteristic values.* □

The following theorem states some properties of ϑ_i, $i = 1, \ldots, n$, where μ_i, $i = 1, \ldots, n$, are the similarity invariants that are associated with LQG balancing (e.g. Jonckheere and Silverman [10]), and σ_i, $i = 1, \ldots, n$, are the Hankel singular values (e.g. Glover [6], Moore [14]).

Theorem 3.2 ([16])

a. $\gamma > \vartheta_i \geq \mu_i$, $i = 1, \ldots, n$.

b. Each ϑ_i is a non-increasing function of γ.

c. Each ϑ_i is a continuous function of γ.

d. $\lim_{\gamma \to \infty} \vartheta_i = \mu_i$, $i = 1, \ldots, n$.

e. If $\gamma = 1$, then $\vartheta_i = \sigma_i$, $i = 1, \ldots, n$. □

In Mustafa and Glover [16, 17, 18] it is argued that if $\vartheta_k \gg \vartheta_{k+1}$, then the state components x_1 up to x_k are more difficult both to control and to filter in an $\mathcal{H}_\infty$ sense than x_{k+1} up to x_n.

Corresponding to the partitioning of the state in the first k components and the last $n-k$ components, the partitioning of the matrices is done as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = (\, C_1 \quad C_2 \,), \tag{20}$$

$$x^1 = (x_1, \ldots, x_k)^T, \quad x^2 = (x_{k+1}, \ldots, x_n)^T, \quad \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix},$$

where $\Sigma_1 = \operatorname{diag}(\vartheta_1, \ldots, \vartheta_k)$ and $\Sigma_2 = \operatorname{diag}(\vartheta_{k+1}, \ldots, \vartheta_n)$. The resulting reduced order system is of the form

$$\begin{aligned} \dot{x} &= A_{11}x + B_1 u + B_1 d \\ y &= C_1 x + v \\ z &= \begin{pmatrix} C_1 x \\ u \end{pmatrix} \end{aligned} \tag{21}$$

Theorem 3.3 ([17]) *The reduced order system (21) is again $\mathcal{H}_\infty$ balanced and the normalized $\mathcal{H}_\infty$ controller for system (21) is the reduced order normalized $\mathcal{H}_\infty$ controller of the full order system (13).* □

For $\gamma > 1$ it is also possible to reduce the system (13) via the normalized coprime factors of the scaled transfer matrix $G(s) = C(sI - A)^{-1}B$, i.e., the transfer matrix $\beta G(s)$, where $\beta := \sqrt{1 - \gamma^{-2}}$. This gives us the same reduced order system. For details about this we refer to Mustafa and Glover [17].

In Scherpen and Van der Schaft [28] an alternative way to consider LQG balancing is given. This alternative way may be derived from Weiland [33]. A similar kind of reasoning, using a different pair of energy functions may be used to achieve the $\mathcal{H}_\infty$ characteristic values ϑ_i, $i = 1, \ldots, n$, for $\gamma \neq 1$. To explain this alternative way we consider the minimal system

(A, B, C). For $\gamma \neq 1$, we define Q^- as

$$Q^-(x_0) := \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty)=0, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} ((1-\gamma^{-2}) \parallel y(t) \parallel^2 + \parallel u(t) \parallel^2) dt.$$

Furthermore, if $\gamma > 1$ we define Q^+ as

$$Q^+(x_0) := \min_{\substack{u \in L_2(0, \infty) \\ x(\infty)=0, x(0)=x_0}} \frac{1}{2}\int_{0}^{\infty} (\parallel y(t) \parallel^2 + \frac{1}{1-\gamma^{-2}} \parallel u(t) \parallel^2) dt,$$

while if $\gamma < 1$, we define Q^+ as (instead of minimizing over u, we maximize over u)

$$Q^+(x_0) := \max_{\substack{u \in L_2(0, \infty) \\ x(\infty)=0, x(0)=x_0}} \frac{1}{2}\int_{0}^{\infty} (\parallel y(t) \parallel^2 + \frac{1}{1-\gamma^{-2}} \parallel u(t) \parallel^2) dt.$$

$Q^-(x_0)$ is called the $\mathcal{H}_\infty$-*past energy* and $Q^+(x_0)$ the $\mathcal{H}_\infty$-*future energy* of the system in the state x_0.

Theorem 3.4 *$Q^-(x_0) = \frac{1}{2}x_0^T Y_\infty^{-1} x_0$ and $Q^+(x_0) = \frac{1}{2}x_0^T X_\infty x_0$, where Y_∞ and X_∞ are the stabilizing positive definite solutions of (16) and (17), respectively.* □

This means that if $\gamma \neq 1$ then for the $\mathcal{H}_\infty$ balanced representation the $\mathcal{H}_\infty$-past and $\mathcal{H}_\infty$-future energy function are $Q^-(x_0) = \frac{1}{2}x_0^T N^{-1} x_0$ and $Q^+(x_0) = \frac{1}{2}x_0^T N x_0$, respectively, with N as in (19). The importance of the state $\tilde{x} = (0, \ldots, 0, x_i, 0, \ldots, 0)$ in terms of $\mathcal{H}_\infty$-past and $\mathcal{H}_\infty$-future energy may be measured by the $\mathcal{H}_\infty$ characteristic values ϑ_i. For large ϑ_i the influence of the state $\tilde{x}$ on the $\mathcal{H}_\infty$-future energy is large while the influence on the $\mathcal{H}_\infty$-past energy is small. Hence if $\vartheta_k \gg \vartheta_{k+1}$, the state components x_{k+1} to x_n are not important from this energy point of view and may be removed to reduce the number of state components of the model.

4 Review on the $\mathcal{H}_\infty$ problem for a nonlinear system

The $\mathcal{H}_\infty$ problem for nonlinear systems has been studied by several authors (see e.g. Ball *et al.* [1], Isidori and Astolfi [9], Başar and Bernhard [2], Van der Schaft [21, 22, 23]). In this section we treat the $\mathcal{H}_\infty$ problem for a special class of systems and follow the work of Van der Schaft [21, 22, 23]. Consider the following smooth nonlinear control system (which is a special form of the class of systems that is considered in the literature mentioned above)

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + g(x)d \\ y &= h(x) + v \\ z &= \begin{pmatrix} h(x) \\ u \end{pmatrix} \end{aligned} \tag{22}$$

Denote by $\tilde{d} := \begin{pmatrix} d \\ v \end{pmatrix}$ the exogenous inputs (disturbances and/or reference signals). Let γ be a fixed positive constant. The $\mathcal{H}_\infty$ suboptimal control problem (for disturbance attenuation level γ) is to find a compensator

$$\begin{aligned} \dot{\hat{x}} &= k(\hat{x}, y) \\ u &= m(\hat{x}, y) \end{aligned} \tag{23}$$

where $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ are local coordinates for a manifold M_c (the state space of the compensator), with $k(0,0) = 0$ and $m(0,0) = 0$, such that the closed-loop system (22), (23) has L_2-gain from $\tilde{d}$ to z less than or equal to γ. In other words, the closed-loop system (22), (23) should be such that there exists a nonnegative constant K which depends on $x(0)$, $\hat{x}(0)$ and which is zero for $x(0) = 0$, $\hat{x}(0) = 0$, such that

$$\int_0^T \| z(t) \|^2 \, dt \leq \gamma^2 \int_0^T \| \tilde{d}(t) \|^2 \, dt + K, \tag{24}$$

for all $\tilde{d}(\cdot)$ and $T \geq 0$, with $z(\cdot)$ denoting the closed-loop response for the initial condition $x(0)$, $\hat{x}(0)$. If we assume that the system (22) is zero-state observable with respect to the outputs z and that a smoothness condition is fulfilled, then (24) implies asymptotic stability of the closed-loop system (22), (23) (see e.g. Van der Schaft [22]).

