Balancing For Nonlinear Systems

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Abstract

We present a method of balancing for nonlinear systems which is an extension of balancing for linear systems in the sense that it is based on the input and output energy of a system. It is a local result, but gives 'broader' results then we obtain by just linearizing the system. Furthermore the relation with balancing of the linearization is dealt with. We propose to use the method as a tool for nonlinear model reduction and investigate some of the properties of the reduced system.

Keywords: balancing, nonlinear systems, Hamilton-Jacobi equations, Hankel singular values, model reduction.

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1 Introduction

Balancing for linear systems is a well known subject on which there has been a lot of research in the last decade. It started with a paper of Moore [11] in 1981, where balancing is introduced with the aim of using it as a tool for model reduction. If a linear system is in balanced form the Hankel singular values are a measure for the importance of state components. This means that the influence of the corresponding state component on the output and input energy is measured by a Hankel singular value. If a Hankel singular value is relatively small the influence of the corresponding state component on the output and input energy is respectively low and high, and thus this state component can be deleted in order to obtain a reduced-order model. There also has been research on the optimality of this model reduction method (Glover [3]), the properties of the reduced system (Pernebo and Silverman [16]) and other ways of balancing, i.e. LQG-balancing (Opdenacker and Jonckheere [15]) and $\mathcal{H}_\infty$-balancing (Mustafa and Glover [12, 13]).

In this paper we give a set up of balancing for nonlinear systems. The intuitive idea behind model reduction for linear systems can be extended to nonlinear systems. Again, as in the linear case, the importance of state components can be measured in terms of the input and output energy. Instead of the Hankel singular values we define for nonlinear systems singular value functions, measuring again the importance of a state component. If a singular value function is relatively small then the corresponding state component is not important, and thus can be deleted in the nonlinear model.

In section 2 we give a very brief review on balancing for linear systems. Section 3 contains properties of the input and output energy functions for nonlinear systems. These properties are instrumental in the set up for balancing of nonlinear systems. In section 4 we go into balancing for nonlinear systems and define the singular value functions. We propose a procedure to bring a nonlinear system in balanced form. We also consider the linearized version of this procedure and conclude that it matches with the linear theory. Furthermore we study model reduction based on the concept of balancing in section 5. Finally in section 6 we give some conclusions.

Throughout this paper we will use a fairly standard notation. We denote by x^Tx or $\| x \|^2$ the squared norm of a vector $x(t)$, $x : \mathbb{R} \to \mathbb{R}^n$. We say that $u : (-\infty, 0) \to \mathbb{R}^m$ is in $L_2(-\infty, 0)$ if $\int_{-\infty}^0 \| u(t) \|^2 \, dt < \infty$. By $\frac{\partial L}{\partial x}(x)$ we denote the row-vector of partial derivatives of a differentiable function $L : \mathbb{R}^n \to \mathbb{R}$. Furthermore we denote by $x(t_2) = \varphi(t_2, t_1, x_1, u)$ the solution on time t_2 of the system $\dot{x} = f(x) + g(x)u$ with initial condition $x(t_1) = x_1$ and input $u : [t_1, t_2] \to \mathbb{R}^m$.

2 Review of balancing for linear systems

Consider a linear system:

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{1}$$

where $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. We assume throughout (1) is stable, controllable and observable.

Definition 2.1 The controllability and observability function of a linear system are defined as

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \| u(t) \|^2 \, dt \tag{2}$$

respectively

$$L_o(x_0) = \frac{1}{2}\int_0^\infty \| y(t) \|^2 \, dt, \qquad x(0) = x_0, \qquad u(t) \equiv 0, \qquad 0 \le t < \infty \tag{3}$$

The value of the controllability function at x_0 is the minimum amount of control energy required to reach the state x_0 and the value of the observability function at x_0 is the amount of output energy generated by the state x_0. The following results are well known (cf. [11]):

Theorem 2.2 *For system (1) we have* $L_c(x_0) = \frac{1}{2}x_0^T W^{-1} x_0$ *where* $W = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$ *is the controllability gramian and* $L_o(x_0) = \frac{1}{2}x_0^T M x_0$ *where* $M = \int_0^\infty e^{A^Tt}C^TCe^{At}dt$ *is the observability gramian. Furthermore W and M are the unique positive definite solutions of the following Lyapunov equations:*

$$AW + WA^T = -BB^T \tag{4}$$

$$A^TM + MA = -C^TC \tag{5}$$

Theorem 2.3 *There exists a state space transformation* $x = S\bar{x}$ *for system (1) such that the transformed system*

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \qquad y = \bar{C}\bar{x} \tag{6}$$

is in balanced form, i.e.:

$$\bar{W} = \bar{M} = \Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix} \quad \text{with} \quad \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \tag{7}$$

are the controllability and observability gramian of the transformed system (6), where $\bar{W} = \bar{M} = S^{-1}WS^{-T} = S^TMS$. *Here the* σ_i*'s, i=1,..,n, are the Hankel singular values, i.e. the singular values of the Hankel operator of the system (see [3]).*

For system (6) the controllability and observability function are respectively $\bar{L}_c(\bar{x}_0) = \frac{1}{2}\bar{x}_0^T\Sigma^{-1}\bar{x}_0$ and $\bar{L}_c(\bar{x}_0) = \frac{1}{2}\bar{x}_0^T\Sigma\bar{x}_0$. For small σ_i the amount of control energy required to reach the state $\tilde{x} = (0\dots0\ x_i\ 0\dots0)$ is large while the output energy generated by this state $\tilde{x}$ is small. Hence if $\sigma_k >> \sigma_{k+1}$, the state components x_{k+1} to x_n are not important from this energy point of view and can be removed to reduce the number of state components of the model. As we already stated in the introduction these results as well as the optimality of this model reduction method in terms of optimal approximation and the properties of the reduced system can be found in [3, 16]. Another way to characterize the Hankel singular values can be found in [1, 2].

