On Nonlinear Control of Euler-Lagrange Systems: Disturbance Attenuation Properties*

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Abstract

In this brief note we analyse the disturbance attenuation properties of some asymptotically stabilizing nonlinear controllers for Euler-Lagrange systems reported in the literature. Our objective with this study is twofold: first, to compare the performance of these schemes from a perspective different from stabilizability; second, to quantify the basic tradeoff between robust stability and robust performance for these designs. We consider passivity-based and feedback linearization schemes developed for the control of DC-to-DC converters and rigid robots. For the DC-to-DC problem we show that for both controllers there exists a lower bound to the achievable attenuation level, i.e. a lower bound to the $\mathcal{L}_2$-gain of the closed loop operator from disturbance to regulated output, which is independent of the design parameters. Also, for the passivity based scheme we obtain an upper bound for the disturbance attenuation, which is insured provided we sacrifice the convergence rate. For rigid robots we show that both approaches yield arbitrarily good disturbance attenuation without compromising the convergence rate.

Keywords: Euler-Lagrange Systems, disturbances, nonlinear control, passivity based control, feedback linearization.

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1 Introduction

A lot of research in recent years has been devoted to the problem of developing control algorithms for mathematical models of physical systems. In view of its practical interest many researchers have concentrated on systems described by Euler-Lagrange equations, and in particular, in mechanical, electromechanical and power electronic systems. Several alternative approaches have been taken to design asymptotically stabilizing controllers for these systems. For instance, passivity concepts have been invoked to control robots, e.g., [16], [14], [10], [3], induction motors [6], and DC-to-DC converters [13]. Feedback linearization is another technique that is used to control these systems, e.g., [15], [9], [12].

In applications these systems are typically subject to external disturbances. For instance, load variations (e.g., frictions) affect the performance in robots and induction motors, while the regulated voltage in converter devices is perturbed by fluctuations in the external voltage source. Our main motivation in this paper is to analyse and compare the disturbance attenuation properties of some nonlinear controllers for Euler-Lagrange systems reported in the literature. Further, we want to investigate how, –if at all–, improving this performance indicator affects the robust stability measure. Towards this end, we use the tools provided by the recent theoretical research on the analysis of the $\mathcal{L}_2$-gain of nonlinear systems, e.g., [7], [17].

The remaining of the paper is organized as follows. In Section 2 we consider DC-to-DC "boost" converters. First, we present the circuit and its (averaged) model as well as the passivity-based and feedback linearization controllers reported in [13] and [12], respectively. Then, we prove some of the disturbance attenuation properties of these schemes. In Section 3 we carry out this $\mathcal{L}_2$ gain analysis for rigid robots. We wrap up the paper with some concluding remarks.

2 DC-to-DC Converters

2.1 Model and Stabilization Problem

We consider the switch–regulated "boost" converter circuit of Figure 1.

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The differential equations describing the circuit are given by

$$\begin{aligned}\dot{x}_1 &= -(1-v)\,\frac{1}{L}x_2 + \frac{E+\omega}{L}\\ \dot{x}_2 &= (1-v)\,\frac{1}{C}x_1 - \frac{1}{RC}x_2\end{aligned}$$

where x_1 and x_2 represent, respectively, the input inductor current and the output capacitor voltage variables; $E > 0$ represents the nominal constant value of the external voltage source and ω is an unknown disturbance, which satisfies $|\omega| < E$; v, which takes values in the discrete set $\{0, 1\}$, denotes the switch position function, and acts as a control input. The regulated output is x_2 which should be driven to some constant desired value $E_o > E$.