For the full dynamic output feedback problem, necessary conditions for the solvability of the $\mathcal{H}_\infty$ suboptimal control problem have been found generalizing the necessary and sufficient conditions for linear systems obtained in Doyle *et al.* [4] (see e.g. Ball *et al.* [1] and Van der Schaft [22]). We restrict our attention to compensators of the form

$$\begin{aligned} \dot{\hat{x}} &= k(\hat{x}) + l(\hat{x})y, \qquad k(0) = 0 \\ u &= m(\hat{x}), \qquad\qquad\quad m(0) = 0 \end{aligned} \tag{25}$$

Suppose that there exists a compensator (25) such that the closed-loop system (22), (25) has L_2-gain $\leq \gamma$ (from $\tilde{d}$ to z), in the sense that there exists a smooth solution $V(x, \hat{x}) \geq 0$ to the Hamilton-Jacobi inequality

$$\begin{aligned} &\tfrac{\partial V}{\partial x}(x,\hat{x})[f(x) + g(x)m(\hat{x})] + \tfrac{\partial V}{\partial \hat{x}}(x,\hat{x})[k(\hat{x}) + l(\hat{x})h(x)] + \\ &\tfrac{1}{2}\tfrac{1}{\gamma^2}\tfrac{\partial V}{\partial x}(x,\hat{x})g(x)g(x)^T\tfrac{\partial^T V}{\partial x}(x,\hat{x}) + \tfrac{1}{2}\tfrac{1}{\gamma^2}\tfrac{\partial V}{\partial \hat{x}}(x,\hat{x})l(\hat{x})l(\hat{x})^T\tfrac{\partial^T V}{\partial \hat{x}}(x,\hat{x}) \\ &+\tfrac{1}{2}h(x)^T h(x) + \tfrac{1}{2}m(\hat{x})^T m(\hat{x}) \leq 0, \quad V(0,0) = 0. \end{aligned} \tag{26}$$

This is the (differential) dissipation inequality associated to the closed-loop system having L_2-gain $\leq \gamma$. Let us consider the equation

$$\frac{\partial V}{\partial \hat{x}}(x, \hat{x}) = 0$$

and suppose this equation has a smooth solution $\hat{x} = F(x)$, $F(0) = 0$. Define $P(x) := V(x, F(x)) \geq 0$, then substitution of $\hat{x} = F(x)$ into (26) and completing the squares yields that $P(x)$ is a nonnegative smooth solution to the Hamilton-Jacobi inequality

$$\frac{\partial P}{\partial x}(x)f(x) - \frac{1}{2}(1 - \frac{1}{\gamma^2})\frac{\partial P}{\partial x}(x)g(x)g^T(x)\frac{\partial^T P}{\partial x}(x) + \frac{1}{2}h^T(x)h(x) \leq 0, \tag{27}$$

with $P(0) = 0$. Now define $Q(x) := \frac{1}{\gamma^2}V(x,0) \geq 0$. Substitution of $\hat{x} = 0$ in (26) and completing the squares yields that $Q(x)$ is a nonnegative smooth solution to the Hamilton-Jacobi inequality

$$\frac{\partial Q}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial Q}{\partial x}(x)g(x)g^T(x)\frac{\partial^T Q}{\partial x}(x) - \frac{1}{2}(1 - \frac{1}{\gamma^2})h^T(x)h(x) \leq 0, \tag{28}$$

with $Q(0) = 0$. These inequalities are important for the necessary conditions for the $\mathcal{H}_\infty$ suboptimal control problem, as is summarized in the following theorem.

Theorem 4.1 (e.g. [22]) *Suppose the $\mathcal{H}_\infty$ suboptimal control problem for $\gamma > 0$ is solvable by a compensator (25) in the sense that there exists a smooth solution $V(x,\hat{x}) \geq 0$ to the Hamilton-Jacobi inequality (26). Assume that $\frac{\partial^2 V}{\partial \hat{x}^2}(x,\hat{x}) > 0$, and that the equation $\frac{\partial V}{\partial \hat{x}}(x,\hat{x}) = 0$ has a smooth solution $\hat{x} = F(x)$, $F(0) = 0$, with $F : M \to M_c$. Then the Hamilton-Jacobi inequalities (27) and (28) have smooth nonnegative solutions $P(x)$ and $Q(x)$, respectively, for which the coupling condition*

$$P(x) \leq \gamma^2 Q(x) \tag{29}$$

for all x near 0 holds. □

Remark 4.2 The existence of smooth solutions to (27) and (28) with equality may be pursued by an iterative procedure under some additional technical conditions (see Van der Schaft [21], Remark 19). □

However, the necessary conditions of Theorem 4.1 are not sufficient for the solvability of the nonlinear $\mathcal{H}_\infty$ problem. Recently another approach has been taken to the full $\mathcal{H}_\infty$ suboptimal control problem (e.g. Başar and Bernhard [2], Van der Schaft [22]). Using methods of the theory of differential games, it has been shown that under suitable technical conditions the $\mathcal{H}_\infty$ suboptimal control problem for (22) is solved by the controller

$$\begin{aligned} \dot{\hat{x}} = \quad & f(\hat{x}) - (1 - \frac{1}{\gamma^2}) g(\hat{x}) g^T(\hat{x}) \frac{\partial^T P}{\partial x}(\hat{x}) \\ & + \gamma^2 \left(\frac{\partial^2 S}{\partial x^2}(\hat{x}, t) \right)^{-1} \frac{\partial^T h}{\partial x}(\hat{x})(y(t) - h(\hat{x})) \\ u = \quad & -g^T(\hat{x}) \frac{\partial^T P}{\partial x}(\hat{x}), \end{aligned} \tag{30}$$

where $S(x,t) = R(x,t) - P(x)$. The function $P \geq 0$ is the solution to the Hamilton-Jacobi equation (27) with $P(0) = 0$, such that the vector field

$$f(x) - (1 - \frac{1}{\gamma^2}) g(x) g^T(x) \frac{\partial^T P}{\partial x}(x)$$

is asymptotically stable. $R(x,t)$ is the solution (cf. Van der Schaft [22]) to the non-stationary Hamilton-Jacobi equation

$$\begin{aligned} & \frac{\partial R}{\partial t} + \frac{\partial R}{\partial x} f(x) + \frac{1}{2} \frac{1}{\gamma^2} \frac{\partial R}{\partial x} g(x) g^T(x) \frac{\partial^T R}{\partial x} + \frac{1}{2}(1 - \gamma^2) h^T(x) h(x) + \\ & \gamma^2 h^T(x) y(t) + \frac{\partial R}{\partial x} g(x) u(t) - \frac{1}{2}\gamma^2 \parallel y(t) \parallel^2 + \frac{1}{2} \parallel u(t) \parallel^2 = 0, \end{aligned} \tag{31}$$

with $u(t)$ given as the output of (30), and such that the solution $x(t)$ of

$$\dot{x} = f(x) + \frac{1}{\gamma^2} g(x) g^T(x) \frac{\partial^T R}{\partial x}(x,t)$$

satisfies $\lim_{t \to -\infty} x(t) = 0$. (This means that

$$-R(x,t) = \min_{\substack{\tilde{d} \in L_2(-\infty, 0) \\ x(-\infty) = 0, x(t) = x}} \int_{-\infty}^{t} \|z(\tau)\|^2 - \gamma^2 \|\tilde{d}(\tau)\|^2 d\tau.)$$

Thus, if there exist such solutions P and R to (27) and (31), respectively, with $S(x,t)$ having a unique maximum $\hat{x}(t)$, then under suitable technical conditions the controller (30) solves the $\mathcal{H}_\infty$ suboptimal control problem.

If system (22) is linear, the controller (30) gives the central controller of Doyle *et al.* [4] (see equation (18)). In the nonlinear case (30) is called the nonlinear *central controller*. The nonlinear central controller (30) is in general an infinite dimensional controller. We observe that in the linear case the central controller may be computed off-line, since then $\frac{\partial^2 S}{\partial x^2}(x,t)$ is the constant matrix L^{-1}, where L is given by (15). There also exists a class of nonlinear systems for which $\frac{\partial^2 R}{\partial x^2}(x,t)$, and, hence, $\frac{\partial^2 S}{\partial x^2}(x,t)$ may be computed off-line, and thus the nonlinear central controller reduces to a finite-dimensional controller. This class of systems is treated in Van der Schaft [23]. Assume that the nonlinear system (22) has the property that

- $g(x)$ does not depend on x
- $h(x)$ is linear in x
- a smooth function $V(x)$ can be found such that
  - ⋆ $f(x) + \frac{1}{\gamma^2} g(x) g^T(x) \frac{\partial^T V}{\partial x}(x)$ is affine in x
  - ⋆ $\frac{1}{2}(1-\gamma^2) h^T(x) h(x) + \frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V}{\partial x}(x) g(x) g^T(x) \frac{\partial^T V}{\partial x}(x) + \frac{\partial V}{\partial x}(x) f(x)$ at most of order 2 in x
  - ⋆ $\frac{\partial V}{\partial x}(x) g(x)$ is affine in x

Hence, the vector field $g(x)$ is a constant matrix, and $h(x)$ can be written as $h(x) = Cx$, with C a constant matrix. This means that the only nonlinearity in (22) appears in the vector field $f(x)$, which is restricted by the last property. Then, as in the linear case, it follows that the solution $R(x,t)$ of (31) can be written as $R(x,t) = \bar{R}(x,t) + V(x)$, with $\bar{R}(x,t) = \bar{m}(t) + \bar{l}(t)x + \frac{1}{2}x^T \bar{Q}(t) x$, and $\bar{Q}(t) = \frac{\partial^2 \bar{R}}{\partial x^2}(x,t)$ can be computed off-line (without knowing $y(t)$ and $u(t)$). Therefore the Hessian $\frac{\partial^2 R}{\partial x^2}(x,t)$ can also be computed off-line, and thus, in this special case, the nonlinear central controller reduces to a finite-dimensional controller.