3 The controllability and observability function of nonlinear systems

Consider a smooth, i.e., C^∞, nonlinear system of the form

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x) \tag{8}$$

where $u = (\, u_1 \ \dots \ u_m \,) \in \mathbb{R}^m$, $y = (\, y_1 \ \dots \ y_p \,) \in \mathbb{R}^p$ and $x = (\, x_1 \ \dots \ x_n \,)$ are local coordinates for a smooth state space manifold denoted by M. Throughout we assume that the system has an equilibrium. Without loss of generality we take this equilibrium in 0, i.e. $f(0) = 0$ and we also take $h(0) = 0$. The controllability and observability function, respectively L_c and L_o, of system (8) are defined in the same way as in section 2 for linear systems. Again the value of the controllability function at x_0 is the minimum amount of control energy required to reach the state x_0 and the value of the observability function at x_0 is the amount of output energy generated by x_0.

Definition 3.1 The controllability and observability function of a nonlinear system are defined as

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \| u(t) \|^2 \, dt \tag{9}$$

respectively

$$L_o(x_0) = \frac{1}{2}\int_0^{\infty} \| y(t) \|^2 \, dt, \qquad x(0) = x_0, \qquad u(t) \equiv 0, \qquad 0 \le t < \infty \tag{10}$$

These functions do not necessarily exist, i.e. are finite. In particular, L_o can be infinite if the system is unstable and if x_0 can not be reached from 0, then by convention $L_c(x_0)$ will be infinite. In this section we assume throughout that L_c and L_o are *finite*. Also, for the rest of this paper we assume L_c and L_o are *smooth* functions of x.

Theorem 3.2 *If 0 is an asymptotically stable equilibrium of $f(x)$ on a neighborhood W of 0, then for all $x \in W$, $L_o(x)$ is the unique smooth solution of*

$$\frac{\partial L_o}{\partial x}(x)f(x) + \frac{1}{2}h^T(x)h(x) = 0, \qquad L_o(0) = 0 \tag{11}$$

under the assumption that (11) has a smooth solution on W. Furthermore for all $x \in W$, $L_c(x)$ is the unique smooth solution of

$$\frac{\partial L_c}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial L_c}{\partial x}(x)g(x)g^T(x)\frac{\partial^T L_c}{\partial x}(x) = 0, \qquad L_c(0) = 0 \tag{12}$$

under the assumption that (12) has a smooth solution $\bar{L}_c$ on W and that 0 is an asymptotically stable equilibrium of $-(f(x) + g(x)g^T(x)\dfrac{\partial^T \bar{L}_c}{\partial x}(x))$ on W.

Proof Assume (11) has on W as smooth solution $\bar{L}_o(x)$. Then $\frac{1}{2}h^T(x)h(x) = -\frac{\partial \bar{L}_o}{\partial x}(x)f(x)$. Since L_o is defined as in definition 3.1 we have:

$$\begin{aligned} L_o(x_0) &= \tfrac{1}{2}\int_0^{\infty} \| y(t) \|^2 \, dt = \tfrac{1}{2}\int_0^{\infty} h^T(x(t))h(x(t))dt = -\int_0^{\infty} \tfrac{\partial \bar{L}_o}{\partial x}(x(t))f(x(t))dt \\ &= -\int_0^{\infty} \tfrac{\partial}{\partial t}\bar{L}_o(x(t))dt = -\bar{L}_o(x(\infty)) + \bar{L}_o(x(0)) = \bar{L}_o(x_0), \qquad \forall x_0 \in W \end{aligned}$$

since $x(0) = x_0$ and $x(\infty) = 0$ by the asymptotic stability of $f(x)$. Hence part 1 is proven. For part 2 we assume (12) has on W a smooth positive definite solution $\bar{L}_c(x)$. Then $\frac{\partial \bar{L}_c}{\partial x}(x)f(x) = -\frac{1}{2}\frac{\partial \bar{L}_c}{\partial x}(x)g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x)$. As in definition 3.1 we consider the inputs u such that $x(0) = x_0 \in W$ and $x(-\infty) = 0$, then

$$\begin{aligned} \tfrac{d}{dt}\bar{L}_c(x) &= \tfrac{\partial \bar{L}_c}{\partial x}(x)\dot{x} = \tfrac{\partial \bar{L}_c}{\partial x}(x)f(x) + \tfrac{\partial \bar{L}_c}{\partial x}(x)g(x)u = -\tfrac{1}{2}\tfrac{\partial \bar{L}_c}{\partial x}(x)g(x)g^T(x)\tfrac{\partial^T \bar{L}_c}{\partial x}(x) + \tfrac{\partial \bar{L}_c}{\partial x}(x)g(x)u \\ &= \tfrac{1}{2}u^Tu - \tfrac{1}{2}u^Tu + \tfrac{\partial \bar{L}_c}{\partial x}(x)g(x)u - \tfrac{1}{2}\tfrac{\partial \bar{L}_c}{\partial x}(x)g(x)g^T(x)\tfrac{\partial^T \bar{L}_c}{\partial x}(x) \\ &= \tfrac{1}{2} \| u \|^2 - \tfrac{1}{2} \| u - g^T(x)\tfrac{\partial^T \bar{L}_c}{\partial x}(x) \|^2, \end{aligned}$$

and thus

$$\begin{aligned} \bar{L}_c(x_0) &= \int_{-\infty}^{0} \tfrac{d}{dt}\bar{L}_c(x(t))dt = \tfrac{1}{2}\int_{-\infty}^{0} \| u(t) \|^2 \, dt - \tfrac{1}{2}\int_{-\infty}^{0} \| u(t) - g^T(x(t))\tfrac{\partial^T \bar{L}_c}{\partial x}(x(t)) \|^2 \, dt \\ &\le \tfrac{1}{2}\int_{-\infty}^{0} \| u(t) \|^2 \, dt, \qquad \forall x_0 \in W \end{aligned}$$

Hence $\bar{L}_c(x_0)$ is a lower bound for $\frac{1}{2}\int_{-\infty}^{0} \| u(t) \|^2 \, dt$. It is clear that for $u = g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x)$, $\bar{L}_c$ is equal to this lower bound. By the asymptotic stability of $-(f(x) + g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ on W this latter input is such that $x(-\infty) = 0$. Therefore for all $x_0 \in W$

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \| u(t) \|^2 \, dt = \bar{L}_c(x_0) \qquad \blacksquare$$

Remark 3.3 Equation (11) is a nonlinear Lyapunov type of equation and equation (12) is a Hamilton-Jacobi equation associated with an optimal control problem. If we take for system (8) a linear system we see that the Lyapunov equation for the observability gramian is given by (11) and that multiplying the Lyapunov equation for the controllability gramian from the left and the right by the inverse controllability gramian results in an equation which is given by (12).