A Pulse Width Modulation (PWM) policy regulating the switch position function v, may be specified as follows,

$$v(t) = \begin{cases} 1 \text{ for} & t_k \le t < t_k + \mu(t_k)T \\ 0 \text{ for} & t_k + \mu(t_k)T \le t < t_k + T \end{cases}$$

where t_k represents a sampling instant defined by $t_{k+1} = t_k + T,\ k = 0, 1, \ldots,$; the parameter $T > 0$ is the fixed sampling period, also called the duty cycle. The duty ratio function, $\mu(\cdot)$, ranging on the closed interval $[0, 1]$, is the control input to the average PWM model given by [13]

$$\begin{aligned} \dot{z}_1 &= -u\,\frac{1}{L} z_2 + \frac{E + \omega}{L} \\ \dot{z}_2 &= u\,\frac{1}{C} z_1 - \frac{1}{RC} z_2 \end{aligned} \tag{2.1}$$

where $u := 1 - \mu$, and we denote by z_1 and z_2 the average input current and the average output capacitor voltage, respectively. As discussed in [13] this model accurately describes the behavior of the converter provided the switching is sufficiently fast and the capacitor voltage is bounded away from zero, i.e., $x_2 \ge \epsilon > 0$.

2.2 Control Laws

In this subsection we recall two control laws proposed in the literature to regulate (2.1). In the absence of external disturbances, i.e., when $\omega \equiv 0$, they both achieve (local) asymptotic stabilization, (that is, they insure that for suitable initial conditions $z_2 \to E_o = \text{const} > E$ with internal stability).

A. Passivity-based controller

In [13] the following (nonlinear dynamic state feedback) controller that preserves passivity of the closed loop was proposed

$$\dot{z}_{2d} = -\frac{1}{RC}\left\{ z_{2d} - \frac{E_o^2}{E z_{2d}}\left[E + R_1\left(z_1 - \frac{E_o^2}{RE}\right)\right]\right\},\ z_{2d}(0) > 0 \tag{2.2}$$

$$u = -\frac{1}{z_{2d}}\left[E + R_1\left(z_1 - \frac{E_o^2}{RE}\right)\right] \tag{2.3}$$

where $R_1 > 0$ is a design parameter that injects the damping required for asymptotic stability.

B. Feedback linearizing controller

In [12] a (nonlinear static state feedback) controller that linearizes the input-output behaviour of the system was proposed as follows. Consider the circuits total energy that is given by $H := \frac{1}{2}(L z_1^2 + C z_2^2)$. Applying the control

$$u = \frac{1}{(\frac{E}{L} + \frac{2}{RC} z_1) z_2}\left\{ \left(\frac{2}{R^2 C} - \frac{a_1}{R} + \frac{a_2 C}{2}\right) z_2^2 + \left(a_1 E + \frac{a_2 L}{2} z_1 \right) z_1 + \frac{E^2}{L} - a_2 H_d \right\} \tag{2.4}$$

where $a_1, a_2 > 0$ are the design parameters, and

$$H_d := \frac{E_o^2}{2}\left(C + \frac{L}{R^2 E^2} E_o^2\right).$$

yields in the new coordinates $[H, \dot{H}]$ the closed loop linear model given by

$$\ddot{H} + a_1 \dot{H} + a_2 H = a_2 H_d \tag{2.5}$$

Notice that H_d is chosen such that as $H \to H_d$ we have $z_2 \to E_o$ as desired.

2.3 Disturbance Attenuation Properties

In this subsection we will study the disturbance attenuation capabilities of the two controllers given above. Towards this end, we will evaluate some bounds on the achievable $\mathcal{L}_2$ gain of the closed loop operator from the external disturbance ω to the regulated output z_2. The qualifier "achievable" stems from the fact that these bounds are independent of the controller parameters.

2.3.1 Preliminary Lemma

First, we present a lemma which establishes an $\mathcal{L}_2$ gain property of the passivity based controller. This lemma will be instrumental for the analysis below.

Lemma 2.1 Consider the system (2.1) in closed loop with the controller (2.2), (2.3). Then, the $\mathcal{L}_2$ gain[1] of the operator $T_{\omega\tilde{z}_2} : \omega \mapsto \tilde{z}_2$, where $\tilde{z}_2 := z_2 - z_{2d}$, can be made arbitrarily small with a suitable choice of the design parameter R_1.