5 The $\mathcal{H}_\infty$ balanced representation

Consider the smooth nonlinear system (see (1))

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{32}$$

and let γ be a fixed positive constant. Assume that the system is zero-state observable. Similar to HJB balancing that is treated in Scherpen and Van der Schaft [28], we consider a certain past and future energy function, but the form is dependent on γ. We distinguish between $\gamma > 1$, $\gamma < 1$ and $\gamma = 1$. The special case $\gamma = 1$ will be treated later. First we define the following energy functions.

Definition 5.1 If $\gamma \neq 1$, then the $\mathcal{H}_\infty$*-past energy function* of a nonlinear system (32) is defined as

$$Q_\gamma^-(x_0) = \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty)=0, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} ((1 - \frac{1}{\gamma^2}) \|y(t)\|^2 + \|u(t)\|^2)dt. \tag{33}$$

If $\gamma > 1$, then the $\mathcal{H}_\infty$*-future energy function* of a nonlinear system (32) is defined as

$$Q_\gamma^+(x_0) = \min_{\substack{u \in L_2(0, \infty) \\ x(0)=x_0, x(\infty)=0}} \frac{1}{2}\int_{0}^{\infty} (\|y(t)\|^2 + \frac{\gamma^2}{\gamma^2 - 1} \|u(t)\|^2)dt. \tag{34}$$

If $\gamma < 1$, then the $\mathcal{H}_\infty$*-future energy function* of a nonlinear system (32) is defined as

$$Q_\gamma^+(x_0) = \max_{\substack{u \in L_2(0, \infty) \\ x(0)=x_0, x(\infty)=0}} \frac{1}{2}\int_{0}^{\infty} (\|y(t)\|^2 + \frac{\gamma^2}{\gamma^2 - 1} \|u(t)\|^2)dt. \tag{35}$$

□

The $\mathcal{H}_\infty$-future energy functions only differ in the kind of optimization: minimization and maximization, respectively. We derive the following theorem, which relates the above $\mathcal{H}_\infty$

energy functions to the necessary conditions for solvability of the $\mathcal{H}_\infty$ problem stated in Section 3.

Theorem 5.2 *Suppose $\gamma \neq 1$. Assume that the following Hamilton-Jacobi-Bellman equation has a smooth solution Q_γ*

$$\frac{\partial Q_\gamma}{\partial x}(x)f(x) - \frac{1}{2}(1-\frac{1}{\gamma^2})\frac{\partial Q_\gamma}{\partial x}(x)g(x)g(x)^T\frac{\partial^T Q_\gamma}{\partial x}(x) + \frac{1}{2}h^T(x)h(x) = 0 \tag{36}$$

with $Q_\gamma(0) = 0$, such that

$$f(x) - (1-\frac{1}{\gamma^2})g(x)g(x)^T\frac{\partial^T Q_\gamma}{\partial x}(x) \tag{37}$$

is asymptotically stable. Then Q_γ^+ exists and is the unique smooth non-negative solution of (36), such that (37) is asymptotically stable.

Similarly, assume that the following Hamilton-Jacobi-Bellman equation has a smooth solution $\bar{Q}_\gamma$

$$\frac{\partial \bar{Q}_\gamma}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial \bar{Q}_\gamma}{\partial x}(x)g(x)g^T(x)\frac{\partial^T \bar{Q}_\gamma}{\partial x}(x) - \frac{1}{2}(1-\frac{1}{\gamma^2})h(x)^T h(x) = 0 \tag{38}$$

with $\bar{Q}_\gamma(0) = 0$, such that

$$-\left(f(x) + g(x)g(x)^T\frac{\partial^T \bar{Q}_\gamma}{\partial x}(x)\right) \tag{39}$$

is asymptotically stable. Then Q_γ^- exists and is the unique smooth non-negative solution of (38), such that (39) is asymptotically stable.

Proof Let Q_γ be a solution to (36), (37). Then

$$\frac{\partial Q_\gamma}{\partial x}(x)f(x) = \frac{1}{2}(1-\frac{1}{\gamma^2})\frac{\partial Q_\gamma}{\partial x}(x)g(x)g(x)^T\frac{\partial^T Q_\gamma}{\partial x}(x) - \frac{1}{2}h^T(x)h(x).$$

Consider inputs u such that $x(0) = x_0$ and $x(\infty) = 0$, then

$$\begin{aligned}\frac{d}{dt}Q_\gamma(x) &= \frac{\partial Q_\gamma}{\partial x}(x)f(x) + \frac{\partial Q_\gamma}{\partial x}(x)g(x)u \\ &= \frac{1}{2}(1-\frac{1}{\gamma^2})\frac{\partial Q_\gamma}{\partial x}(x)g(x)g(x)^T\frac{\partial^T Q_\gamma}{\partial x}(x) - \frac{1}{2}h^T(x)h(x) \\ &\quad +\frac{\partial Q_\gamma}{\partial x}(x)g(x)u\end{aligned}$$

We treat the cases $\gamma > 1$ and $\gamma < 1$ separately. First suppose $\gamma > 1$. Define $\bar{\epsilon} := (1 - \frac{1}{\gamma^2})^{\frac{1}{2}}$, then

$$\frac{d}{dt}Q_\gamma(x) = \frac{1}{2} \parallel \bar{\epsilon} g^T(x)\frac{\partial^T Q_\gamma}{\partial x}(x) + \frac{1}{\bar{\epsilon}}u \parallel^2 - \frac{1}{2}\frac{1}{\bar{\epsilon}^2} \parallel u \parallel^2 - \frac{1}{2} \parallel y \parallel^2$$

and thus

$$\begin{aligned}Q_\gamma(x_0) &= -\int_0^\infty \frac{d}{dt}Q_\gamma(x(t))dt = \frac{1}{2}\int_0^\infty (\parallel y(t) \parallel^2 + \frac{1}{\bar{\epsilon}^2} \parallel u(t) \parallel^2 \\ &\quad - \parallel \bar{\epsilon} g^T(x(t))\frac{\partial^T Q_\gamma}{\partial x}(x(t)) + \frac{1}{\bar{\epsilon}}u(t) \parallel^2)dt \\ &\leq \frac{1}{2}\int_0^\infty (\parallel y(t) \parallel^2 + \frac{1}{\bar{\epsilon}^2} \parallel u(t) \parallel^2)dt\end{aligned}$$

Hence $Q_\gamma(x_0)$ is a lower bound for $Q_\gamma^+(x_0)$, and this bound is attained for the input $u = -\bar{\epsilon}^2 g(x)g^T(x)\frac{\partial^T Q_\gamma}{\partial x}(x)$, thus Q_γ^+ is equal to Q_γ. By the asymptotic stability of $(f(x) - \bar{\epsilon}^2 g(x)g^T(x)\frac{\partial^T Q_\gamma}{\partial x}(x))$ this latter input is indeed such that $x(\infty) = 0$. Therefore $Q_\gamma^+(x_0) = Q_\gamma(x_0)$, and the first part of the theorem is proved in the case $\gamma > 1$.

