

Minimality and Local State Decompositions of a Nonlinear State Space Realization using Energy Functions*

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Abstract

In this paper a set of sufficient conditions is developed in terms of controllability and observability functions under which a given state space realization of a formal power series is minimal. Specifically, it is shown that positivity of these functions, in addition to a stability requirement and a few technical conditions, implies minimality. In doing so, connections are established between Hamilton-Jacobi type optimal control theory and the well known necessary and sufficient conditions for minimality in terms of Kalman type rank conditions on the accessibility and observability distributions. Using the nonlinear analogue of the Kalman decomposition, connections are then established between minimality, singular value functions, balanced realizations, and various notions of reachability and observability for nonlinear systems.

Keywords: minimal realizations, controllability and observability energy functions, formal power series, nonlinear systems

1 Introduction

The problem of determining when the dimension of a state space realization for a given input-output map is minimal is a fundamental problem in systems. It connects to many other topics in realization theory like controllability and observability properties, similarity invariants, balanced realizations and model reduction. The theory is quite complete in the case of linear systems. For example, it is well known that minimality is equivalent to joint controllability and observability, and for stable systems, this is further equivalent to the positive definiteness of the controllability and observability Gramians. These Gramian matrices naturally appear in balanced realization theory and optimal control problems. In the nonlinear setting, minimality theory is not nearly as well developed. For example, there are several existing theories for minimality depending on the exact nature in which the input-output mapping is described, i.e., in terms of a set of input-output differential equations (see [20] and the references therein), a Volterra series [6, 11, 12] or a formal power series/Chen-Fliess functional expansion [6]. At present, the exact connections between

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these different approaches are not completely understood. Furthermore, motivated by the linear case, we might expect that minimality should have connections to the nonlinear extensions of the Gramians, which have been developed for nonlinear balancing [3, 4, 14, 15, 16]. But these connections are also largely unknown at present.

The primary purpose of this paper is to develop a set of sufficient conditions in terms of controllability and observability functions under which a given state space realization of a formal power series is minimal. Specifically, it will be shown that positivity of these so called *energy functions*, plus a few technical conditions, implies minimality. Of course there exists well known necessary and sufficient conditions for minimality in terms of Kalman type rank conditions on the accessibility and observability distributions [6, 11]. So the novelty of the approach taken here is in establishing a connection between these differential geometric type minimality conditions and properties of energy functions, which are connected with Hamilton-Jacobi type optimal control theory. Then, using the nonlinear analogue of the Kalman decomposition, we establish connections between minimality, singular value functions and the various notions of reachability and observability for nonlinear systems which preliminary appeared in [16].

The paper is organized as follows. In Section 2, the background material pertaining to all the relevant subjects is briefly reviewed, specifically: the definitions and known properties of controllability and observability functions, balanced realizations for nonlinear systems, and minimality theory for state space realizations of formal power series. In Section 3 we then develop relationships between positivity of the energy functions and the accessibility/observability rank conditions. This section concludes with the main result of the paper involving minimality. Then in Section 4 we introduce as an application of the new minimality results, the decomposition material. Section 5 concludes with two examples: an academic example that is meant to demonstrate the main ideas of the paper, and an example with more physical relevance.

Notation

The mathematical notation used throughout is fairly standard. Vector norms are represented by $\|x\| = \sqrt{x^T x}$ for $x \in \mathbb{R}^n$. $L_2(a, b)$ represents the set of Lebesgue measurable functions, possibly vector-valued, with finite L_2 norm $\|x\|_{L_2} = \sqrt{\int_a^b \|x(t)\|^2 dt}$. If $L : \mathbb{R}^n \mapsto \mathbb{R}$ is a differentiable function, then its partial derivative $\frac{\partial L}{\partial x}$ will be the row vector of partial derivatives $\frac{\partial L}{\partial x_i}$ where $i = 1, \dots, n$. Furthermore, $x(t_2) = \phi(t_2, t_1, x_1, u)$ denotes the solution at time t_2 of the system $\dot{x} = f(x) + g(x)u$ with initial condition $x(t_1) = x_1$ and input $u : [t_1, t_2] \rightarrow \mathbb{R}^m$. A condition about 0 means that this conditions holds for a neighborhood of 0. Finally, $x(-\infty)$ is an abbreviation for $\lim_{t \rightarrow -\infty} x(t)$.

2 Background

2.1 Controllability and observability functions for stable nonlinear systems

Controllability and observability functions play an important role in balancing and model reduction for stable nonlinear systems [14]. In this section we give a brief review of the results that are important for the minimality theory presented in Section 3.

Consider a smooth, i.e., C^∞ , nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

where $u = (u_1, \dots, u_m) \in \mathbb{R}^m$, $y = (y_1, \dots, y_p) \in \mathbb{R}^p$ and $x = (x_1, \dots, x_n)$ are local coordinates for a smooth state space manifold denoted by M . Throughout we assume that the system has an equilibrium. Without loss of generality we take this equilibrium to be at 0, i.e. $f(0) = 0$, and we also take $h(0) = 0$.

Definition 2.1 [14] The *controllability* and *observability function* of a system (1) are defined as

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty) = 0, x(0) = x_0}} \frac{1}{2} \int_{-\infty}^0 \|u(t)\|^2 dt \tag{2}$$

$$L_o(x_0) = \frac{1}{2} \int_0^\infty \|y(t)\|^2 dt, \quad x(0) = x_0, \quad u(t) \equiv 0, \quad 0 \leq t < \infty, \quad (3)$$

respectively. \square

The value of the controllability function at x_0 is the minimum amount of control energy required to reach the state x_0 , and the value of the observability function at x_0 is the amount of output energy generated by x_0 . Obviously, $L_c(x)$ and $L_o(x)$ are non-negative.

It is assumed throughout that L_c and L_o are *finite* and *smooth* functions of x .