Proof
For ease of reference we define the following, more compact, matrix representation of (2.1),

$$\mathcal{D}\dot{z} - u\mathcal{J}z + \mathcal{R}z = \mathcal{E} \tag{2.6}$$

where

$$\mathcal{D} := \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} \quad ; \quad \mathcal{J} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad ; \quad \mathcal{R} := \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{R} \end{bmatrix} \quad ; \quad \mathcal{E} := \begin{bmatrix} E+\omega \\ 0 \end{bmatrix}$$

It can easily be verified that the closed loop is described by (2.2), (2.3) and

$$\mathcal{D}\dot{\tilde{z}} + u\mathcal{J}\tilde{z} + \begin{bmatrix} R_1 & 0 \\ 0 & \frac{1}{R} \end{bmatrix}\tilde{z} = \begin{bmatrix} \omega \\ 0 \end{bmatrix} \tag{2.7}$$

where $\tilde{z} := z - [\frac{E_o^2}{RE}, z_{2d}]^T$.
We find it convenient now to write the equations in state space form

$$\dot{x} = f(x) + g(x)\omega. \tag{2.8}$$

where we have defined $x = [x_1, x_2, x_3] := [\tilde{z}_1, \tilde{z}_2, z_{2d}]$, and $f(x)$, $g(x)$ are obtained from (2.2), (2.3), and (2.7). The equilibrium of this system is given by $f(x_0) = 0$, $x_0 = [0, 0, E_o]$. Since we are interested in the operator $T_{\omega\tilde{z}_2}$ we take as "output" signal $y = h(x) := x_2$. We know that (e.g. [17], Theorem 2) if for $\gamma > 0$ there exists a smooth nonnegative solution $V(x)$ to the following Hamilton-Jacobi inequality

$$\frac{\partial V}{\partial x}(x)f(x) + \frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V}{\partial x}(x)g(x)g(x)^T\frac{\partial^T V}{\partial x}(x) + \frac{1}{2}h^T(x)h(x) \leq 0, \quad V(x_0) = 0, \tag{2.9}$$

[1] We recall that the $\mathcal{L}_2$ gain of an operator $T : \mathcal{L}_2^m \to \mathcal{L}_2^n$ is defined as

$$\|T\|_{i2} \triangleq \sup_{u \in \mathcal{L}_2} \frac{\|Tu\|_2}{\|u\|_2}$$

where $\|\cdot\|_{i2}$ is the $\mathcal{L}_2$ induced norm, and $\|\cdot\|_2$ the $\mathcal{L}_2$ norm, see e.g. [5]

then the $\mathcal{L}_2$ gain of $T_{\omega\tilde{z}_2}$ is smaller than or equal to γ. Now, we look at the quadratic function $V_d := \frac{1}{2}\tilde{z}^T\mathcal{D}\tilde{z}$ whose derivative satisfies

$$\dot{V}_d = -\tilde{z}\begin{bmatrix} R_1 & 0 \\ 0 & \frac{1}{R} \end{bmatrix}\tilde{z} < 0 \quad \forall\, \tilde{z} \neq 0 \tag{2.10}$$

and plug $V = K_1 V_d$, with K_1 a positive constant, in equation (2.9) to obtain

$$-K_1 R_1 \tilde{z}_1^2 - \frac{1}{R}K_1\tilde{z}_2^2 + \frac{1}{2}\frac{1}{\gamma^2}K_1^2\tilde{z}_1^2 + \frac{1}{2}\tilde{z}_2^2 \leq 0$$

Obviously, if $K_1 \geq \frac{1}{2}R$ and $\frac{1}{2}\frac{1}{\gamma^2}K_1^2 - K_1R_1 \leq 0$, then V satisfies the inequality (2.9). Choose $K_1 = \frac{1}{2}R$, then the inequality holds for all $\gamma^2 \geq \frac{R}{4R_1}$, and γ can be made arbitrarily small by choosing R_1 arbitrarily large. This concludes the proof.

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From the lemma above we see that increasing the damping (R_1) we decrease the effect of the disturbances on the signal $\tilde{z}_2$. On the other hand, it follows from (2.10) that

$$\dot{V}_d \leq -\alpha V_d, \quad \alpha := \frac{\min(R_1, \frac{1}{R})}{\max(L, C)} > 0$$

Hence, the convergence rate of $\tilde{z}$ to zero is also improved by pumping up this gain. From these observations one might be tempted to try a high-gain design, which a more careful analysis reveals not to be a good idea. To see this notice that $\tilde{z}_2 \to 0$ does not imply that $z_2 \to E_o$ as desired, unless $z_{2d} \to E_o$ as well. To study the behavior of the latter consider the signal $\eta := \frac{1}{2}(z_{2d}^2 - E_o^2)$, which satisfies

$$RC\dot{\eta} = -2\eta + \frac{R_1 E_o^2}{E}\tilde{z}_1$$

This equation clearly shows that increasing the damping will induce a "peaking" in η, and consequently a slower convergence of $z_{2d} \to E_o$.