Now suppose $\gamma < 1$, and define $\tilde{\epsilon} := (\frac{1}{\gamma^2} - 1)^{\frac{1}{2}}$. Then

$$\frac{d}{dt}Q_\gamma(x) = -\frac{1}{2} \parallel \tilde{\epsilon} g^T(x)\frac{\partial^T Q_\gamma}{\partial x}(x) - \frac{1}{\tilde{\epsilon}}u \parallel^2 + \frac{1}{2}\frac{1}{\tilde{\epsilon}^2} \parallel u \parallel^2 - \frac{1}{2} \parallel y \parallel^2$$

and thus

$$\begin{aligned}Q_\gamma(x_0) &= -\int_0^\infty \frac{d}{dt}Q_\gamma(x(t))dt = \frac{1}{2}\int_0^\infty (\parallel y(t) \parallel^2 - \frac{1}{\tilde{\epsilon}^2} \parallel u(t) \parallel^2 \\ &\quad + \parallel \tilde{\epsilon} g^T(x(t))\frac{\partial^T Q_\gamma}{\partial x}(x(t)) - \frac{1}{\tilde{\epsilon}}u(t) \parallel^2)dt \\ &\geq \frac{1}{2}\int_0^\infty (\parallel y(t) \parallel^2 - \frac{1}{\tilde{\epsilon}^2} \parallel u(t) \parallel^2)dt\end{aligned}$$

Hence $Q_\gamma(x_0)$ is an upper bound for $Q^+_\gamma(x_0)$, and this bound is attained for the input $u = \tilde{\epsilon}^2 g(x)g^T(x)\frac{\partial^T Q_\gamma}{\partial x}(x)$. This latter input is such that $x(\infty) = 0$ by the asymptotic stability of $(f(x) + \tilde{\epsilon}^2 g(x)g^T(x)\frac{\partial^T Q_\gamma}{\partial x}(x))$. Therefore also $Q^+_\gamma(x_0) = Q_\gamma(x_0)$ in the case $\gamma < 1$.

The second part of the theorem (which is simpler, since we do not have to distinguish between $\gamma > 1$ and $\gamma < 1$) is proved by similar arguments. ■

Remark 5.3 If we assume that $Q^+_\gamma(x)$ exists and is smooth for $\gamma \neq 1$, then we know by the same arguments as in Scherpen [24] (taken from Optimal Control theory), that (36) has a smooth solution, such that (37) is asymptotically stable.

Similarly, if we assume that $Q^-_\gamma(x)$ exists and is smooth, then (38) has a smooth solution, such that (39) is asymptotically stable. □

Note that the Hamilton-Jacobi-Bellman equations (36) and (38) that give a characterization of the $\mathcal{H}_\infty$ past and future energy functions equal the Hamilton-Jacobi inequalities (27) and (28) of the $\mathcal{H}_\infty$ suboptimal control problem with equality, respectively.

Theorem 5.4 *Let $\gamma \neq 1$. Assume that Q^+_γ exists and is smooth on some neighborhood W of 0. Then $Q^+_\gamma(x_0) > 0$ for $x_0 \in W$, $x_0 \neq 0$. Similarly, assume that Q^-_γ exists and is smooth on W. Then $Q^-_\gamma(x_0) > 0$ for $x_0 \in W$, $x_0 \neq 0$.*

Proof First assume that $\gamma > 1$. Let $\hat{\epsilon} = 1 - \frac{1}{\gamma^2} > 0$. By the form (34) of Q^+_γ, $Q^+_\gamma(x) \geq 0$ for $x \in W$, $x \neq 0$. We know that the minimum is attained for $u = -\hat{\epsilon} g(x)g^T(x)\frac{\partial Q^+_\gamma}{\partial x}(x)$, and thus

$$
\begin{aligned}
Q^+_\gamma(x_0) \quad &= \frac{1}{2}\int_0^\infty \quad [\hat{\epsilon}\frac{\partial Q^+_\gamma}{\partial x}(x(t))g(x(t))g(x(t))^T\frac{\partial^T Q^+_\gamma}{\partial x}(x(t)) \\
&\qquad +h(x(t))^T h(x(t))]dt
\end{aligned}
$$

Now let $x_0 \neq 0$. If $\hat{\epsilon}\frac{\partial Q^+_\gamma}{\partial x}(x(t))g(x(t))g(x(t))^T\frac{\partial^T Q^+_\gamma}{\partial x}(x(t)) + h(x(t))^T h(x(t)) = 0$ for $0 \leq t < \infty$, then $u(t) = 0$ and $h(x(t)) = 0$ for all t, $0 \leq t < \infty$. However, by the zero-state observability of system (32) this implies that $x(t) = 0$ for all $0 \leq t < \infty$ and this contradicts $x_0 \neq 0$. Hence $Q_\gamma^+(x_0) > 0$, $\forall x_0 \in W$, $x_0 \neq 0$.

Suppose $\gamma < 1$. Then $Q_\gamma^+(x_0) > 0$, $x_0 \neq 0$, follows from Proposition 3.4 in Van der Schaft [22].

The second part of the theorem is proved via a similar argument. ■

We may consider $\gamma \to \infty$. The following result follows straightforwardly from the formulas.

Theorem 5.5 *Assume that the past and future energy functions, (7) and (8), respectively, of Section 2 exist and are smooth. Furthermore, assume that the $\mathcal{H}_\infty$ past and future energy functions, (33) and (34), respectively, exist and are smooth for $\gamma > 1$. Then*

$$\lim_{\gamma \to \infty} Q_\gamma^-(x) = K^-(x), \qquad \lim_{\gamma \to \infty} Q_\gamma^+(x) = K^+(x)$$

□

The following theorem may be found in Van der Schaft [21, 22].

Theorem 5.6 ([22], Prop. 3.4) *Assume that $\gamma < 1$ and that Q_γ^+ exists and is smooth on W. Then system (32) is locally asymptotically stable. If Q_γ^+ is proper, then (32) is asymptotically stable on W.* □

Until now we have considered systems (32) with $\gamma \neq 1$. The following theorem shows that as γ tends to 1, we obtain a relation with the observability and controllability functions of Section 2.

Theorem 5.7 *Assume that for γ in a neighborhood of 1 (36) and (38) have smooth solutions, such that (37) and (39), respectively, are asymptotically stable. Furthermore, assume that Q_γ^+ and Q_γ^- are continuous in $\gamma = 1$. Then*

$$\lim_{\gamma \uparrow 1} Q_\gamma^+(x) = L_o(x), \qquad \lim_{\gamma \uparrow 1} Q_\gamma^-(x) = L_c(x),$$

where $L_o(x)$ and $L_c(x)$ are the observability and controllability function, respectively, of system (32).

Additionally, assume that (32) is asymptotically stable. Then

$$\lim_{\gamma \downarrow 1} Q_\gamma^+(x) = L_o(x), \qquad \lim_{\gamma \downarrow 1} Q_\gamma^-(x) = L_c(x).$$

Proof By Theorem 5.2 it follows that Q_γ^+ and Q_γ^- exist and are smooth. The first part is immediately obtained from the Hamilton-Jacobi-Bellman equations (36) and (38) and Theorem 5.6. The second part follows from the Hamilton-Jacobi-Bellman equations, and from the asymptotic stability of f. ■

Remark 5.8 Theorem 5.7 also may be easily obtained by Definition 5.1. In (33) it immediately follows that $Q_\gamma^- \to L_c$ for $\gamma \to 1$. For Q_γ^+ it follows directly that for $\gamma \to 1$ the minimum of (34) and maximum of (35), respectively, are obtained for $u \equiv 0$. □

Define $Q_1^+ := L_o$ and $Q_1^- := L_c$. As in the linear case (Theorem 3.2), we are also able to say something about the dependence of the $\mathcal{H}_\infty$-past and future energy function on γ. Assume that Q_γ^+ exists and is smooth for all γ. Consider the Hamiltonian $H_\gamma : T^*M \to \mathbb{R}$ given as

$$H_\gamma(x,p) = p^T f(x) - \frac{1}{2}(1 - \frac{1}{\gamma^2})p^T g(x)g(x)^T p + \frac{1}{2}h(x)^T h(x).$$

Then by (36)

$$H_\gamma(x, \frac{\partial^T Q_\gamma^+}{\partial x}(x)) = 0, \quad Q_\gamma^+(0) = 0, \quad \frac{\partial^T Q_\gamma^+}{\partial x}(0) = 0.$$

It can be verified (e.g. Van der Schaft [21, 22]) that the following n-dimensional submanifold of T^*M

$$N_\gamma = \{(x,p) \in T^*M \mid p = \frac{\partial^T Q_\gamma^+}{\partial x}(x)\} \tag{40}$$

is an invariant manifold of the Hamiltonian vector field X_{H_γ} through $(0, 0)$, meaning that $X_{H_\gamma}(x, p)$ is tangent to N at every point $(x, p) \in N$. Let us assume throughout the rest of this chapter that

- The linearization of X_{H_γ} at $(0, 0)$ has no purely imaginary eigenvalues

(such a vector field is called *hyperbolic*).