Theorem 2.2 [14] *If $f(x)$ is asymptotically stable on a neighborhood W of 0, then for all $x \in W$, $L_o(x)$ is the unique smooth solution of the following Lyapunov type equation:*

$$\frac{\partial L_o}{\partial x}(x)f(x) + \frac{1}{2}h^T(x)h(x) = 0, \quad L_o(0) = 0. \quad (4)$$

Furthermore for all $x \in W$, $L_c(x)$ is the unique smooth solution of the following Hamilton-Jacobi equation:

$$\frac{\partial L_c}{\partial x}(x)f(x) + \frac{1}{2} \frac{\partial L_c}{\partial x}(x)g(x)g^T(x) \frac{\partial^T L_c}{\partial x}(x) = 0, \quad L_c(0) = 0 \quad (5)$$

with $-(f(x) + g(x)g^T(x) \frac{\partial^T L_c}{\partial x}(x))$ asymptotically stable on W . \square

Remark 2.3 If we assume that $f(x)$ is asymptotically stable and that (4) has a smooth solution, it then follows that L_o , as in (3), exists, i.e., is finite, [14]. See [1] for more results about the existence and the continuity of L_o (in [1] L_o also plays an important role in the context of stability and invariance). Furthermore, if we assume that (5) has a smooth solution L_c that is anti-stabilizing (i.e., $-(f(x) + g(x)g^T(x) \frac{\partial^T L_c}{\partial x}(x))$ is asymptotically stable), it follows that L_c , as in (2), exists [14]. \square

Theorem 2.4 [14] *Assume $f(x)$ is asymptotically stable on a neighborhood W of 0 and (5) has a smooth solution \bar{L}_c on W . Then $\bar{L}_c(x) > 0$ for $x \in W$, $x \neq 0$, if and only if $-(f(x) + g(x)g^T(x) \frac{\partial^T \bar{L}_c}{\partial x}(x))$ is asymptotically stable on W .* \square

For the analysis in this paper the definitions of local *reachability*, (*strong*) *accessibility*, and *observability* are needed. We refer to standard references like [5, 6, 11, 13]. These definitions are usually given in the context where only piecewise constant inputs are admissible. However, the effects of approximations of more general inputs by piecewise constant inputs has been considered in earlier work [18], and statements about these properties holding for larger classes of inputs can be found in [17, 19]. For clarity we give the definition of a special case of observability, though also well-known, it is less standard.

Definition 2.5 Consider the system (1).

- The system is *zero-state observable* if any trajectory where $u(t) \equiv 0, y(t) \equiv 0$ implies $x(t) \equiv 0$, i.e., for all $x \in M$, $h(\varphi(t, 0, x, 0)) = 0, t \geq 0 \Rightarrow \varphi(t, 0, x, 0) = 0, t \geq 0$.
- The system (1) is *locally zero-state observable*, if there exists a neighborhood W of 0 such that for all $x \in W$, $h(\varphi(t, 0, x, 0)) = 0$, for all $t \geq 0 \Rightarrow \varphi(t, 0, x, 0) = 0$ for all $t \geq 0$. \square

The following theorem is closely related to results that appear in [5, 13]. It reveals an important relationship between zero-state observability and positive definiteness of the observability function.

Theorem 2.6 [14] *Assume $f(x)$ is asymptotically stable on a neighborhood W of 0. If the system (1) is zero-state observable on W , then $L_o(x) > 0, \forall x \in W, x \neq 0$.* \square

For local observability one can give well known sufficient conditions in terms of Lie derivatives. Furthermore, the local accessibility and local strong accessibility properties can be characterized by well-known conditions in terms of Lie brackets (e.g., [11]). The only extension needed here is the definition of the zero-observability space.

Definition 2.7 Consider the system (1).

- The *zero-observation space* \mathcal{O}_0 is the linear space of functions on M containing h_1, \dots, h_p and all repeated Lie derivatives $L_f^k h_j$, $j \in 1, \dots, p$, $k = 1, 2, \dots$
- The *zero-observability co-distribution* $d\mathcal{O}_0$ is given by $d\mathcal{O}_0(q) = \text{span}\{dH(q) \mid H \in \mathcal{O}_0\}$, where $q \in M$.
□

It is well-known that for the accessibility distribution, C , the strong accessibility distribution, C_0 , and the observation space, \mathcal{O} , there exist rank conditions implying (strong) accessibility and observability. For zero-observability a similar rank condition exists. As a consequence, local zero-state observability implies local observability at 0. Furthermore, it follows that local strong accessibility at x_0 implies local accessibility at x_0 .

2.2 A more general observability function for stable systems

Local zero-state observability is certainly more restrictive than local observability. The previous results in a more general observability setting require the input to play a role. Given the system (f, g, h) , the corresponding homogeneous system is denoted by (f, g_h, h) , where $g_h(x) = g(x) - g(0)$. Thus, under our general assumptions $(f(0), g_h(0), h(0)) = (0, 0, 0)$. It is easily shown that (f, g, h) and its homogeneous counterpart always have the same observability spaces, and thus have basically the same observability properties. Consider the following definition.

Definition 2.8 [3, 4] The *natural observability function* for the system (1) is defined as

$$L_o^N(x_0) = \max_{\substack{u \in B_\alpha \\ x(0)=x_0, x(\infty)=0}} \frac{1}{2} \int_0^\infty \|\tilde{y}(t)\|^2 dt, \quad (6)$$

where $B_\alpha := \{u \in L_2[0, \infty) : \|u\|_{L_2} \leq \alpha\}$ with $\alpha \geq 0$ a fixed real number, and \tilde{y} is the output response of the corresponding homogeneous system. □

Clearly $L_o^N(x_0)$ is the maximum output energy one could expect from initializing the homogeneous system at $x(0) = x_0$ and applying any input with energy bounded by α . When $\alpha = 0$, we have the observability function given in Definition 2.1. A defining equation for L_o^N analogous to equation (4) exists. We refer to [3, 4] for the details. A smooth solution to this equation implies the existence of L_o^N , and as is also the case for L_o , the converse can be stated. The following theorem reviews the relation between observability and positivity of L_o^N .

Theorem 2.9 [3, 4] *Suppose 0 is an asymptotically stable equilibrium of the system (f, g, h) on a neighborhood W of 0 and $h(0) = 0$. If the system (f, g, h) is observable with respect to B_α then $L_o^N(x) > 0$ when $x \in W$, $x \neq 0$.* □

2.3 Balanced realizations

Balanced realizations play an important role in a variety of realization and control problems. The classic linear case was first introduced by Moore in [10]. The extension to the nonlinear case appears in [14, 15].

Consider a nonlinear system of the form (1) with smooth and well-defined controllability and observability function L_c and L_o , respectively, as in Definition 2.1. Additionally, assume

1. $f(x)$ is asymptotically stable on some neighborhood Y of 0.
2. the system is zero-state observable on Y .
3. $\frac{\partial^2 L_c}{\partial x^2}(0) > 0$ and $\frac{\partial^2 L_o}{\partial x^2}(0) > 0$.

From Morse's Lemma, e.g. [8], one can bring the system into input normal form. Furthermore, by applying the Fundamental Theorem of Integral Calculus and smoothness results from [7], the following input-normal/output-diagonal results are obtained.