Remark 3.4 If L_o is a solutions of (11), we can conclude from the negative semi-definiteness of $\frac{\partial L_o}{\partial x}(x)f(x)$ that L_o is decreasing along f. Since 0 is an asymptotically stable equilibrium of f, 0 is a minimum for L_o and hence L_o is *non-negative*. Furthermore, if L_c is a solution of (12), we can conclude from the negative semi-definiteness of $\frac{\partial L_c}{\partial x}(x)(-(f(x)+g(x)g(x)^T\frac{\partial^T L_c}{\partial x}(x))$, that L_c is decreasing along $-(f(x)+g(x)g(x)^T\frac{\partial^T L_c}{\partial x}(x))$. Again, since 0 is an asymptotically stable equilibrium of $-(f(x)+g(x)g(x)^T\frac{\partial^T L_c}{\partial x}(x))$, 0 is a minimum for L_c and hence L_c is *non-negative*.

Remark 3.5 If $A=\frac{\partial f}{\partial x}(0)$ is asymptotically stable then locally about 0 (11) and (12) have smooth solutions, see [17].

Remark 3.6 If we replace the condition that (12) has a smooth solution $\bar{L}_c$ on W and that 0 is an asymptotically stable equilibrium of $-(f(x)+g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ by the condition that (12) has a smooth solution $\bar{L}_c$ that is positive definite, then we get the same result, see theorem 3.8.

Definition 3.7 A *Lyapunov function* on W for system (8) is a positive definite function L such that $\frac{\partial L}{\partial x}(x)f(x)\leq 0$ for all $x\in W$.

Lyapunov functions are well known and can be used to show stability properties of a system. In the following we will do this and we will also use LaSalle's invariance principle. See for example [9].

Theorem 3.8 *Assume 0 is an asymptotically stable equilibrium of f on W and (12) has a smooth solution $\bar{L}_c$ on W. Then $\bar{L}_c(x_0)>0$ for $x_0\in W$, $x_0\neq 0$, if and only if 0 is an asymptotically stable equilibrium of $-(f(x)+g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ on W.*

Proof Assume $\bar{L}_c(x_0)>0$ for $x_0\in W$, $x_0\neq 0$. We know that on W

$$\frac{\partial \bar{L}_c}{\partial x}(x)(-(f(x)+g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))=-\frac{1}{2}\frac{\partial \bar{L}_c}{\partial x}(x)g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x)\leq 0$$

Hence $\bar{L}_c$ is a Lyapunov function for $-(f(x)+g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ which therefore is stable on W. To prove asymptotic stability we need to find the maximal invariant set of $-(f(x)+g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ in $V:=\{x|\frac{\partial \bar{L}_c}{\partial x}(x)g(x)=0\}$. This is the same as finding the maximal invariant set of $-f(x)$ in V. By (12) we know that $V=\{x|\frac{\partial \bar{L}_c}{\partial x}(x)f(x)=0\}=\{x|\frac{d}{dt}\bar{L}_c(x(t))=0\}$. Since f is asymptotically stable and $\bar{L}_c$ positive definite on W we conclude from this that the maximal invariant set in V is $\{0\}$ and LaSalle's invariance principle thus implies that $-(f(x)+g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ is asymptotically stable on W.

For the if part of the theorem we use theorem 3.2. This states that $\bar{L}_c=L_c$ on W, where L_c is the controllability function of system (8). Furthermore we know from the proof of theorem 3.2 that the minimum is taken for $u=g^T(x)\frac{\partial^T L_c}{\partial x}(x)$. Hence

$$L_c(x_0)=\frac{1}{2}\int_{-\infty}^{0}\frac{\partial L_c}{\partial x}(x(t))g(x(t))g^T(x(t))\frac{\partial^T L_c}{\partial x}(x(t))dt$$

Let now $x_0 \neq 0$. If $\frac{\partial L_c}{\partial x}(x(t))g(x(t)) = 0$ for $-\infty \leq t \leq 0$ then $u(t) = 0$, for all t, $-\infty \leq t \leq 0$. However, since f is asymptotically stable, we cannot have $x(-\infty) = 0$ and $x(0) = x_0 \neq 0$. Hence we have a contradiction and thus there exists a t, $-\infty \leq t \leq 0$, such that $\frac{\partial L_c}{\partial x}(x(t))g(x(t))g^T(x(t))\frac{\partial^T L_c}{\partial x}(x(t)) > 0$. This implies that $L_c(x_0) > 0, \forall x_0 \in W, x_0 \neq 0$. ∎

For the following definition see e.g. [17].

Definition 3.9 The system (8) is reachable from x_0 if for any $\bar{x} \in M$ there exists a $\bar{t} \geq 0$ and input u such that $\bar{x} = \varphi(\bar{t}, 0, x_0, u)$.
The system (8) is zero-state observable if any trajectory such that $u(t) \equiv 0, y(t) \equiv 0$ implies $x(t) \equiv 0$, i.e., for all $x \in M$, $h(\varphi(t,0,x,0)) = 0, t \geq 0 \Rightarrow \varphi(t,0,x,0) = 0, t \geq 0$.

The following theorems are related to some results in [6] and [17].

Theorem 3.10 *If the system (8) is zero-state observable and (11) has a smooth positive definite solution $\tilde{L}_o$, then the system $\dot{x} = f(x)$ is locally asymptotically stable. If $\tilde{L}_o$ is proper (i.e. for each $c > 0$ the set $\{x \in M | 0 \leq \tilde{L}_o(x) \leq c\}$ is compact), then $\dot{x} = f(x)$ is globally asymptotically stable.*

Proof $\frac{\partial \tilde{L}_o}{\partial x}(x)f(x) = -\frac{1}{2}h^T(x)h(x) \leq 0$ and by the zero-state observability $\frac{\partial \tilde{L}_o}{\partial x}(x)f(x) = 0 \Rightarrow x = 0$ for $\dot{x} = f(x)$. Global asymptotic stability now follows by LaSalle's invariance principle. ∎

Theorem 3.11 *Assume 0 is an asymptotically stable equilibrium of $f(x)$ on a neighborhood W of 0. If the system (8) is zero-state observable and (11) has the smooth solution L_o on W, then $L_o(x_0) > 0$, $\forall x_0 \in W$, $x_0 \neq 0$.*