Theorem 5.9 *Assume that Q_γ^+ and Q_γ^- exist and are smooth functions of x for all $\gamma > 0$. Then Q_γ^+ and Q_γ^- are jointly continuously differentiable (C^1) functions of x and γ.*

Proof By Remark 5.3 Q_γ^+ is the solution of (36), such that (37) is asymptotically stable. Since X_{H_γ} is hyperbolic, $N_\gamma \subset T^*M$ is the stable submanifold of X_{H_γ}. Now consider the vector field on $T^*M \times \mathbb{R}$

$$\begin{aligned} \dot{x}_i &= \frac{\partial H_\gamma}{\partial p_i}(x, p) \\ \dot{p}_i &= -\frac{\partial H_\gamma}{\partial x_i}(x, p), \qquad i = 1, ..., n \\ \dot{\gamma} &= 0 \end{aligned} \tag{41}$$

Since f, g, and h are assumed to be smooth, the Hamiltonian vector field X_{H_γ} is also smooth. This means that the center stable manifold of (41) (see e.g. Carr [3]) is continuous on a neighborhood of every point (x, p, γ). Denote the center stable manifold by W^{cs}. Then it is clear from the particular form of the dynamics (41) that

$$W^{cs} \cap \{\gamma = a\}, \quad a \in \mathbb{R}$$

is an invariant manifold, which in fact is equal to N_a (N_γ is given by (40)). Thus W^{cs} is foliated by N_γ, $\gamma \in \mathbb{R}$, and by the continuity of W^{cs} it follows that N_γ depends (locally) continuously on γ (see e.g. Vanderbauwhede and Van Gils [30, 29]). Therefore, it follows that the gradient vector of Q_γ^+ depends continuously on x and γ, and thus Q_γ^+ is a continuously differentiable function of x and γ. Similar arguments hold for Q_γ^- using the unstable invariant manifold that is defined similarly to (40), and the center unstable manifold of (41). ■

Remark 5.10 The assumption that X_{H_γ} is hyperbolic is not very restrictive, since it is automatically fulfilled for γ larger than the optimal value of γ (see Van der Schaft [21]). □

From Theorem 5.9 we derive the following result:

Theorem 5.11 *Assume that Q_γ^+ and Q_γ^- exist and are smooth for all $\gamma > 0$. Then Q_γ^+ is a non-increasing function of γ and Q_γ^- is a non-decreasing function of γ.*

Proof Q_γ^+ satisfies equation (36) such that (37) is asymptotically stable. By Theorem 5.9 Q_γ^+ is continuously differentiable in γ, and thus we may differentiate equation (36) with respect to γ. Denote

$$\left(\frac{\partial Q_\gamma^+}{\partial x}(x)\right)' := \frac{\partial}{\partial \gamma}\left(\frac{\partial Q_\gamma^+}{\partial x}(x)\right)\left(= \frac{\partial}{\partial x}\left(\frac{\partial Q_\gamma^+}{\partial \gamma}(x)\right)\right).$$

Differentiating equation (36) with respect to γ yields

$$\left(\frac{\partial Q_\gamma^+}{\partial x}(x)\right)'\left(f(x) - (1 - \frac{1}{\gamma^2})g(x)g(x)^T\frac{\partial^T Q_\gamma^+}{\partial x}(x)\right) =$$
$$\frac{1}{\gamma^3}\frac{\partial Q_\gamma^+}{\partial x}(x)g(x)g(x)^T\frac{\partial^T Q_\gamma^+}{\partial x}(x) \geq 0,$$

since $\gamma > 0$. By the asymptotic stability of (37) it follows that $(Q_\gamma^+)' \leq 0$. Thus, Q_γ^+ is a non-increasing function of γ.

Q_γ^- fulfills equation (38) such that (39) is asymptotically stable. By Theorem 5.9 Q_γ^- is continuously differentiable in γ, and thus we may differentiate equation (38) with respect to γ. This yields

$$\left(\frac{\partial Q_\gamma^-}{\partial x}(x)\right)'\left(f(x) + g(x)g(x)^T\frac{\partial^T Q_\gamma^+}{\partial x}(x)\right) = \frac{1}{\gamma^3}h(x)^T h(x) \geq 0$$

since $\gamma > 0$. By the asymptotic stability of (39) it follows that $(Q_\gamma^-)' \geq 0$. Thus, Q_γ^- is a non-decreasing function of γ. ■

Now we consider nonlinear systems of the form (32) with $\mathcal{H}_\infty$ future and $\mathcal{H}_\infty$ past energy function respectively Q_γ^+ and Q_γ^- as in Definition 5.1 and with the following standing assumptions:

1. $\gamma \neq 1$.

2. Q_γ^+ and Q_γ^- exist and are smooth on a neighborhood Y of 0.

3. $\dfrac{\partial^2 Q_\gamma^+}{\partial x^2}(0) > 0$ and $\dfrac{\partial^2 Q_\gamma^-}{\partial x^2}(0) > 0$.

4. The system is zero-state observable on Y.

By similar arguments as formulated in Scherpen [24] we can transform Q_γ^+ and Q_γ^- for any $\gamma \neq 1$ into a special form. For notational convenience we formulate these again for Q_γ^+ and Q_γ^- for a $\gamma > 0$.

Lemma 5.12 *There exists a coordinate transformation $x = \zeta(\bar{x})$, $\zeta(0) = 0$, such that in the new coordinates $\bar{x} = \zeta^{-1}(x)$ the function $Q_\gamma^-(x)$ is of the form*

$$Q_\gamma^-(\zeta(\bar{x})) = \frac{1}{2}\bar{x}^T\bar{x}. \tag{42}$$

Furthermore in the new coordinates $\bar{x} = \zeta^{-1}(x)$ we can write $Q_\gamma^+(x)$ in form

$$Q_\gamma^+(\zeta(\bar{x})) = \frac{1}{2}\bar{x}^T H(\bar{x})\bar{x} \quad \textit{where} \quad H(0) = \frac{\partial^2 Q_\gamma^+}{\partial x^2}(0), \tag{43}$$

where $H(\bar{x})$ is a $n \times n$ symmetric matrix with entries which are smooth functions of $\bar{x}$.

Proof See Scherpen [24]. ■

Theorem 5.13 *Consider system (32) and assume there exists a neighborhood V of 0 where the number of distinct eigenvalues of $H(\bar{x})$ is constant for $\bar{x} \in V$. On a neighborhood U of zero there exists a coordinate transformation $x = \varrho(z)$, $\varrho(0) = 0$, such that in the new coordinates $z \in W := \varrho^{-1}(U)$ the function $Q_\gamma^-(x)$ is of the form*

$$\tilde{Q}_\gamma^-(z) := Q_\gamma^-(\varrho(z)) = \frac{1}{2} z^T z, \tag{44}$$

while in the new coordinates $Q_\gamma^+(x)$ is of the form

$$\tilde{Q}_\gamma^+(z) := Q_\gamma^+(\varrho(z)) = \frac{1}{2} z^T \begin{pmatrix} \kappa_1(z) & & 0 \\ & \ddots & \\ 0 & & \kappa_n(z) \end{pmatrix} z, \tag{45}$$

where $\kappa_1(z) \geq \cdots \geq \kappa_n(z)$ are smooth functions of z, called the $\mathcal{H}_\infty$ singular value functions.

Proof See Scherpen [24]. ■

Remark 5.14 For linear systems the $\mathcal{H}_\infty$ singular value functions κ_i, $i = 1, \ldots, n$ are constant and are equal to the squared $\mathcal{H}_\infty$ characteristic values of Theorem 3.1. □

If we take into account the coupling condition (29), we obtain a similar result as in Theorem 3.2.a in the linear case .