Theorem 2.10 [14] Consider system (1) with certain technical conditions (see [7, 8]). Then there exists a neighborhood U of 0 and a coordinate transformation $x = \psi(z)$, $\psi(0) = 0$, such that in the new coordinates $z \in W := \psi^{-1}(U)$ the functions L_c , and L_o are of the form

$$\check{L}_c(z) := L_c(\psi(z)) = \frac{1}{2}z^T z, \quad \check{L}_o(z) := L_o(\psi(z)) = \frac{1}{2}z^T \begin{pmatrix} \tau_1(z) & & 0 \\ & \ddots & \\ 0 & & \tau_n(z) \end{pmatrix} z,$$

where $\tau_1(z) \geq \dots \geq \tau_n(z)$ are the smooth singular value functions. \square

Remark 2.11 For a linear system the singular value functions τ_i , $i = 1, \dots, n$ are constant and equal to the squared Hankel singular values. \square

The form of the controllability and observability function is not yet entirely balanced. For that we need a simple additional coordinate transformation. We refer to [14] for the details on this matter.

2.4 Minimal realizations via formal power series

In this section we briefly review a theory of minimal state space realizations for input-output systems that can be represented by a formal power series (Chen-Fliess functional expansion). A detailed treatment may be found in [6]. Ultimately this leads to the well-known rank conditions, which are necessary and sufficient conditions for a realization to be minimal.

Let S be a given input-output map represented by a convergent generating series

$$S : u \rightarrow y(t) = \sum_{\eta \in I^*} c(\eta) E_\eta[u](t, t_0), \quad (7)$$

where I^* is the set of multi-indices for the index set $I = \{0, 1, \dots, m\}$, $c(\eta) \in \mathbb{R}^p$, and

$$E_{i_k \dots i_0}[u](t, t_0) = \int_{t_0}^t u_{i_k}(\tau) E_{i_{k-1} \dots i_0}[u](\tau, t_0) d\tau \quad (8)$$

for $t \in [t_0, T]$ with $E_\emptyset(t, t_0)[u] := 1$ and $u_0(t) := 1$. The mapping S can then also be represented by a formal power series in noncommuting monomials $\mathcal{Z} = \{z_0, z_1, \dots, z_m\}$ via $c = \sum_{\eta \in I^*} c(\eta) z_\eta$, where $z_\eta = z_{i_k} \dots z_{i_0}$ when $\eta = (i_k \dots i_0)$. Now define the sets:

$\mathbb{R} \langle \mathcal{Z} \rangle$: the set of polynomials in \mathcal{Z} over \mathbb{R} ;

$\mathbb{R}^p \ll \mathcal{Z} \gg$: the set of formal power series in \mathcal{Z} over \mathbb{R}^p .

The (block) Hankel mapping associated with c is defined as the \mathbb{R} -vector space morphism $\mathcal{H} : \mathbb{R} \langle \mathcal{Z} \rangle \rightarrow \mathbb{R}^p \ll \mathcal{Z} \gg$, uniquely specified by the generalized shifting property $[\mathcal{H}(z_\zeta)](\eta) = c(\eta \zeta)$, where $\eta, \zeta \in I^*$. In this context we have the following definition.

Definition 2.12 The *Lie rank* of a formal power series c is defined as $\rho_L(c) := \dim(\mathcal{H}(\mathcal{L}(\mathcal{Z})))$, where $\mathcal{L}(\mathcal{Z})$ denotes the smallest Lie algebra containing \mathcal{Z} . \square

An analytic state space realization (f, g, h) defined locally about x_0 is said to *realize* a formal power series c if

$$c(i_k \dots i_0) = L_{X_{i_0}} L_{X_{i_1}} \dots L_{X_{i_k}} h(x_0) \quad (9)$$

for every $(i_k \dots i_0) \in I^*$, where $X_i \in \{f, g_1, \dots, g_m\}$. It is well known that if a certain growth condition on the coefficients $\{c(\eta)\}_{\eta \in I^*}$ is satisfied, then there exists a realization of c if and only if the Lie rank of c is finite. A realization (f, g, h) about x_0 of a formal power series c is *minimal* if its dimension is less than or equal to the dimension of any other realization of c . The following results characterize minimality.

Theorem 2.13 An analytic realization (f, g, h) about x_0 of a formal power series c is minimal if and only if its dimension is equal to the Lie rank $\rho_L(c)$. \square

Theorem 2.14 An analytic realization (f, g, h) about x_0 of a formal power series c is minimal if and only if $\dim C(x_0) = n$ and $\dim dO(x_0) = n$. \square

3 Minimality and energy functions

3.1 The controllability function and the accessibility rank condition

In this section we develop connections between the controllability function and the accessibility rank condition in order to apply Theorem 2.14. It is assumed throughout that the system (1) is asymptotically stable on a neighborhood Y of 0.

The following relation is easily deduced (following the lines of the proof of Theorem 13 in [13]).

$$L_c(x_0) = L_r(x_0) := \inf_{\substack{u \in L_2(-\bar{t}, 0) \\ \bar{t} \geq 0 \\ x(-\infty) = 0, x(0) = x_0}} \frac{1}{2} \int_{-\bar{t}}^0 \|u(t)\|^2 dt, \quad (10)$$

and thus reachability from x_0 implies well-definedness of L_r for all $x \in M$, and likewise for L_c . However, reachability is not implied from a well-defined and positive definite L_c . For our application it is sufficient (as observed from Theorem 2.4) to consider only the anti-stabilizability of the solution of the Hamilton-Jacobi equation (5), which is a condition that can be seen as reachability from 0 in infinite time (so called asymptotic reachability from 0). This notion is formally defined below.

Definition 3.1 A system (1) is said to be *asymptotically reachable from x_0* on a neighborhood W of x_0 if $\forall x \in W$ there exists a $u \in L_2(0, \infty)$ such that $\varphi(\tau, 0, x_0, u) \in W$ for $\tau \geq 0$, and $\lim_{t \rightarrow \infty} \varphi(t, 0, x_0, u) = x$. A system (1) is said to be *locally asymptotically reachable from x_0* if there exists a neighborhood W of x_0 such that the system is asymptotically reachable from x_0 on every neighborhood $V \subset W$ of x_0 . \square

Clearly, this notion of asymptotic reachability corresponds to the notion of anti-stabilizability, which is related to the positivity and finiteness of L_c in Remark 2.3 and Theorem 2.4. In the following theorem, we obtain the relation between local asymptotic reachability from x_0 and local accessibility from x_0 .