Proof By theorem 3.2 and definition 3.1 we have $\forall x_0 \in W$

$$L_o(x_0) = \frac{1}{2}\int_0^\infty \| y(t) \|^2 \, dt, \qquad x(0) = x_0$$

Zero-state observability implies that for some $\tau > 0$ we have $h(\varphi(t, 0, x_0, 0)) \neq 0$ for $0 \leq t < \tau$. Hence for $x_0 \neq 0$

$$L_o(x_0) = \frac{1}{2}\int_0^\infty h^T(x(t))h(x(t))dt > 0$$ ∎

4 Balancing for nonlinear systems

In the rest of this paper we consider nonlinear systems of the form (8) with controllability and observability function L_c respectively L_o as given in definition 3.1 and with the following standing assumptions:

1. 0 is an asymptotically stable equilibrium of $f(x)$ on some neighborhood Y of 0
2. $A = \frac{\partial f}{\partial x}(0)$ is asymptotically stable
3. the system is zero-state observable and reachable from 0 on Y
4. L_o exists on Y
5. (11) and (12) have smooth solutions on Y
6. 0 is an asymptotically stable equilibrium of $-(f(x) + g(x)g^T(x)\frac{\partial^T L_c}{\partial x}(x))$ on Y
7. $\frac{\partial^2 L_c}{\partial x^2}(0) > 0$ and $\frac{\partial^2 L_o}{\partial x^2}(0) > 0$

The reachability from 0 implies that L_c exists on Y, i.e. is finite. Assumption 6 can be replaced by $\det\left(\frac{\partial^2 L_c}{\partial x^2}(0)\right) \neq 0$ and $\det\left(\frac{\partial^2 L_o}{\partial x^2}(0)\right) \neq 0$, since we already know that both are non-negative definite matrices. By section 3 we know that these assumptions imply among other things that L_o is the smooth positive definite solution of (11) and L_c the smooth positive definite solution of (12). They also imply that (A, B) is controllable and that (C, A) is observable, where $B = g(0)$ and $C = \frac{\partial h}{\partial x}(0)$.

Lemma 4.1 *There exists a coordinate transformation $x = \phi(\bar{x})$, $\phi(0) = 0$ (defined on a neighborhood of 0), such that $L_c(x)$ in the new coordinates $\bar{x} = \phi^{-1}(x)$ is of the following form:*

$$L_c(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T\bar{x} \tag{13}$$

Furthermore we can write $L_o(x)$ in the new coordinates $\bar{x} = \phi^{-1}(x)$ in the following form:

$$L_o(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T M(\bar{x})\bar{x} \quad \textit{where} \quad M(0) = \frac{\partial^2 L_o}{\partial x^2}(0) \tag{14}$$

with $M(\bar{x})$ a $n \times n$ symmetric matrix with entries which are smooth functions of $\bar{x}$.

Proof Since 0 is a minimum of the observability and controllability function we have beside $L_o(0) = 0$ and $L_c(0) = 0$ that $\frac{\partial L_o}{\partial x}(0) = 0$ and $\frac{\partial L_c}{\partial x}(0) = 0$. Therefore we can apply Morse's Lemma (see lemma 2.2 in [10]) to L_c. In this case it means that there exists local coordinates $\bar{x} = (\,\bar{x}_1 \;\; \ldots \;\; \bar{x}_n\,)$ such that $x = \phi(\bar{x})$, $\phi(0) = 0$ and such that $L_c(x)$ in the new coordinates $\bar{x} = \phi^{-1}(x)$ has the form (13). Finally (14) follows by repeated application of lemma 2.1 from [10]. ∎

Lemma 4.2 *If there exists a neighborhood V of 0 where the number of distinct eigenvalues of $M(\bar{x})$ is constant for $\bar{x} \in V$, then on V the eigenvalues $\lambda_i(\bar{x})$, $i = 1, .., n$, are smooth functions of $\bar{x}$, as well as the associated eigenvectors.*

Proof We can conclude this from Theorem 5.13a in [8]. ∎

Theorem 4.3 *Consider system (8) and assume the condition of lemma 4.2 is fulfilled. On a neighborhood U of zero there exists a coordinate transformation $x = \psi(z)$, $\psi(0) = 0$, such that $L_c(x)$ in the new coordinates $z \in W := \psi^{-1}(U)$ is of the following form:*

$$\tilde{L}_c(z) := L_c(\psi(z)) = \frac{1}{2}z^T z \tag{15}$$

while L_o is for the new coordinates of the following form:

$$\tilde{L}_o(z) := L_o(\psi(z)) = \frac{1}{2}z^T \begin{pmatrix} \tau_1(z) & & 0 \\ & \ddots & \\ 0 & & \tau_n(z) \end{pmatrix} z \tag{16}$$

where $\tau_1(z) \geq \ldots \geq \tau_n(z)$ are smooth functions of z, called the <u>singular value functions</u>.

Proof From lemma 4.1 we know that there exists a transformation $x = \phi(\bar{x})$, $\phi(0) = 0$ such that in the new coordinates L_c and L_o are of the form (13) respectively (14). By lemma 4.2 we know that on V the eigenvalues of $M(\bar{x})$ and the associated eigenvectors are smooth functions of $\bar{x}$. Furthermore we know that $M(0) > 0$ which means that $M(0)$ is diagonalizable. By the smoothness of the eigenvalues and eigenvectors this implies that $M(\bar{x})$ is diagonalizable on V. Indeed, since $M(\bar{x})$ is symmetric, we can write $M(\bar{x}) = T(\bar{x})\Lambda(\bar{x})T^T(\bar{x})$ where

$$\Lambda(\bar{x}) = \begin{pmatrix} \lambda_1(\bar{x}) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(\bar{x}) \end{pmatrix} \quad \textit{with} \quad \lambda_1(\bar{x}) \leq \ldots \leq \lambda_n(\bar{x})$$