Corollary 5.15 *Assume that the coupling condition (29) is fulfilled for Q_γ^+ and Q_γ^-, i.e. $Q_\gamma^+(x) \leq \gamma^2 Q_\gamma^-(x)$. Then $\kappa_i(0, \ldots, 0, z_i, 0, \ldots, 0) \leq \gamma^2$, $i = 1, \ldots, n$.* □

Like in Section 3.1 the form of the past and future energy function in (44) and (45) is not yet entirely the form we want. For that we need an additional coordinate transformation. We take as smooth coordinate transformation $\bar{z}_i = \beta_i(z_i) := \kappa_i(0, \ldots, 0, z_i, 0, \ldots, 0)^{\frac{1}{4}} z_i$, $i = 1, \ldots, n$

and hence $\bar{z} = \beta(z) := (\beta_1(z_1), \ldots, \beta_n(z_n))$ on $\bar{z} \in \bar{W} := \beta(W)$. Define $\bar{Q}_\gamma^-(\bar{z}) := \tilde{Q}_\gamma^-(\beta^{-1}(\bar{z}))$ and $\bar{Q}_\gamma^+(\bar{z}) := \tilde{Q}_\gamma^+(\beta^{-1}(\bar{z}))$. Then (44) and (45) become

$$\bar{Q}_\gamma^-(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} \vartheta_1(\bar{z}_1)^{-1} & & 0 \\ & \ddots & \\ 0 & & \vartheta_n(\bar{z}_n)^{-1} \end{pmatrix} \bar{z}, \tag{46}$$

and

$$\bar{Q}_\gamma^+(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} \vartheta_1(\bar{z}_1)^{-1}\kappa_1(\beta^{-1}(\bar{z})) & & 0 \\ & \ddots & \\ 0 & & \vartheta_n(\bar{z}_n)^{-1}\kappa_n(\beta^{-1}(\bar{z})) \end{pmatrix} \bar{z}, \tag{47}$$

respectively, where $\vartheta_i(\bar{z}_i) = \kappa_i(0, \ldots, 0, \xi_i^{-1}(\bar{z}_i), 0, \ldots, 0)^{\frac{1}{2}}$ for $i = 1, \ldots, n$. It follows that

$$\bar{Q}_\gamma^-(0, \ldots, 0, \bar{z}_i, 0, \ldots, 0) = \frac{1}{2}\bar{z}_i^2 \vartheta_i(\bar{z}_i)^{-1}$$

$$\bar{Q}_\gamma^+(0, \ldots, 0, \bar{z}_i, 0, \ldots, 0) = \frac{1}{2}\bar{z}_i^2 \vartheta_i(\bar{z}_i)$$

for $i = 1, \ldots, n$. In terms of the past and future energy we infer from $\kappa_i(\beta^{-1}(\bar{z})) > \kappa_{i+1}(\beta^{-1}(\bar{z}))$ that the state component $\bar{z}_i$ is more important than the state component $\bar{z}_{i+1}$ on $\bar{W}$.

Definition 5.16 A nonlinear system (32) is in $\mathcal{H}_\infty$ *balanced* form if its past and future energy functions are of the form (46) and (47), respectively. □

This means that we can bring system (32) in a HJB balanced form by a coordinate transformation of the form

$$x = \varpi(\bar{z}) := \varrho(\beta^{-1}(\bar{z})), \tag{48}$$

where ϱ is as in Theorem 5.13. For a linear system this means that the system is in the $\mathcal{H}_\infty$ balanced form, since then $\bar{Q}_\gamma^-(\bar{z}) = \frac{1}{2}\bar{z}^T Y_\infty^{-1} \bar{z}$ and $\bar{Q}_\gamma^+(\bar{z}) = \frac{1}{2}\bar{z}^T X_\infty \bar{z}$ with $N = Y_\infty = X_\infty$ as in Theorem 3.1.

For completeness we also treat the linearization of the $\mathcal{H}_\infty$ balancing method. Therefore, linearize the system (32) at $x = 0$ and $u = 0$. This yields a system (A, B, C) as mentioned in Section 3. By linearizing the $\mathcal{H}_\infty$ balancing method we mean that every system and every coordinate transformation that appear during the $\mathcal{H}_\infty$ balancing procedure is linearized.

Theorem 5.17 *If we linearize the $\mathcal{H}_\infty$ balancing method for nonlinear systems at $x = 0$ and $u = 0$, we obtain the $\mathcal{H}_\infty$ balancing method for linear systems as is treated in Section 3.*

Proof Follows directly from Taylor series expansions at 0. ■

Conversely, consideration of the linearization yields a sufficient condition for locally $\mathcal{H}_\infty$ balancing the nonlinear system:

Theorem 5.18 *Assume that the linearization of system (32) is minimal. If the $\mathcal{H}_\infty$ characteristic values $\vartheta_1, \ldots, \vartheta_n$ of Theorem 3.1 satisfy $\vartheta_i \neq \vartheta_j$ for $i \neq j$, then locally the nonlinear system (32) may be brought into $\mathcal{H}_\infty$ balanced form.*

Proof See [27]. ■

In Section 4 we obtained a class of nonlinear systems (including the linear systems) for which $\frac{\partial^2 R}{\partial x^2}(t, x)$ may be computed off-line, and thus in this case the controller (30) reduces to a finite-dimensional controller. For these systems we may obtain the $\mathcal{H}_\infty$ singular value functions from the solutions R and P of the equations (27) and (31), and then the $\mathcal{H}_\infty$ singular value functions are a measure for the difficulties both to control and filter in the $\mathcal{H}_\infty$ sense the corresponding state component. This relation is similar to the relation between HJB balancing and the nonlinear version of the LQG problem, see Scherpen and Van der Schaft [28].

In the linear case Mustafa and Glover [18] also treat balancing of the normalized coprime factorization of the scaled plant (see Section 3) if $\gamma > 1$. This results in a relation between the Hankel singular values of the normalized coprime factorization of the scaled plant and the $\mathcal{H}_\infty$ characteristic values. In the nonlinear case it is also possible to find such relation. Assume $\gamma > 1$ and define $\bar{\epsilon} := \sqrt{1-\gamma^{-2}}$. Consider the scaled system

$$
\begin{aligned}
\dot{x} &= f(x) + \bar{\epsilon} g(x) u \\
y &= h(x)
\end{aligned}
\tag{49}
$$

We consider the normalized right coprime representation of this system, which exists if we make the appropriate asumptions, and which is given by (see Scherpen and Van der Schaft [28])

$$
\begin{aligned}
&\dot{x} = \left(f(x) - \bar{\epsilon}^2 g(x) g(x)^T \tfrac{\partial^T \check{L}_o}{\partial x}(x) \right) + \bar{\epsilon} g(x) w, \\
&\begin{cases} y = h(x) \\ u = -\bar{\epsilon} g(x)^T \tfrac{\partial^T \check{L}_o}{\partial x}(x) + w \end{cases}
\end{aligned}
\tag{50}
$$

$\check{L}_o$ is the observability function of system (50) and fulfills (see Section 2)

$$
\frac{\partial \check{L}_o}{\partial x}(x) f(x) - \frac{1}{2}\bar{\epsilon}^2 \frac{\partial \check{L}_o}{\partial x}(x) g(x) g(x)^T \frac{\partial^T \check{L}_o}{\partial x}(x) + \frac{1}{2} h(x)^T h(x) = 0, \tag{51}
$$

with $\check{L}_o(0) = 0$. Now, assume that the controllability function of system (50) exists and is smooth. Then the controllability function $\check{L}_c$ of system (50) fulfills

$$
\begin{aligned}
&\frac{\partial \check{L}_c}{\partial x}(x) \left(f(x) - \bar{\epsilon}^2 g(x) g(x)^T \frac{\partial^T \check{L}_o}{\partial x}(x) \right) + \\
&\frac{1}{2}\bar{\epsilon}^2 \frac{\partial \check{L}_c}{\partial x}(x) g(x) g(x)^T \frac{\partial^T \check{L}_c}{\partial x}(x) = 0,
\end{aligned}
\tag{52}
$$

with $\check{L}_c(0) = 0$. Now consider the $\mathcal{H}_\infty$ future and the $\mathcal{H}_\infty$ past energy functions Q_γ^+ and Q_γ^- of system (32), then we have the following relation.