Theorem 3.2 *Assume that the accessibility distribution C has constant dimension about x_0 . Then local asymptotic reachability from x_0 implies that the system is locally accessible from x_0 .*

Proof Suppose that the system is not locally accessible from x_0 , then we know from standard results in the literature (e.g., [11]) that $\dim C(x_0) = k < n$. Hence from Proposition 3.12 in [11] there must exist a neighborhood V of x_0 and local coordinates x_1, \dots, x_n such that the sub-manifold $S_{x_0} = \{q \in V | x_i(q) = x_i(x_0), i = k+1, \dots, n\}$ contains $R_T^{\bar{V}}(x_0)$ for any neighborhood $\bar{V} \subset V$ of x_0 and for all $T > 0$. This implies that all $q \in V$ such that $q \notin S_{x_0}$ are not asymptotically reachable from x_0 on V , and thus the local asymptotic reachability from x_0 is contradicted. \blacksquare

Our main aim now is to relate the positive definiteness and finiteness of the controllability function to the accessibility rank condition. Note that having $L_c(x)$ finite on W implicitly implies that $\varphi(\tau, -\infty, 0, u) \in W$ for all $\tau \leq 0$. This, combined with Remark 2.3 and Theorem 2.4 gives rise to the following corollary:

Corollary 3.3 *Assume that the accessibility distribution C has constant dimension about 0, and assume that f is locally asymptotically stable. If there exists a neighborhood W of 0 such that the controllability function $L_c(x)$ is smooth, finite and satisfies $L_c(x) > 0$ for $x \in V$, $x \neq 0$, for all $V \subset W$, then $\dim C(0) = n$. \square*

Remark 3.4 The above corollary is restricted by local requirements on L_c , since we need local asymptotic reachability from 0 in order to use Theorem 3.2. Only asymptotic reachability on a neighborhood W of 0 does not suffice. An example of a smooth system that is asymptotically reachable on a neighborhood W of 0 and that is not locally accessible is easy to construct. However, if we assume that the system (1) is *analytic*, then we can relax the local requirements on L_c to requirements on a neighborhood W of 0. This is due to the fact that asymptotic reachability from x_0 implies local accessibility from x_0 for analytic systems, e.g., [17]. Analyticity is actually not a strong restriction in our setting, since it is also a standing assumption for the realization theory in Section 2.4. \square

The analysis in Remark 3.4 results in the following corollary:

Corollary 3.5 *Let the system (1) be analytic. Assume that the accessibility distribution C has constant dimension about 0, and assume that f is asymptotically stable on a neighborhood W of 0. If the controllability function $L_c(x)$ is smooth, finite and satisfies $L_c(x) > 0$ for $x \in W$, $x \neq 0$, then $\dim C(0) = n$. \square*

So far, the focus has been on the concept of local accessibility. However, for the state space analysis presented in Section 4, we use the nonlinear counterpart of the Kalman decomposition, and thus we need to use the concept of local strong accessibility (see Definition 2.5). The local strong accessibility version of Theorem 3.2 is given below.

Theorem 3.6 *Assume that the strong accessibility distribution C_0 has constant dimension about x_0 . Then local asymptotic reachability from x_0 implies that the system is locally strongly accessible from x_0 .*

Proof Suppose that the system is not locally strongly accessible from x_0 , then we know from standard results in the literature (e.g., [11]) that $\dim C_0(x_0) = k < n$. Hence from Proposition 3.22 in [11] there are two possibilities:

- (i) If $f(x_0) \in C_0(x_0)$, then the proof here follows similar to the proof of Theorem 3.2.
- (ii) If $f(x_0) \notin C_0(x_0)$, then by continuity $f(q) \notin C_0(q)$ for all $q \in \tilde{U}$, $\tilde{U} \subset U$ is a neighborhood of x_0 , and $\dim C(q) = \dim C_0(q) + 1$ for all $q \in \tilde{U}$. In this case, one can select the coordinates $\tilde{x}_{k+1}, \dots, \tilde{x}_n$ in such a way that $S_{x_0}^T = \{q \in \tilde{U} \mid \tilde{x}_{k+1}(q) = T, \tilde{x}_{k+2}(q) = \dots = \tilde{x}_n(q) = 0\}$ contains $R^{\tilde{U}}(x_0, T)$ for any $T > 0$. Again, we have two cases: (a) If $\dim C_0(x_0) < n - 1$, then this implies that all $q \in \tilde{U}$ such that $q \notin S_{x_0}^T$ are not asymptotically reachable from x_0 on V , and thus the local asymptotic reachability from x_0 is contradicted. (b) If $\dim C_0(x_0) = n - 1$, then all $q \in \tilde{U}$ such that $\tilde{x}_n = -K$, $K > 0$, are not asymptotically reachable from x_0 on \tilde{U} . This concludes the proof. \blacksquare

This theorem gives rise to corollaries similar to Corollary 3.3 and 3.5, except with accessibility replaced by strong accessibility.

3.2 The observability function and the observability rank condition

For the observability counterpart of the previous section we consider the observability functions as defined in (3) and (6). It is assumed throughout that the system (1) is asymptotically stable on a neighborhood Y of 0. We start with the observability function in (6) for which observability with respect to the input class B_α plays an important role. The corresponding results for the observability function (3) then follow as a special case when $\alpha = 0$.

Lemma 3.7 *Let $L_o^N(x)$ be the natural observability function (6) for some fixed $\alpha > 0$. Assume that $L_o^N(x)$ is smooth and finite for system (1) on a neighborhood W of 0. Then $L_o^N(x) > 0$ for $x \in W$, $x \neq 0$, implies that the system (1) is locally observable at 0 with respect to B_α .*

Proof Assume that system (1) is not locally observable at 0 with respect to B_α . Then the corresponding homogeneous system is also not locally observable at 0 with respect to B_α . Hence there exists an initial state $x_a \neq 0$ such that $h(\tilde{\varphi}(t, 0, 0, u)) = h(\tilde{\varphi}(t, 0, x_a, u))$, $t \geq 0$, $\forall u \in B_\alpha$, where $\tilde{\varphi}(\cdot)$ denotes the solution to homogeneous system. By definition of the natural observability function, we have that $L_o^N(0) = 0$ and by the positivity of L_o^N it follows that $L_o^N(x_a) > 0$. However, from equation (6) it follows immediately that the maximum over $u \in B_\alpha$ for both states 0 and x_a results in the same optimal input u . This implies that $L_o^N(0) = L_o^N(x_a)$, and yields the desired contradiction to prove the lemma. \blacksquare

Motivated by the minimality conditions of Theorem 2.14, we next obtain the following corollary, which follows straightforwardly from the previous lemma and some standard results from [6, 11].

Corollary 3.8 *Assume that the observability co-distribution $d\mathcal{O}$ has constant dimension about 0. If the natural observability function (6) is smooth, finite and satisfies $L_o^N(x) > 0$ for $x \in W$, $x \neq 0$, then $\dim d\mathcal{O}(0) = n$. \square*

Now, if we let $\alpha = 0$, then we obviously return to the observability function of (3), and the observability with respect to the input class B_α becomes zero-state observability. The following special case of Corollary 3.8 is useful in Section 4.