2.3.2 Lower Bounds

It is important to remark that the "ideal" disturbance attenuation property of the passivity based controller established in lemma 2.1 is with respect to the output signal $\tilde{z}_2$, while the actual regulated output of the system is z_2. Unfortunately, in the following theorem we prove that for neither one of the controllers we can actually obtain arbitrarily small disturbance attenuation for z_2.

Proposition 2.2 Consider the system (2.1) in closed loop with the passivity-based controller (2.2), (2.3). Then, the $\mathcal{L}_2$ gain of the operator $T_{\omega z_2} : \omega \mapsto z_2$ satisfies the lower bound

$$\|T_{\omega z_2}\|_{i2} \geq \frac{E_o}{2}$$

On the other hand, for the linearizing controller (2.4) we have the lower bound

$$\|T_{\omega z_2}\|_{i2} > \frac{L^3E_o^3}{E^3CR^2 + 2LEE_o^2}$$

Proof

The proof makes use of the linearization of the closed loop systems, and Proposition 6 of [17], which proves that if the linearized system has $\mathcal{L}_2$ gain $> (\geq)\gamma$ then the original nonlinear system also has $\mathcal{L}_2$ gain $> (\geq)\gamma$.

Let us first consider the passivity based control scheme described by (2.7), (2.2), and (2.3). We are interested in the output function $z_2 = \tilde{z}_2 + z_{2d}$. Define x, $f(x)$, and $g(x)$ as in the proof of Lemma 2.1, and take now as the output function $h(x) = x_2 + x_3$. Linearizing about the equilibrium $x_0 = [0, 0, E_o]$ gives

$$A := \frac{\partial f}{\partial x}(x_0) = \begin{bmatrix} -\frac{R_1}{L} & -\frac{E}{LE_o} & 0 \\ \frac{E}{CE_o} & -\frac{1}{RC} & 0 \\ \frac{R_1 E_o}{CRE} & 0 & -\frac{2}{CR} \end{bmatrix} \quad B := g(x_0) = \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix} \quad C := \frac{\partial h}{\partial x}(x_0) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

Some simple calculations yield the transfer function of this system as

$$\hat{G}(s) := C(sI - A)^{-1}B = \frac{1}{d(s)}\left(\frac{E}{LCE_o}(s + \frac{2}{CR}) + \frac{R_1 E_o}{LCRE}(s + \frac{1}{RC})\right)$$

where

$$d(s) := \det(sI - A) = (s + \frac{2}{CR})\left(s^2 + (\frac{1}{RC} + \frac{R_1}{L})s + \frac{E_o^2 R_1 + 2E^2 R}{LRCE_o^2}\right)$$

By definition of the $\mathcal{L}_2$ norm of a linear time invariant system, we know that

$$\|G\|_{i2} = \sup_w |\hat{G}(jw)| \geq |\hat{G}(0)| = \frac{E_o}{2}.$$

This completes the proof of the first part.

The second part can be proven in a similar way. Hence, consider the model (2.1), with the control (2.4) in state space form

$$\dot{z} = f_1(z) + g_1(z)\omega.$$

where $z := [z_1, z_2]$. Furthermore, we are interested in the output z_2, and thus let $h_1(z) = z_2$. It is readily checked that the (unforced systems) equilibria $\bar{z} = [\bar{z}_1, E_o]$ satisfy

$$\bar{z}_1 = \frac{E_o^2}{ER}$$

Define $A_1 := \frac{\partial f_1}{\partial z}(\bar{z})$, $B_1 := g_1(\bar{z})$, and $C_1 := \frac{\partial h_1}{\partial z}(\bar{z})$. Then

$$\hat{G}_1(s) := C_1(sI - A_1)^{-1}B_1 = \frac{K}{s^2 + a_1 s + a_2}$$

where

$$K := \frac{(E^4 LR^2 + a_1 L^2 RE^2 E_o^2 + a_2 L^3 E_o^4)}{E_o(E^3 CR^2 + 2LEE_o^2)}$$