Theorem 5.19 *The solutions $\check{L}_o$ and $\check{L}_c$ of (51) and (52) are related to the solutions Q_γ^+ and Q_γ^- of (36) and (38) by $Q_\gamma^+ = \check{L}_o$ and $Q_\gamma^- = \bar{\epsilon}^2(\check{L}_c - \check{L}_o)$.*

Proof Obviously (51) and (36) are the same equations and hence $Q_\gamma^+ = \check{L}_o$. From (52) we obtain

$$\tfrac{\partial \check{L}_c}{\partial x}(x)f(x) + \tfrac{1}{2}\bar{\epsilon}^2 \tfrac{\partial(\check{L}_c - \check{L}_o)}{\partial x}(x)g(x)g(x)^T \tfrac{\partial^T(\check{L}_c - \check{L}_o)}{\partial x}(x) -$$

$$\tfrac{1}{2}\bar{\epsilon}^2 \tfrac{\partial \check{L}_o}{\partial x}(x)g(x)g(x)^T \tfrac{\partial^T \check{L}_o}{\partial x}(x) = 0.$$

If we subtract (51) from this, we obtain

$$\tfrac{\partial(\check{L}_c - \check{L}_o)}{\partial x}(x)f(x) + \tfrac{1}{2}\bar{\epsilon}^2 \tfrac{\partial(\check{L}_c - \check{L}_o)}{\partial x}(x)g(x)g(x)^T \tfrac{\partial^T(\check{L}_c - \check{L}_o)}{\partial x}(x) -$$

$$\tfrac{1}{2}h(x)^T h(x) = 0$$

and with (38) this yields $\check{L}_c(x) - \check{L}_o(x) = \frac{1}{\bar{\epsilon}^2} Q_\gamma^-(x)$. ■

Now, assume that $\check{L}_c(x)$ and $\check{L}_o(x)$ are of a form that is similar to the forms (44) and (45), respectively (see Scherpen and Van der Schaft [28]), i.e.,

$$\check{L}_c(x) = \frac{1}{2}x^T x$$

$$\check{L}_o(x) = \frac{1}{2}x^T \begin{pmatrix} \bar{\tau}_1(x) & & 0 \\ & \ddots & \\ 0 & & \bar{\tau}_n(x) \end{pmatrix}$$

where $\bar{\tau}_1(x), \ldots, \bar{\tau}_n(x)$, for $x \in U$ denote the graph singular values of system (49). Furthermore, denote the transformation that is necessary to transform system (32) into the form of Theorem 5.13 by $z = \beta(x)$, $\beta(0) = 0$, for $z \in Y := \beta(U)$.

Theorem 5.20 *There exist a neighborhood U of 0 such that for all $z \in Y = \beta(U)$*

$$\vartheta_i(z) = \frac{\bar{\tau}_i(\beta^{-1}(z))}{\bar{\epsilon}^2(1 - \bar{\tau}_i(\beta^{-1}(z))}, \qquad i = 1, \ldots, n.$$

Proof See Scherpen and Van der Schaft [28]. ■

Hence $\tau_1(\beta^{-1}(z)) \geq \cdots \geq \tau_n(\beta^{-1}(z))$ is equivalent with $\vartheta_1(z) \geq \cdots \geq \vartheta_n(z)$. Thus model reduction based on these graph singular value functions and model reduction based on the $\mathcal{H}_\infty$ singular value functions results in the same reduced order model for $z \in Y$.

6 Model reduction

Model reduction of linear systems based on the $\mathcal{H}_\infty$ characteristic values is treated in Mustafa and Glover [17, 18], see Section 3. Similarly, model reduction of nonlinear systems based on the $\mathcal{H}_\infty$ singular value functions defined in Section 5, amounts to deleting state components that correspond to small $\mathcal{H}_\infty$ singular value functions.

Consider system (32) after the transformation (48), which brings it into $\mathcal{H}_\infty$ balanced form:

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}) + \bar{g}(\bar{z})u \\ y &= \bar{h}(\bar{z}) \end{aligned} \tag{53}$$

for $\bar{z} \in \bar{W}$. Model reduction can be performed by truncation. If for $k < n$ the $\mathcal{H}_\infty$ singular value functions fulfill $\kappa_k(z) > \kappa_{k+1}(z)$, $z \in W$, then the state components $\bar{z}_1, \ldots, \bar{z}_k$ are more important in terms of the $\mathcal{H}_\infty$ past an $\mathcal{H}_\infty$ future energy than the state components $\bar{z}_{k+1}, \ldots, \bar{z}_n$ and the model is reduced by setting $\bar{z}_{k+1} = \cdots = \bar{z}_n = 0$. Partitioning the system in a corresponding way, with $\bar{z}^1 = (\bar{z}_1, \ldots, \bar{z}_k)$, $\bar{z}^b = (\bar{z}_{k+1}, \ldots, \bar{z}_n)$ and $\bar{z} = (\bar{z}^a, \bar{z}^b)$ we obtain the reduced order system

$$\begin{aligned} \dot{\bar{z}}^a &= \tilde{f}(\bar{z}^a) + \tilde{g}(\bar{z}^a)u \\ y &= \tilde{h}(\bar{z}^a) \end{aligned} \tag{54}$$

with $\tilde{f}(z^a) = \bar{f}(\bar{z}^a, 0)$, $\tilde{g}^a(\bar{z}^a) = \bar{g}(\bar{z}^a, 0)$, and $\tilde{h}(\bar{z}^a) = \bar{h}(\bar{z}^a, 0)$. Assume the $\mathcal{H}_\infty$-past and $\mathcal{H}_\infty$-future energy function of this system are respectively $\tilde{Q}_\gamma^-$ and $\tilde{Q}_\gamma^+$. By the form of $\bar{Q}_\gamma^-$, see (46), and the form of the corresponding Hamilton-Jacobi-Bellman equation (38) we conclude that $\tilde{Q}_\gamma^-(\bar{z}^a) = \bar{Q}_\gamma^-(\bar{z}^a, 0)$, but in general we can not conclude the same for $\tilde{Q}_\gamma^+$.

Theorem 6.1 *If* $\frac{\partial \bar{Q}_\gamma^+}{\partial \bar{z}^b}(\bar{z}^a, 0)\bar{f}_b(\bar{z}^a, 0) = 0$ *and* $\frac{\partial \bar{Q}_\gamma^+}{\partial \bar{z}^b}(\bar{z}^a, 0)\bar{g}_b(\bar{z}^a, 0) = 0$ *then the reduced system (54) is again in* $\mathcal{H}_\infty$ *balanced form, and the future energy function of (54) is* $\tilde{Q}_\gamma^+(\bar{z}^a) = \bar{Q}_\gamma^+(\bar{z}^a, 0)$.

Proof This follows directly from the Hamilton-Jacobi-Bellman equation (36). ■

Remark 6.2 For linear systems the conditions of this theorem are always fulfilled, since $\frac{\partial \bar{Q}_\gamma^+}{\partial \bar{z}^b}(\bar{z}^a, 0) = 0$ by the quadratic form of Q_γ^+. □

Now, we consider the system (22) together with its controller (30). Assume we have transformed (32) into the balanced form (53), then (22) takes the form

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}) + \bar{g}(\bar{z})u + \bar{g}(\bar{z})d \\ y &= \bar{h}(\bar{z}) + v \\ z &= \begin{pmatrix} \bar{h}(\bar{z}) \\ u \end{pmatrix} \end{aligned} \tag{55}$$

for $\bar{z} \in \bar{W}$. Furthermore, assume $\bar{R}(t, \bar{z})$ is the solution to the non-stationary Hamilton-Jacobi equation (31) of system (55). Obviously, if

$$\frac{\partial^2 \bar{R}}{\partial \bar{z}^2}(t, \bar{z}) = \gamma^2 \frac{\partial^2 \bar{Q}_\gamma^-}{\partial \bar{z}^2}(\bar{z})$$

and thus $\frac{\partial^2 \bar{R}}{\partial \bar{z}^2}$ can be computed off-line, and if the conditions of Theorem 6.1 are fulfilled, then reducing the order of the central controller (30) results in the central controller for the reduced order system. We have the following theorem for the nonlinear $\mathcal{H}_\infty$ controller (30):

Theorem 6.3 *If the conditions of Theorem 6.1 are fulfilled and if* $\frac{\partial^2 \bar{R}}{\partial \bar{z}^a \bar{z}^b}(t, \bar{z}^a, 0) = 0$*, then reducing the order of the controller of the form (30) that solves the* $\mathcal{H}_\infty$ *suboptimal control problem of the system (55), results in a reduced order controller that solves the* $\mathcal{H}_\infty$ *suboptimal control problem for the reduced order system and is of the form (30).*

Proof This follows from (30). ■

Remark 6.4 For linear systems the condition $\frac{\partial^2 \bar{R}}{\partial \bar{z}^a \bar{z}^b}(t, \bar{z}^a, 0) = 0$ is always fulfilled. □

Hence, model reduction based on the $\mathcal{H}_\infty$ singular value functions does not always lead to a reduced order model that has the same properties as the full order system. We gave additional sufficient conditions under which some of the properties of the full order model also hold for the reduced order model.