Corollary 3.9 *Assume that the zero-observability co-distribution $d\mathcal{O}_0$ has constant dimension about 0. If the observability function (3) is smooth, finite and satisfies $L_o(x) > 0$, $x \in W$, $x \neq 0$, then $\dim d\mathcal{O}_0(0) = n$.*

Remark 3.10 It is interesting to compare the results of this section to those of the previous section. They do not completely follow along similar or “dual” lines. Specifically, the results related to the observability functions as given by (3) and (6) are given in terms of the zero-state observability and observability rank condition, respectively. Starting with the rank conditions the converse of Corollary 3.8 and Corollary 3.9 also hold by applying Theorem 2.6. However, for the controllability function, we are considering asymptotic reachability which implies local accessibility, which in turn can be related to the accessibility rank condition. The reverse direction is far less obvious in this case, however, because accessibility from 0 is not sufficient for asymptotic reachability from 0. If asymptotic reachability can somehow be assumed for a given system, then the converse of Corollary 3.3 and Corollary 3.5 would follow for the controllability function.

3.3 Sufficient conditions for minimality

Briefly summarized below is a main result of the paper.

Theorem 3.11 *Assume that the observability co-distribution $d\mathcal{O}$ (or the zero-observability co-distribution $d\mathcal{O}_0$, respectively) and the accessibility distribution C of a system (f, g, h) each have constant dimension about 0. Furthermore, assume that the analytic system (f, g, h) is a realization of the formal power series c , and that $f(x)$ is asymptotically stable. Then, if $0 < L_c(x) < \infty$ and $0 < L_o^N(x) < \infty$ (or $0 < L_o(x) < \infty$, respectively) for $x \in W$, $x \neq 0$, then (f, g, h) is a minimal realization of c . \square*

These conditions are not necessary due to the fact that, contrary to the linear case, accessibility and controllability are not equivalent in general. Only under additional assumptions can a converse result be obtained.

4 Local state decompositions

For linear systems it is well known that the Hankel singular values are independent of the chosen state space realization and only depend on the input-output behavior of the system. In fact, they are the singular values of the Hankel operator of the system (e.g., Glover [2]). If we consider a non-minimal linear state space system with controllability Gramian W and observability Gramian M , the non-zero eigenvalues of MW correspond exactly to the squared Hankel singular values, and the number of zero eigenvalues of MW equals the difference between the state space dimension of the given system and the state space dimension of any minimal representation. In this section we extend these observations to the nonlinear setting. We are interested in the *Hankel* structure of the system and the related nonlinear balancing concept presented in Section 2.3. Since the system Hankel operator corresponds to the mapping from past inputs to future outputs (where the input is zero for positive time) we consider the controllability function as defined in (2) and the observability function as defined in (3).

Consider the nonlinear system (1) and assume that it is locally asymptotically stable. In this section we do *not* assume local zero-state observability, and hence the observability function is not necessarily positive definite. Furthermore, we do *not* assume that $-(f(x) + g(x)g(x)^T \frac{\partial^T L_c}{\partial x}(x))$ is locally asymptotically stable (or in other words: we do *not* assume asymptotic reachability from 0), and thus the controllability function need not be finite for all x .

One can use Frobenius’ Theorem to construct the zero-state observable ‘part’ of the system. In order to be able to do the same for the asymptotically reachable ‘part’ of the system, one must consider the part of the

state space system that is asymptotically reachable from 0, i.e., where

$$-\left(f(x) + g(x)g^T(x)\frac{\partial L_c}{\partial x}(x)\right) \quad (11)$$

is asymptotically stable. In the linear case this part equals the controllable part of the system. In the nonlinear case, the converses of Theorem 3.2 and Theorem 3.6 are not always true. So, in order to be able to construct a decomposition analogous to the known nonlinear generalization of the Kalman decomposition (e.g., Theorem 3.51 in [11]), we must consider the strongly accessible part of the system.

Theorem 4.1 *Assume that the distributions C_0 , $\ker d\mathcal{O}_0$ and $C_0 + \ker d\mathcal{O}_0$ all have constant dimension and that $C_0 + \ker d\mathcal{O}_0$ is involutive. Then one can find local coordinates $x = (x^1, x^2, x^3, x^4)$ such that $C_0 = \text{span}\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$ and $\ker d\mathcal{O}_0 = \text{span}\{\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^4}\}$. The system takes the form*

$$\dot{x}^1 = f^1(x^1, x^3) + \sum_{j=1}^m g_j^1(x^1, x^2, x^3, x^4)u_j \quad (12)$$

$$\dot{x}^2 = f^2(x^1, x^2, x^3, x^4) + \sum_{j=1}^m g_j^2(x^1, x^2, x^3, x^4)u_j \quad (13)$$

$$\dot{x}^3 = f^3(x^3) \quad (14)$$

$$\dot{x}^4 = f^4(x^3, x^4) \quad (15)$$

$$y = h(x^1, x^3). \quad (16)$$

Proof The proof is similar to that given in Theorem 3.51 of [11], which uses Frobenius' Theorem. The primary difference is that here we deal with the zero-observable part instead of the observable part. Therefore, for this proof it is enough to observe that the co-distribution $d\mathcal{O}_0$ is invariant for the dynamics $\dot{x} = f(x)$ since $L_f d\mathcal{O}_0 \subset d\mathcal{O}_0$. Hence $\ker d\mathcal{O}_0 = \text{span}\{\frac{\partial}{\partial x^2}\}$ is an invariant distribution for $\dot{x} = f(x)$. Since $\ker d\mathcal{O}_0 \subset \ker dh$, the theorem is proven. ■

Remark 4.2 Another way to view the difference between the decomposition above and that given by Theorem 3.51 in [11] is in the form of the input vector field in (12). For zero-state observability, the input vector field does not matter, while for the more general concept of observability it *may* matter. That means that x_1 and x_3 are zero-state observable, and thus observable, and that x_2 and x_4 are not zero-state observable, but they still may be observable! However, since we are interested only in the Hankel structure, and specifically in the singular value functions of the nonlinear system, the above decomposition is the most suitable. □

Let n_i be the dimension of x^i , $i = 1, 2, 3, 4$, and let Y be a neighborhood of 0 where the decomposition above is valid. Then clearly (12), (14) and (16) form the zero-state observable part of the system, while (12) and (13) is the strongly accessible part of the system. To assure that for (12), (14) and (16) the observability function exists, we assume that in these local coordinates equation (4) in Theorem 2.2 has a smooth solution for $(x^1, 0, x^3, 0) \in Y$. Furthermore, note that $(f^3(x^3)^T, f^4(x^3, x^4)^T)^T$ is asymptotically stable, and by the form of (14) and (15) it is impossible for $-(f(x) + g(x)g(x)^T \frac{\partial^T L_c}{\partial x}(x))$ to be asymptotically stable on Y . To assure that for (12) and (13) the controllability function exists, we assume that in these local coordinates equation (5) has an anti-stabilizing solution as in Theorem 2.2 for $(x^1, x^2, 0, 0) \in Y$. In fact, the assumption on the existence of the controllability function for the strongly accessible part of the system implies that *the part of the system that is asymptotically reachable from 0 corresponds exactly to the strongly accessible part of the system.*

Theorem 4.3 *If the above assumptions on existence of solutions and anti-stabilizing solutions to the equations (4) and (5), respectively, on parts of the state space are fulfilled, then:*

1. $L_o(x^1, x^2, x^3, x^4) > 0$ whenever $(x^1, x^4, x^3, x^4) \in Y$, and $(x^1, x^3) \neq (0, 0)$.
2. $L_o(0, x^2, 0, x^4) = 0$ for all $(0, x^2, 0, x^4) \in Y$.