$\lambda_i(\bar{x}), i = 1,..,n$, are the eigenvalues of $M(\bar{x})$ and $T(\bar{x})$ is the corresponding matrix of eigenvectors with $T(\bar{x})$ an orthogonal matrix, i.e. $T^T(\bar{x})T(\bar{x}) = I$, $\bar{x} \in V$. Now we can rewrite (14) as:

$$L_o(\phi(\bar{x})) = \tfrac{1}{2}\bar{x}^T T(\bar{x})\Lambda(\bar{x})T^T(\bar{x})\bar{x}, \quad \bar{x} \in V$$

Define a new coordinate transformation $z = \nu(\bar{x}) := T^T(\bar{x})\bar{x}$. In these coordinates we get:

$$L_o(\phi(\nu^{-1}(z))) = \tfrac{1}{2}z^T \Lambda(\nu^{-1}(z))z, \quad z \in W := \nu(V).$$

From (13) and $\bar{x} = T(\bar{x})z$ we get:

$$L_c(\phi(\nu^{-1}(z))) = \frac{1}{2}z^T T^T(\bar{x})T(\bar{x})z = \frac{1}{2}z^T z$$

Define $\tau_i(z) := \lambda_i(\nu^{-1}(z))$, $i = 1,..,n$, $\psi := \phi \circ \nu^{-1}$ and $U := \phi^{-1}(V)$, then the theorem is proven. ■

Remark 4.4 For a linear system the singular value functions τ_i, $i = 1,..,n$ are constants and are the squared Hankel singular values.

The form of the controllability and observability function in (15) and (16) is not yet entirely balanced. For that we need another additional coordinate transformation. We take as transformation $\bar{z}_i = \eta_i(z_i) := \tau_i(0,..,0,z_i,0,..0)^{\frac{1}{4}} z_i$, $i = 1,..,n$ and hence $\bar{z} = \eta(z) := (\,\eta_1(z_1) \quad \ldots \quad \eta_n(z_n)\,)$ on $\bar{z} \in \bar{W} := \eta(W)$. Since $\tilde{L}_o(z) > 0$ we have that $\tau_i(0,..,0,z_i,0,..,0) > 0$, $i = 1,..,n$, for $z \in W$, $z \neq 0$ and therefore η is a well defined transformation. Define $\bar{L}_c(\bar{z}) := \tilde{L}_c(\eta^{-1}(\bar{z}))$ and $\bar{L}_o(\bar{z}) := \tilde{L}_o(\eta^{-1}(\bar{z}))$. Then (15) and (16) become respectively:

$$\bar{L}_c(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} \sigma_1(\bar{z}_1)^{-1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n(\bar{z}_n)^{-1} \end{pmatrix} \bar{z} \tag{17}$$

$$\bar{L}_o(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} \sigma_1(\bar{z}_1)^{-1}\tau_1(\eta^{-1}(\bar{z})) & & 0 \\ & \ddots & \\ 0 & & \sigma_n(\bar{z}_n)^{-1}\tau_n(\eta^{-1}(\bar{z})) \end{pmatrix} \bar{z} \tag{18}$$

where $\sigma_i(\bar{z}_i) = \tau_i(0,..,0,\eta_i^{-1}(\bar{z}_i),0,..,0)^{\frac{1}{2}}$ for $i = 1,..,n$. Now $\bar{L}_c(0,..,0,\bar{z}_i,0,..,0) = \frac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i)^{-1}$ and $\bar{L}_o(0,..,0,\bar{z}_i,0,..,0) = \frac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i)$ for $i = 1,..,n$. This corresponds with the linear theory, since in that case σ_i is constant for $i = 1,..,n$, and thus the σ_i's are the Hankel singular values. We know $\tau_1(\bar{z}) \geq \ldots \geq \tau_n(\bar{z})$ for $\bar{z} \in \bar{W}$. In energy terms we have for $\tau_i(\bar{z}) > \tau_{i+1}(\bar{z})$ that the state variable $\bar{z}_i$ is more important then the state variable $\bar{z}_{i+1}$ on $\bar{W}$.

Similar to the concept of balancing for linear systems we call the nonlinear system *balanced* if it has a controllability and observability function of the form of (17) and (18). This means that we can balance system (8) by a coordinate transformation of the form $x = \chi(\bar{z}) := \psi(\eta^{-1}(\bar{z}))$ where ψ is as in theorem 4.3, which results in:

$$\dot{\bar{z}} = \bar{f}(\bar{z}) + \bar{g}(\bar{z})u, \qquad y = \bar{h}(\bar{z}) \tag{19}$$

Remark 4.5 For $\tau_i(z) > 0$ for all $i = 1,..,n$, $z \in \hat{W} \subseteq W$, we can transform (15) and (16) by the coordinate transformation $\breve{z}_i = \kappa_i(z) := \tau_i(z)^{-\frac{1}{2}}z_i$ for $i = 1,..,n$, $z \in \hat{W}$. In the new coordinates, $\breve{z} = \kappa(z) = (\kappa_1(z),\ldots,\kappa_n(z))$, for $\breve{z} \in \breve{W} := \kappa(\hat{W})$, we have the following controllability and observability function:

$$\breve{L}_c(\breve{z}) := \tilde{L}_c(\kappa^{-1}(\breve{z})) = \frac{1}{2}\breve{z}^T \begin{pmatrix} \tau_1(\kappa^{-1}(\breve{z}))^{-1} & & 0 \\ & \ddots & \\ 0 & & \tau_n(\kappa^{-1}(\breve{z}))^{-1} \end{pmatrix} \breve{z} \tag{20}$$

$$\breve{L}_o(\breve{z}) := \tilde{L}_o(\kappa^{-1}(\breve{z})) = \frac{1}{2}\breve{z}^T \breve{z} \tag{21}$$

Now we can follow a same kind of reasoning as above to get a form which we also could call balanced by the same type of arguments. If we change the role of L_c and L_o in theorem 4.3 then the controllability and observability function in the new coordinates $\breve{z}$, $\breve{z} \in \breve{W}$, are respectively $\breve{L}_c$ and $\breve{L}_o$ of the form (20) and (21).