Similarly to HJB balancing (see Scherpen and Van der Schaft [28]) $\mathcal{H}_\infty$ balancing may be a tool for nonlinear (closed-loop) model reduction based on the normalized $\mathcal{H}_\infty$ control problem.

7 Conclusions

We introduced $\mathcal{H}_\infty$ balancing for nonlinear systems. This balancing method is related to the nonlinear $\mathcal{H}_\infty$ control problem, and may be seen as a way to analyze the nonlinear system in terms of the corresponding $\mathcal{H}_\infty$ past and future energy function. The method is basically concerned with the structure of the closed-loop system. We may use $\mathcal{H}_\infty$ balancing as a tool for model reduction of the nonlinear system, as well as for the controller that solves the normalized $\mathcal{H}_\infty$ control problem.

We also investigated the monotonicity (with regard to γ) of the Hamilton-Jacobi equations that appear in this balancing method. From this we obtained the relation with balancing of stable nonlinear systems and HJB balancing (see [24] and [28], respectively).

Model reduction based on $\mathcal{H}_\infty$ balancing for nonlinear systems is only based on the intuitive idea that a pair of energy functions form a measure for the importance of the different state components. No estimate for the error between full order and reduced order models has been obtained in this paper. In the case of other balancing methods for nonlinear systems there also is not such an estimate. These errors need to be studied further in order to obtain a better idea about the accuracy of balancing as a tool for model reduction of nonlinear systems.

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References

[1] J. Ball, J.W. Helton, M. Walker, $\mathcal{H}_\infty$ control for nonlinear systems via output feedback, *IEEE Trans. Aut. Contr.*, AC-38 (1993).

[2] T. Başar, P. Bernhard, *$\mathcal{H}_\infty$-optimal control and related minimax design problems*, (Birkhäuser, Boston, 1990).

[3] J. Carr, *Applications of center manifold theory*, Appl. Math. Sci. Vol. 35, (Springer Verlag, New York, 1981).

[4] J.C. Doyle, K. Glover, P.P. Khargonekar, B.A. Francis, State space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems, *IEEE Trans. Aut. Contr.*, AC-34 (1989) 831-846.

[5] S.T. Glad, Robustness of nonlinear state feedback - a survey, *Automatica*, 23 (1987) 425-435.

[6] K. Glover, All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds, *Int. J. of Contr.*, 39 (1984) 1115-1193.

[7] K. Glover, D. Mustafa, Derivation of the maximum entropy $\mathcal{H}_\infty$-controller and a state-space formula for its entropy, *Int. J. of Contr.*, 50 (1989) 899-916.

[8] O.B. Hijab, *Minimum energy estimation*, Doctoral dissertation, University of California, Berkeley (1980).

[9] A. Isidori, A. Astolfi, Disturbance attenuation and $\mathcal{H}_\infty$ control via measurement feedback in nonlinear systems, *IEEE Trans. Aut. Contr.*, AC-37 (1992) 1283-1293.

[10] E.A. Jonckheere, L.M. Silverman, A new set of invariants for linear systems - Applications to reduced order compensator design, *IEEE Trans. Aut. Contr.*, AC-28 (1983) 953-964.

[11] J.P. LaSalle, Stability theory and invariance principles, in: L. Cesari, J.K. Hale, J.P. LaSalle, Eds., *Dynamical Systems, An International Symposium, Vol.1*, Academic Press New York (1976) 211-222.

[12] E.B. Lee, L. Markus, *Foundations of optimal control theory*, (John Wiley & Sons, New York, 1967).

[13] D.C. McFarlane, K. Glover, *Robust controller design using normalized coprime factor descriptions*, Lect. Notes Contr. Inf. Sci. No. 138 (Springer Verlag, Berlin, 1990).

[14] B.C. Moore, Principal component analysis in linear systems: Controllability, observability and model reduction, *IEEE Trans. Aut. Contr.*, AC-26 (1981) 17-32.

[15] R.E. Mortensen, Maximum likelihood recursive nonlinear filtering, *J. Optimization Theory and Appl.*, 2 (1968) 386-394.

[16] D. Mustafa, $\mathcal{H}_\infty$-characteristic values, *Proc. 28th IEEE Conf. on Dec. and Contr.*, (1989) 1483-1487.

[17] D. Mustafa, K. Glover, *Minimum Entropy $\mathcal{H}_\infty$ Control*, Lect. Notes Contr. Inf. Sci. No. 146, (Springer-Verlag, Berlin, 1990).

[18] D. Mustafa, K. Glover, Controller reduction by $\mathcal{H}_\infty$-balanced truncation, *IEEE Trans. Aut. Contr.*, AC-36 (1991) 668-682.

[19] H. Nijmeijer, A.J. van der Schaft, *Nonlinear Dynamical Control Systems*, (Springer-Verlag, New York, 1990).

[20] Ph. Opdenacker, E.A. Jonckheere, LQG balancing and reduced LQG compensation of symmetric passive systems, *Int. J. Contr.*, 41 (1985) 73-109.

[21] A.J. van der Schaft, L_2-gain analysis of nonlinear systems and nonlinear state feedback $\mathcal{H}_\infty$ control, *IEEE Trans. Aut. Contr.*, AC-37 (1992) 770-784.

[22] A.J. van der Schaft, Nonlinear state space $\mathcal{H}_\infty$ control theory, in: H.L. Trentelman, J.C. Willems, Eds., *Essays on Control: Perspectives in the theory of its applications*, Birkhäuser, Boston (1993) 153-190.

[23] A.J. van der Schaft, Nonlinear systems which have finite-dimensional $\mathcal{H}_\infty$ suboptimal central controllers, *Proc. 32nd IEEE Conf. on Dec. and Contr.* Vol 1 (1993) 202-203.

[24] J.M.A. Scherpen, Balancing for nonlinear systems, *Syst. & Contr. Letters*, 21 (1993) 143-153.

[25] J.M.A. Scherpen, Balancing for nonlinear systems, *Proc. Eur. Contr. Conf. 1993*, Vol.4 (1993) 1838-1843.

[26] J.M.A. Scherpen, Balancing for unstable nonlinear systems, *Proc. 32nd IEEE Conf. on Dec. and Contr.*, Vol 1 (1993) 14-19.

[27] J.M.A. Scherpen, *Balancing for nonlinear systems*, doctoral dissertation, Universtity of Twente (1994).

[28] J.M.A. Scherpen, A.J. van der Schaft, Normalized coprime factorizations and balancing for unstable nonlinear systems, (Memorandum No. 1155, Faculty of Applied Mathematics, University of Twente, 1993), to appear in *Int. J. of Contr.*

[29] A. Vanderbauwhede, Centre manifolds, normal forms and elementary bifurcations, *Dynamics Reported*, Vol 2 (1989) 89-169.

[30] A. Vanderbauwhede, S.A. van Gils, Center manifolds and contractions on a scale of Banach spaces, *J. Funct. Anal.*, Vol 72 (1987) 209-224.

[31] S. Weiland, Balanced representations and approximation of linear systems, *Proc. 28th IEEE Conf. on Dec. and Contr.*, (1989) 1334-1336.

[32] S. Weiland, Balancing general quadratic measures on external signals, in Progress in Systems and Control Theory; Realization and Modelling in System Theory; *Proc. MTNS-89*, Vol 1 (1990) 133-140.

[33] S. Weiland, *Theory of approximation and disturbance attenuation for linear systems*, Doctoral dissertation, University of Groningen (1991).

[34] J.C. Willems, Dissipative dynamical systems, Part I: General theory, *Arch. Rat. Mech. Anal.*, Vol 45 (1972) 321-351.

[35] H.K. Wimmer, Monotonicity of maximal solutions of algebraic Riccati equations, *Syst. & Contr. Letters*, 5 (1985) 317-319.