3. $L_c(x^1, x^2, x^3, x^4)$ is infinite whenever $(x^1, x^2, x^3, x^4) \in Y$, $(x^3, x^4) \neq (0, 0)$.

4. $0 < L_c(x^1, x^2, 0, 0) < \infty$ for all $(x^1, x^2, 0, 0) \in Y$, $(x^1, x^2) \neq (0, 0)$.

Proof Proof of 1 and 2: It is clear that $h(0, x^2(\tau), 0, x^4(\tau)) = 0$ for all $\tau \geq 0$. By the form of (12) and (14) we obtain that

$$L_o(0, x^2, 0, x^4) = \frac{1}{2} \int_0^\infty h(0, x^2(\tau), 0, x^4(\tau))^T h(0, x^2(\tau), 0, x^4(\tau)) d\tau = 0,$$

for $u \equiv 0$, and for all $(0, x^2, 0, x^4) \in Y$. Again, by the form of (12), (14), and (16) we then have

$$L_o(x^1, x^2, x^3, x^4) = \frac{1}{2} \int_0^\infty h(x^1(\tau), x^3(\tau))^T h(x^1(\tau), x^3(\tau)) d\tau = \check{L}_o(x^1, x^3)$$

for $u \equiv 0$, where \check{L}_o is the observability function of (12), (14), and (16). By assumption $\check{L}_o = L_o$ exists and is smooth. By Theorem 2.6 $L_o(x^1, x^2, x^3, x^4) = \check{L}_o(x^1, x^3) > 0$ for $(x^1, x^3) \neq (0, 0)$.

Proof of 3 and 4: The controllability function L_c must satisfy equation (2). Since the system formed by (14) and (15) is asymptotically stable, it follows immediately that $L_c(x^1, x^2, x^3, x^4) = \infty$ for all $(x^1, x^2, x^3, x^4) \in Y$, with $(x^3, x^4) \neq (0, 0)$. By Theorem 2.2 $L_c(x^1, x^2, 0, 0) < \infty$ for all $(x^1, x^2, 0, 0) \in Y$. Furthermore, by Theorem 2.4 it follows that $L_c(x^1, x^2, 0, 0) > 0$ for all $(x^1, x^2, 0, 0) \in Y$, $(x^1, x^2) \neq (0, 0)$. ■

Remark 4.4 L_c is infinite on the subsystem that is *not* strongly accessible. Hence, that subsystem is also *not* asymptotically reachable from 0. This in essence yields another proof of Theorem 3.6. □

Remark 4.5 Now assume that the full system is locally accessible (remember that this is, together with local observability, a condition that implies minimality), but *not* locally strongly accessible. We know from Theorem 4.3 that the states which are not locally strongly accessible force the controllability function L_c to become infinite. Thus, one can conclude, contrary to the linear case, that minimality for a nonlinear system as discussed in the previous sections does *not* insure that the controllability function is finite. □

Remark 4.6 The observability counterpart to Remark 4.5 is similar, but in fact easier to describe. Assume that the full system is locally observable, but *not* locally zero-state observable. We know from Theorem 4.3 that the part of the system which is not locally zero-state observable corresponds to the observability function L_o being zero. Thus one can conclude, again contrary to the linear case, that minimality for a nonlinear system does *not* insure the observability function to be positive. However, we have introduced the natural observability function L_o^N in Section 2.2. For this function to be positive definite, we only need observability with respect to B_α , and not the more restrictive zero-state observability. If observability with respect to B_α is equivalent to observability (which is not very restrictive, since we only require the input to have finite energy), we can repeat the analysis of this section for L_o^N with zero-state observability replaced by observability. The new analysis results in the generalized Kalman decomposition as found in [11], and straightforwardly we obtain similar results as for L_o , with the additional property that for L_o^N the results do coincide with the usual results for the observability function in the linear case. □

If additionally one assumes that $\frac{\partial^2 L_o}{\partial x^1 \partial x^1}(0) > 0$ and $\frac{\partial^2 L_c}{\partial x^1 \partial x^1}(0) > 0$, then it becomes clear from Theorem 4.3 that $L_o(x^1, 0, 0, 0)$ and $L_c(x^1, 0, 0, 0)$ may be transformed into the form of Theorem 2.10. In fact, there exists a local x^1 coordinate transformation $x^1 = \psi(z)$, $\psi(0) = 0$, $(\psi^{-1}(x^1), 0, 0, 0) \in Y$, such that $L_c(\psi(z), 0, 0, 0)$ and $L_o(\psi(z), 0, 0, 0)$ are in the form of Theorem 2.10. Thus this part of the system may be balanced on a neighborhood of 0 with singular value functions $\tau_1(z) \geq \dots \geq \tau_{n_1}(z)$. Furthermore, if we also consider x^2 , then there exist local coordinates $(z^1, z^2) = \phi^{-1}(x^1, x^2)$ such that $L_c(\phi(z^1, z^2), 0, 0) = \frac{1}{2}z^1{}^2 + \frac{1}{2}z^2{}^2$. Now write $L_o(\phi(z^1, z^2), 0, 0) = \frac{1}{2}(z^1{}^T \ z^2{}^T)M(z^1, z^2)(z^1{}^T \ z^2{}^T)^T$. If the assumptions of Theorem 2.10 are fulfilled, one may diagonalize $M(z^1, z^2)$. The functions on the diagonal: $\bar{\tau}_1(z^1, z^2) \geq \dots \geq \bar{\tau}_{n_1+n_2}(z^1, z^2)$ are such that $\bar{\tau}_i(z^1, 0) = \tau_i(z)$, $i = 1, \dots, n_1$, and $\bar{\tau}_j(0, z^2) = 0$, $j = n_1 + 1, \dots, n_1 + n_2$. This is analogous to the linear case, where the unobservable part corresponds to zero Hankel singular values. Note that it is not possible to transform the whole system into the form of Theorem 2.10, since $L_c(0, 0, x^3, x^4)$ is infinite. This

is still in agreement with the linear theory, since here we are dealing with the ‘inverse of the controllability Gramian’. Hence that part of the system that is not strongly accessible yields an ‘inverse of the controllability Gramian’ that is infinite, and thus a ‘controllability Gramian’ that is singular.