For linear systems the largest Hankel singular value is equal to the Hankel norm of the system. This norm can be expressed in the notation of section 2 as follows:

$$\| G \|_H^2 = \max_{x\in\mathbb{R}^n} \frac{L_o(x)}{L_c(x)} = \max_{x\in\mathbb{R}^n} \frac{x^T M x}{x^T W^{-1} x} = \max_{\tilde{x}\in\mathbb{R}^n} \frac{\tilde{x}^T \Sigma^2 \tilde{x}}{\tilde{x}^T \tilde{x}} = \sigma_1^2$$

where G is the transfer function of the linear system, see [3]. In fact the Hankel-norm for a linear system gives the L_2-gain from past inputs to future outputs. Similarly for nonlinear systems we can consider

$$\max_{\bar{z}\in\bar{W}} \frac{\bar{L}_o(\bar{z})}{\bar{L}_c(\bar{z})} \tag{22}$$

Define $\tau_i^{max} := \max_{z\in W} \tau_i(z)$, $i = 1,..,n$, then

$$\max_{\bar{z}\in\bar{W}} \frac{\bar{L}_o(\bar{z})}{\bar{L}_c(\bar{z})} = \max_{z\in W} \frac{\tilde{L}_o(z)}{\tilde{L}_c(z)} = \max_{z\in W} \frac{\sum_{i=1}^n \tau_i(z) z_i^2}{\sum_{i=1}^n z_i^2} \leq \max_{z\in W} \frac{\sum_{i=1}^n \tau_i^{max} z_i^2}{\sum_{i=1}^n z_i^2} = \tau_1^{max}$$

This means we only get an upper bound for the L_2-gain from past inputs to future outputs.

Now we consider the linearized version. We linearize system (8) and system (19), which is (8) after a coordinate transformation $x = \chi(\bar{z})$. Linearization of system (8) in $x = 0$ and $u = 0$ gives:

$$\dot{\hat{x}} = A\hat{x} + Bu, \qquad y = C\hat{x} \tag{23}$$

where $A = \frac{\partial f}{\partial x}(0)$, $B = g(0)$, and $C = \frac{\partial h}{\partial x}(0)$. This system has as controllability and observability function respectively $M_c(\hat{x})$ and $M_o(\hat{x})$. Linearize (19) in $\bar{z} = 0$ and $u = 0$, then we get:

$$\dot{\hat{z}} = \bar{A}\hat{z} + \bar{B}u, \qquad y = \bar{C}\hat{z} \tag{24}$$

where $\bar{A} = \frac{\partial \bar{f}}{\partial \bar{z}}(0)$, $\bar{B} = \bar{g}(0)$ and $\bar{C} = \frac{\partial \bar{h}}{\partial \bar{z}}(0)$. This system has as controllability and observability function respectively $\bar{M}_c(\hat{z})$ and $\bar{M}_o(\hat{z})$.

Theorem 4.6 *Define $S := \frac{\partial \chi}{\partial \bar{z}}(0)$, then (23) is transformed into (24) by the coordinate transformation $\hat{x} = S\hat{z}$. Therefore $\bar{A} = S^{-1}AS$, $\bar{B} = S^{-1}B$ and $\bar{C} = CS$. A and $\bar{A}$ are asymptotically stable. Furthermore*

$$M_c(\hat{x}) = \frac{1}{2}\hat{x}^T \frac{\partial^2 L_c}{\partial x^2}(0)\hat{x}, \qquad M_o(\hat{x}) = \frac{1}{2}\hat{x}^T \frac{\partial^2 L_o}{\partial x^2}(0)\hat{x} \tag{25}$$

$$\bar{M}_c(\hat{z}) = \frac{1}{2}\hat{z}^T \frac{\partial^2 \bar{L}_c}{\partial \bar{z}^2}(0)\hat{z}, \qquad \bar{M}_o(\hat{z}) = \frac{1}{2}\hat{z}^T \frac{\partial^2 \bar{L}_o}{\partial \bar{z}^2}(0)\hat{z} \tag{26}$$

with

$$\frac{\partial^2 \bar{L}_c}{\partial \bar{z}^2}(0) = \begin{pmatrix} \sigma_1(0)^{-1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n(0)^{-1} \end{pmatrix} \quad \textit{and} \quad \frac{\partial^2 \bar{L}_o}{\partial \bar{z}^2}(0) = \begin{pmatrix} \sigma_1(0) & & 0 \\ & \ddots & \\ 0 & & \sigma_n(0) \end{pmatrix}$$

The gramians are connected as follows:

$$\frac{\partial^2 \bar{L}_c}{\partial \bar{z}^2}(0) = S^T \frac{\partial^2 L_c}{\partial x^2}(0)S \qquad \frac{\partial^2 \bar{L}_o}{\partial \bar{z}^2}(0) = S^T \frac{\partial^2 L_o}{\partial x^2}(0)S \tag{27}$$

Proof This follows by taking the Taylor series expansions in zero. ■

Hence system (23) can be brought in a balanced form by a linear coordinate transformation which is the linearization of the coordinate transformation that is used to balance the nonlinear system (8). The balanced linear system (24) is the linearization of the balanced nonlinear system (19).

5 Model reduction

Model reduction for linear systems is explained briefly in section 2. If we want to reduce system (8) we can use the same kind of reasoning. Hence if for $k < n$ we have $\tau_k(z) > \tau_{k+1}(z)$, $z \in W$, which is actually the same as $\sigma(\bar{z}_k)^{-1}\tau_k(\eta^{-1}(\bar{z})) > \sigma(\bar{z}_{k+1})^{-1}\tau_{k+1}(\eta^{-1}(\bar{z}))$, $\bar{z} \in \bar{W}$, then the state variable $\bar{z}_k$ is more important in terms of energy than the state variable $\bar{z}_{k+1}$. Hence $\bar{z}_1$ until $\bar{z}_k$ are more important than $\bar{z}_{k+1}$ until $\bar{z}_n$ due to the ordering of the singular value functions. A possibility to reduce the number of states in the system (19) is to put $\bar{z}_{k+1} = \ldots = \bar{z}_n = 0$. We will partition the system in a corresponding way as follows:

$$\bar{f}(\bar{z}) = \begin{pmatrix} \bar{f}_a(\bar{z}^a, \bar{z}^b) \\ \bar{f}_b(\bar{z}^a, \bar{z}^b) \end{pmatrix}, \quad \bar{g}(\bar{z}) = \begin{pmatrix} \bar{g}_a(\bar{z}^a, \bar{z}^b) \\ \bar{g}_b(\bar{z}^a, \bar{z}^b) \end{pmatrix}, \quad \bar{h}(\bar{z}) = \bar{h}(\bar{z}^a, \bar{z}^b)$$