5 Examples

The first example is an academic one meant to simply illustrate the basic theory presented in this paper. The second example is physical in nature and reveals some computational issues related to solving Hamilton-Jacobi equations.

Example 5.1 Consider the following system (1), where:

$$f(x) = \begin{pmatrix} -x_1 + x_2^3 \\ -x_2 - x_2^3 \\ -x_3 + x_1x_2^2 + x_3x_2^2 - x_2^3 \end{pmatrix} \quad g(x) = \begin{pmatrix} 0 & -\sqrt{2-2(x_1+x_3)^2+2x_2^2} \\ 0 & \sqrt{2-2(x_1+x_3)^2+2x_2^2} \\ \sqrt{2} & \sqrt{2-2(x_1+x_3)^2+2x_2^2} \end{pmatrix} \quad h(x) = \begin{pmatrix} 2x_1 + 2x_3 \\ \sqrt{2}x_2 \end{pmatrix}$$

This system is asymptotically stable and analytic on a neighborhood of 0. The rank of the accessibility distribution C at 0 is 2 (it is easily seen that the Lie bracket directions are already given by $g_1(0)$, and $g_2(0)$). The accessibility distribution equals in this case the strong-accessibility distribution. The rank of the observability co-distribution $d\mathcal{O}$ at 0 is also 2 (the two directions of the zero-state observability co-distribution in 0 are given by $dh_1(0)$, and $dh_2(0)$). The observability co-distribution equals in this case the zero-observability co-distribution. By Corollary 3.5 and Corollary 3.9 we know now that there exist $x \in \mathbb{R}^3$ such that $L_o(x) = 0$ with $x \neq 0$ and $L_c(x)$ infinite, with x finite. Now, to bring the system in the form of Theorem 4.1 apply the transformation

$$x = Tz = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} z.$$

The system then transforms into

$$\begin{cases} \dot{z}_1 = -z_1 + z_1z_2^2 + u_1\sqrt{2} \\ \dot{z}_2 = -z_2 - z_2^3 + u_2\sqrt{2-2z_1^2+2z_2^2} \\ \dot{z}_3 = -z_3 \end{cases} \quad \begin{cases} y_1 = 2z_1 \\ y_2 = \sqrt{2}z_2 \end{cases}$$

Obviously, z_3 is the non-accessible and non-observable part of the system. By Theorem 4.3, we have that $L_o(0, 0, z_3) = 0$, that $L_o(z_1, z_2, z_3) > 0$ for $(z_1, z_2) \neq (0, 0)$, that $0 < L_c(z_1, z_2, 0) < \infty$, and that $L_c(z_1, z_2, z_3)$ is infinite for $z_3 \neq 0$. This also directly follows from trying to solve the corresponding Hamilton-Jacobi equations (4) and (5). For the (z_1, z_2) subsystem (i.e., the minimal subsystem) note that L_c and L_o are already in the form of Theorem 2.10, i.e.,

$$L_c(z) = \frac{1}{2}z^T z, \quad L_o(z) = \frac{1}{2}z^T \begin{pmatrix} 2 & 0 \\ 0 & 1 + z_1^2 \end{pmatrix} z.$$

The singular value functions are therefore $\tau_1(z) = 2$ and $\tau_2(z) = 1 + z_1^2$. \square

Example 5.2 Consider a frictionless double pendulum (or two-link robot manipulator) with control torque u applied at the first joint, see Figure 1. The dynamics of such a double pendulum may be obtained via the Hamiltonian formalism. We derive the equations for the simple Hamiltonian form in order to be able to consider the associated gradient system, which is of smaller order, and therefore computationally easier to handle, but still captures the physical properties of the system. Furthermore, the frictionless system is only Lyapunov stable, but not asymptotically stable, while the associated gradient system is asymptotically

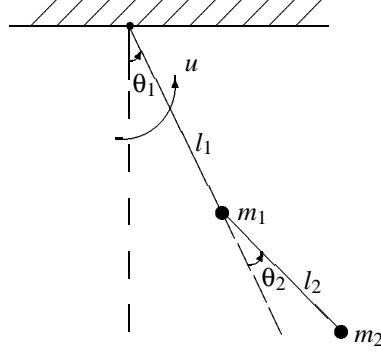


Figure 1: The double pendulum

stable, and thus fulfills the requirements of this paper. Let $\theta = (\theta_1, \theta_2)$ and $\dot{\theta} = (\dot{\theta}_1, \dot{\theta}_2)$. The kinetic co-energy is given by the sum of the kinetic co-energies of the masses m_1 and m_2 , respectively. This yields

$$T(\dot{\theta}) := \frac{1}{2} \dot{\theta}^T M(\theta) \dot{\theta}$$

with $M(\theta)$ as follows

$$M(\theta) = \begin{pmatrix} m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 \cos \theta_2 & m_2 l_2^2 + m_2 l_1 l_2 \cos \theta_2 \\ m_2 l_2^2 + m_2 l_1 l_2 \cos \theta_2 & m_2 l_2^2 \end{pmatrix}.$$

Note that $M(\theta)$ is a positive definite matrix for every θ . Similarly the potential energy V is the sum of the potential energies of the two masses, i.e.,

$$V(\theta) = -m_1 g l_1 \cos \theta_1 - m_2 g l_1 \cos \theta_1 - m_2 g l_2 \cos(\theta_1 + \theta_2). \quad (17)$$

Define $q := \theta$, and $p := M(\theta) \dot{\theta}$, thus $\dot{q} = M(q)^{-1} p$. Furthermore, denote by Q the manifold with local coordinates q_1, q_2 . The Hamiltonian H can be written as

$$H(q, p) = \frac{1}{2} p^T M(q)^{-1} p + V(q), \quad (18)$$

where the kinetic energy in the (q, p) coordinates is given by the Riemannian metric $M(q)$ on Q and $V(q)$ is the potential energy given by (17). We obtain that the output map C is given by $C(q) = q_1$. In the (q, p) coordinates the equations of the double pendulum in simple Hamiltonian form are given by

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= (1 \quad 0) q. \end{aligned} \quad (19)$$

Let $P(q) = M(q)^{-1}$. The associated gradient system is given by

$$\begin{aligned} \dot{x} &= -P(x) \frac{\partial^T V}{\partial x}(x) + P(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= (1 \quad 0) x. \end{aligned} \quad (20)$$

Here, we only consider the case where $l_1 = l_2 = 1$, and $m_1 = m_2 = 1$. Mathematica software was employed to approximately solve the Hamilton-Jacobi equations for the observability and controllability function, L_o

and L_c , for the gradient system (20). Specifically, equations (4) and (5), are solved up to order 4 using an iterative procedure from Lukes [9]. If we write

$$N = \frac{\partial^2 L_o}{\partial x^2}(0), \quad R = P(0), \quad Q = \frac{\partial^2 V}{\partial x^2}(0), \quad H = \frac{\partial C}{\partial x}(0)$$

then

$$\begin{aligned} L_o(x) &= \frac{1}{2}x^T N x + L_o^h(x) \\ -P(x) \frac{\partial^T V}{\partial x}(x) &= -R Q x + f^h(x) \\ \frac{1}{2}C(x)^T C(x) &= \frac{1}{2}x^T H^T H x + \theta^h(x), \end{aligned} \quad (21)$$

where $L_o^h(x)$, $f^h(x)$ and $\theta^h(x)$ contain higher-order terms (beginning with degrees 3, 2 and 3, respectively). The Lyapunov type equation (4) splits into two parts: the first part is the Lyapunov equation of the observability Gramian of the linearized gradient system, while the second part is the higher-order equation

$$\frac{\partial L_o^h}{\partial x}(x) R Q x = \frac{\partial L_o}{\partial x} f^h(x) + \theta^h(x). \quad (22)$$

The m -th order terms $L_o^{(m)}(x)$ of $L_o(x)$ can now be computed inductively for $m \geq 3$. Denote the m -th order terms in the right-hand side of (22) by $K_m(x)$. It follows that

$$\frac{\partial L_o^{(m)}}{\partial x}(x) R Q x = K_m(x), \quad (23)$$

and since $-RQ$ has all eigenvalues in the left half plane, this means that

$$L_o^{(m)}(x) = \int_0^\infty K_m(e^{-RQ t} x) dt. \quad (24)$$

It is easily seen that $K_m(x)$ only depends on $L_o^{(m-1)}$, $L_o^{(m-2)}$, \dots , $L_o^{(2)}$, and therefore (23) determines $L_o^{(m)}$ inductively starting from $L_o^{(2)} = \frac{1}{2}x^T N x$. This procedure can also be followed for the controllability function L_c . It yields for our gradient system the following result

$$\begin{aligned} L_o(x_1, x_2) = & 0.034375x_1^2 + 0.00212286x_1^4 + 0.01875x_1x_2 - \\ & 0.0046596x_1x_2^3 - 0.00168806x_1^2x_2^2 + \\ & 0.00015811x_1^3x_2 + 0.003125x_2^2 - 0.000909133x_2^4, \end{aligned} \quad (25)$$

and

$$\begin{aligned} L_c(x_1, x_2) = & 360x_1^2 - 107.411x_1^4 + 400x_1x_2 - 24.1667x_1x_2^3 - \\ & 191.25x_1^2x_2^2 - 230.595x_1^3x_2 + 120x_2^2 + 21.875x_2^4, \end{aligned} \quad (26)$$

respectively. Examining these functions near the origin (see Figure 2) it is evident that they are strictly



Figure 2: The controllability (left) and observability (right) functions for Example 5.2

positive, and hence, the system is minimal. This corresponds to our physical intuition. Observe that the observability function is quite close to zero at some values. This gives us a kind of measure for “weak” zero-observability. Likewise, for the controllability function we can make a similar observation for “weak” asymptotic reachability. \square

References

- [1] C.I. Byrnes, C.F. Martin, “An integral-invariance principle for nonlinear systems”, *IEEE Trans. Aut. Contr.*, vol. AC-40, pp. 983–994, 1995.
- [2] K. Glover, “All optimal Hankel-norm approximations of linear multivariable systems and their L^∞ -error bounds,” *Int. J. of Contr.*, vol. 39, pp. 1115–1193, 1984.
- [3] W.S. Gray and J. P. Mesko, “Controllability and observability functions for model reduction of nonlinear systems,” *Proc. 1996 Conf. Inf. Sci. & Syst.*, Princeton, NJ, pp. 1244–1249, 1996.
- [4] —, “Observability functions for linear and nonlinear systems,” *Syst. & Contr. Letters*, vol. 38, pp. 99–113, 1999.
- [5] D. Hill and P. Moylan, “The stability of nonlinear dissipative systems,” *IEEE Trans. Aut. Contr.*, vol. AC-21, pp. 708–711, 1976.
- [6] A. Isidori, *Nonlinear Control Systems*, Third Edition, Springer-Verlag, London, 1995.
- [7] T. Kato, *A Short Introduction to Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1982.
- [8] J. Milnor, *Morse Theory*, Princeton University Press, 1963.
- [9] D.L. Lukes, “Optimal regulation of nonlinear dynamical systems,” *SIAM J. Contr.*, vol. 7, pp. 75–100, 1969.
- [10] B.C. Moore, “Principal component analysis in linear systems: controllability, observability and model reduction,” *IEEE Trans. Aut. Contr.*, vol. AC-26, pp. 17–32, 1981.
- [11] H. Nijmeijer and A.J. van der Schaft, *Nonlinear Dynamical Control Systems*, Springer-Verlag, New York, 1990.
- [12] W.J. Rugh, *Nonlinear System Theory: The Volterra-Wiener Approach*, John Hopkins University Press, 1981.
- [13] A.J. van der Schaft, “ L_2 -gain analysis of nonlinear systems and nonlinear state feedback \mathcal{H}_∞ control,” *IEEE Trans. Aut. Contr.*, vol. AC-37, pp. 770–784, 1992.
- [14] J.M.A. Scherpen, “Balancing for nonlinear systems,” *Syst. & Contr. Letters*, vol. 21, pp. 143–153, 1993.
- [15] —, “Balancing for nonlinear systems,” *Proc. 1993 Eur. Contr. Conf.* Groningen, The Netherlands, vol. 4, pp. 1838–1843, 1993.
- [16] —, “On the similarity invariance of balancing for nonlinear systems,” *Preprints IFAC Nonl. Contr. Syst. Des. (NOLCOS)*, Tahoe City, CA, vol. 2, pp. 783–788, 1995.
- [17] E.D. Sontag, “Integrability of certain distributions associated to actions on manifolds and an introduction to Lie-algebraic control,” Rutgers Center SYCON Report 88-04, partly appeared in: *Nonlinear Controllability and Optimal Control*, H.J. Sussmann, ed., pp. 1–41, 1990.
- [18] H.J. Sussman, “Existence and uniqueness of minimal realizations of nonlinear systems,” *Math. Syst. Theory*, vol. 10, pp. 263–284, 1977.
- [19] —, “A general theorem on local controllability,” *SIAM J. Contr. Opt.*, vol. 25, pp. 158–194, 1987.
- [20] Y.F. Zheng, P. Liu, A.S.I. Zinober and C.H. Moog, “What is the dimension of the minimal realization of a nonlinear system?” *Proc. 34th IEEE CDC*, New Orleans, Louisiana, pp. 4239–4244, 1995.