where $\bar{z}^a = (\bar{z}_1, \ldots, \bar{z}_k)$ and $\bar{z}^b = (\bar{z}_{k+1}, \ldots, \bar{z}_n)$. Hence to reduce the system we set $\bar{z}^b = 0$. From (17) it is clear that $\frac{\partial \bar{L}_c}{\partial \bar{z}^b}(\bar{z}^a, 0) = 0$. The Hamilton-Jacobi equations (11) and (12) of system (19) for $\bar{z}^b = 0$, $(\bar{z}^a, 0) \in \bar{W}$, become:

$$\frac{\partial \bar{L}_o}{\partial \bar{z}^a}(\bar{z}^a,0)\bar{f}_a(\bar{z}^a,0) + \frac{\partial \bar{L}_o}{\partial \bar{z}^b}(\bar{z}^a,0)\bar{f}_b(\bar{z}^a,0) + \frac{1}{2}\bar{h}^T(\bar{z}^a,0)\bar{h}(\bar{z}^a,0) = 0 \tag{28}$$

$$\frac{\partial \bar{L}_c}{\partial \bar{z}^a}(\bar{z}^a,0)\bar{f}_a(\bar{z}^a,0) + \frac{1}{2}\frac{\partial \bar{L}_c}{\partial \bar{z}^a}(\bar{z}^a,0)\bar{g}_a(\bar{z}^a,0)\bar{g}_a^T(\bar{z}^a,0)\frac{\partial^T \bar{L}_c}{\partial \bar{z}^a}(\bar{z}^a,0) = 0 \tag{29}$$

The reduced system is the following:

$$\dot{\tilde{z}} = \tilde{f}(\tilde{z}) + \tilde{g}(\tilde{z})u, \qquad \tilde{y} = \tilde{h}(\tilde{z}) \tag{30}$$

where $\tilde{z} = \bar{z}^a$, $\tilde{f}(\tilde{z}) = \bar{f}_a(\bar{z}^a, 0)$, $\tilde{g}(\tilde{z}) = \bar{g}_a(\bar{z}^a, 0)$ and $\tilde{h}(\tilde{z}) = \bar{h}(\bar{z}^a, 0)$, for all $\tilde{z} \in \tilde{W} := \{\tilde{z} | (\tilde{z}, 0) \in \bar{W}\}$. It follows immediately that the controllability function of the reduced system (30), $\tilde{L}_c(\tilde{z})$, is equal to the solution of (29). Hence, $\tilde{L}_c(\tilde{z}) = \bar{L}_c(\bar{z}^a, 0)$. We can not conclude (as in the linear case) that the observability function of the reduced system (30), $\tilde{L}_o(\tilde{z})$, equals the reduced observability function of the original system (19).

Theorem 5.1 *If $\bar{f}_b(\bar{z}^a,0)=0$ or if $\frac{\partial \bar{L}_0}{\partial \bar{z}^b}(\bar{z}^a,0)=0$ for $(\bar{z}^a,0)\in \bar{W}$ then $\tilde{L}_o(\tilde{z})=\bar{L}_o(\bar{z}^a,0)$, the reduced system (30) is in balanced form and has as singular value functions $\tau_1(z^a,0)\geq \ldots \geq \tau_k(z^a,0)$, for $z^a=\eta^{-1}(\bar{z}^a,0)$ and η defined as in section 4.*

Proof For $\bar{f}_b(\bar{z}^a,0)=0$ or $\frac{\partial \bar{L}_0}{\partial \bar{z}^b}(\bar{z}^a,0)=0$ the term $\frac{\partial \bar{L}_o}{\partial \bar{z}^b}(\bar{z}^a,0)\bar{f}_b(\bar{z}^a,0)$ in equation (28) is zero and thus $\tilde{L}_o(\tilde{z})=\bar{L}_o(\bar{z}^a,0)$. Then the reduced system (30) is in balanced form. Since the coordinate transformation $\bar{z}=\eta(z)=(\eta(z_1),\ldots,\eta(z_n))$, defined in section 4, for $\bar{z}^b=0$ becomes $\bar{z}^a=(\eta(z_1),\ldots,\eta(z_k))$ we have that the singular value functions of system (30) are $\tau_1(z^a,0)\geq \ldots \geq \tau_k(z^a,0)$, where $z^a=(z_1,\ldots,z_k)=(\eta_1^{-1}(\bar{z}_1),\ldots,\eta_k^{-1}(\bar{z}_k))$. ∎

Remark 5.2 If we change the role of L_o and L_c as in remark 4.5, then $\tilde{L}_o(\tilde{z})=\breve{L}_o(\breve{z}^a,0)$ and we can not conclude that the controllability function of the reduced system, $\tilde{L}_c$, equals the reduced controllability function of the original system. Here (30) is taken as the representation of the reduced system and $\breve{z}=(\breve{z}^a,\breve{z}^b)^T$ is the partitioning of the state components of the balanced system. In other words: this implies that the roles of L_o and L_c also change in this section.

For the asymptotic stability of the reduced system we can use the first method of Lyapunov, i.e. linearization. Therefore we partition the balanced linearized system (24) of section 5 as we partitioned the nonlinear balanced system above. Hence:

$$\bar{A}=\begin{pmatrix}\bar{A}_{11} & \bar{A}_{12}\\ \bar{A}_{21} & \bar{A}_{22}\end{pmatrix},\qquad \bar{B}=\begin{pmatrix}\bar{B}_1\\ \bar{B}_2\end{pmatrix},\qquad \bar{C}=\begin{pmatrix}\bar{C}_1 & \bar{C}_2\end{pmatrix}$$

where $\bar{A}_{ij}=\frac{\partial \bar{f}_i}{\partial \bar{z}^j}(0)$, $\bar{B}_i=\bar{g}_i(0)$ and $\bar{C}_j=\frac{\partial \bar{h}}{\partial \bar{z}^j}(0)$, $i,j=a,b$. We already assumed the asymptotic stability of A. Therefore also $\bar{A}$ is asymptotically stable.

Theorem 5.3 *The subsystems $(\bar{f}_a(\bar{z}^a,0),\bar{g}_a(\bar{z}^a,0),\bar{h}(\bar{z}^a,0))$ and $(\bar{f}_b(0,\bar{z}^b),\bar{g}_b(0,\bar{z}^b),\bar{h}(0,\bar{z}^b))$ are locally asymptotically stable.*

Proof From theorem 3.2 in [16] we know that both linearized subsystems $(\bar{A}_{ii},\bar{B}_i,\bar{C}_i)$, $i=1,2$, are asymptotically stable. Since $\bar{A}_{11}=\frac{\partial \bar{f}_a}{\partial \bar{z}^a}(0,0)=\frac{\partial \tilde{f}}{\partial \tilde{z}}(0)$ and $\bar{A}_{22}=\frac{\partial \bar{f}_b}{\partial \bar{z}^b}(0,0)$ we conclude that the subsystems are locally asymptotically stable. ∎

Note that the subsystem $(\bar{f}_a(\bar{z}^a,0),\bar{g}_a(\bar{z}^a,0),\bar{h}(\bar{z}^a,0))$ is the reduced system (30) and therefore (30) is locally asymptotically stable. With regard to global asymptotic stability we can state the following:

Theorem 5.4 *If $\bar{f}_b(\bar{z}^a,0)=0$ and $\tilde{L}_o$ is proper on W, the reduced system (30) is asymptotically stable on $\tilde{W}$.*

Proof We know system (19) is zero-state observable. This implies that for all $(\bar{z}^a,0)\in \bar{W}$, $\bar{h}(\varphi(t,0,(\bar{z}^a,0),0))=0$, $t\geq 0$, $\Rightarrow$ $\varphi(t,0,(\bar{z}^a,0),0)=0$, $t\geq 0$. Since $\bar{f}_b(\bar{z}^a,0)=0$ this implies that also the reduced system (30) is zero-state observable. Furthermore by theorem 5.1 we know that the singular value functions of the reduced system are the first k singular value functions of the original system for $\bar{z}^b=0$. Hence

$$\tilde{L}_o(\tilde{z})=\frac{1}{2}\tilde{z}^T\begin{pmatrix}\sigma_1(\tilde{z}_1)^{-1}\tau_1(\eta^{-1}(\tilde{z},0)) & & 0\\ & \ddots & \\ 0 & & \sigma_k(\tilde{z}_k)^{-1}\tau_k(\eta^{-1}(\tilde{z},0))\end{pmatrix}\tilde{z}.$$

Therefore we know that $\tilde{L}_o(\tilde{z})>0$ for $\tilde{z}\in\tilde{W},\tilde{z}\neq 0$. Now we can apply theorem 3.10 and state that the reduced system (30) is asymptotically stable on $\tilde{W}$. ∎

6 Conclusions

We introduced balancing for stable nonlinear systems. The method is an extension of balancing for stable linear systems, since we considered the input and output energy function of a stable nonlinear system in a similar way as we do this for stable linear systems. The properties of the input and output energy functions for a nonlinear system match with the properties of these functions for a linear system. Therefore we used these functions to balance the system about the equilibrium and applied model reduction. In general the nonlinear reduced system will not be balanced again, but we gave some sufficient conditions for which this holds. It is not clear yet how we should interpret the nonlinear reduced system if these conditions are not fulfilled. A nice property of the proposed method is that the linearized version of it gives exactly the method of balancing for linear systems applied to the linearized system.

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References

[1] R. Courant, D. Hilbert, *Methods of Mathematical Physics, Vol. 1* (Interscience Publishers, New York, 1953).

[2] F.R. Gantmacher, *The Theory of Matrices* (New York: Chelsea, 1960).

[3] K. Glover, All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds, *Int. J. of Contr.* 39 (1984) 1115-1193.

[4] K. Glover, J.C. Doyle, A state space approach to $\mathcal{H}_\infty$ optimal control, in: H. Nijmeijer, J.M. Schumacher, Eds., *Three Decades of Mathematical System Theory*, Lect. Notes Contr. Inf. Sci. No. 135 (Springer-Verlag, Berlin, 1989) 179-218.

[5] P. Heuberger, A family of reduced order models, based on open-loop balancing, *Ident. Model. and Contr.*, Delft University Press, 1, 1-10, 1990.

[6] D. Hill, P. Moylan, The stability of nonlinear dissipative systems, *IEEE Trans. Aut. Contr.* AC-21 (1976) 708-711.

[7] E.A. Jonckheere, L.M. Silverman, Singular value analysis of deformable systems, *J. of Circ., Syst. and Sign. Proc.* 1 (1982) 447-470.

[8] T. Kato, *A Short Introduction to Perturbation Theory for Linear Operators*, (Springer-Verlag, New York, 1982).

[9] J.P. LaSalle, Stability theory and invariance principles, in: L. Cesari, J.K. Hale, J.P. LaSalle, Eds., *Dynamical Systems, An International Symposium, Vol.1*, Academic Press New York (1976) 211-222.

[10] J. Milnor, *Morse Theory*, Annals of Math. Stud. 51 (Princeton University Press, 1963).

[11] B.C. Moore, Principle component analysis in linear systems: Controllability, observability and model reduction, *IEEE Trans. Aut. Contr.*, AC-26, (1981) 17-32.

[12] D. Mustafa, K. Glover, *Minimum Entropy $\mathcal{H}_\infty$ Control*, Lect. Notes Contr. Inf. Sci. No. 146 (Springer-Verlag, Berlin, 1990).

[13] D. Mustafa, K. Glover, Controller reduction by $\mathcal{H}_\infty$-balanced truncation, *IEEE Trans. Aut. Contr.*, AC-36 (1991) 668-682.

[14] H. Nijmeijer, A.J. van der Schaft, *Nonlinear Dynamical Control Systems* (Springer-Verlag, New York, 1990).

[15] Ph. Opdenacker, E.A. Jonckheere, LQG balancing and reduced LQG compensation of symmetric passive systems, *Int. J. Contr.*, 41 (1985) 73-109.

[16] L. Pernebo, L.M. Silverman, Model reduction via balanced state space representations, *IEEE Trans. Aut. Contr.*, AC-27 (1982) 382-387.

[17] A.J. van der Schaft, L_2-gain analysis of nonlinear systems and nonlinear state feedback $\mathcal{H}_\infty$ control, *IEEE Trans. Aut. Contr.*, AC-37 (1992) 770-784.