To compute a lower bound for the $\mathcal{L}_2$ gain of this system independent of the design parameters $a_1,\ a_2 > 0$ we have to consider two cases:

- If $a_2 \leq \frac{1}{2}a_1^2$, then

$$\gamma_1 := \|G_1\|_{i2} = \frac{K}{a_2} > \frac{L^3 E_o^3}{E^3 C R^2 + 2LEE_o^2}.$$

- If $a_2 > \frac{1}{2}a_1^2$, then

$$\gamma_2^2 := \|G_1\|_{i2} = \frac{K^2}{a_1^2(a_2 - \frac{1}{4}a_1^2)}.$$

Take $a_2 = ca_1^2$, then for all $c > \frac{1}{2}$

$$\gamma_2^2 > \frac{1}{c - \frac{1}{4}} \left(\frac{cL^3 E_o^3}{E^3 C R^2 + 2LEE_o^2} \right)^2 > \left(\frac{L^3 E_o^3}{E^3 C R^2 + 2LEE_o^2} \right)^2$$

Combining these two cases we get that

$$\|G_1\|_{i2} \geq \min(\gamma_1, \gamma_2) > \frac{L^3 E_o^3}{E^3 C R^2 + 2LEE_o^2}.$$

which concludes the proof of this theorem.

□□□

2.3.3 An Upper Bound for the Passivity-Based Controller

The above proposition gives limits on the achievable disturbance attenuation which depend on the system parameters and desired set point, but are independent of the design parameters. In the following we develop an upper bound on the disturbance attenuation for the passivity based controller.

Again, consider the model of the closed loop system (2.2), (2.3), and (2.7) expressed shortly as in (2.8). Then, we shift the coordinates such that the system has its equilibrium in 0, hence we define $\tilde{z}_{2d} = z_{2d} - E_o$ and consider equation (2.7) together with

$$\begin{aligned} \dot{\tilde{z}}_{2d} &= \frac{1}{CR}(\tilde{z}_{2d} + E_o) + \frac{E_o^2}{CRE(\tilde{z}_{2d} + E_o)}(E + R_1 \tilde{z}_1) \\ y &= \tilde{z}_2 + \tilde{z}_{2d} \; (= z_2 - E_o), \end{aligned} \tag{2.11}$$

To obtain an upper bound for the disturbance attenuation we must find for this system a non-negative solution to the Hamilton-Jacobi inequality (2.9) for a certain γ. In order to find such solution, we split it into two parts. The first part is taken from Lemma 2.1, i.e., define

$$V_1(\tilde{z}) := \frac{K_1}{2}\tilde{z}^T \mathcal{D} \tilde{z},$$

with $K_1 > 0$. Additionally, we consider a term

$$V_2(\tilde{z}_{2d}) := K_2(\frac{1}{3}\tilde{z}_{2d} + \frac{1}{2}E_o)\tilde{z}_{2d}^2$$

with $K_2 > 0$, which is motivated by the fact that $\frac{\partial V_2}{\partial \tilde{z}_{2d}} = K_2(\tilde{z}_{2d} + E_o)\tilde{z}_{2d}$, and therefore helps us to cancel the term $\tilde{z}_{2d}$ in the output. Now define $V(\tilde{z}_1, \tilde{z}_2, \tilde{z}_{2d}) := V_1(\tilde{z}_1, \tilde{z}_2) + V_2(\tilde{z}_{2d})$. Note that $V \geq 0$ for all $\tilde{z}_{2d} \geq -\frac{3}{2}E_o (\Leftrightarrow z_{2d} \geq -\frac{1}{2}E_o)$. If we substitute V into the inequality (2.9), then we obtain

$$-K_1 R_1 \tilde{z}_1^2 - \frac{K_1}{R}\tilde{z}_2^2 - \frac{K_2}{CR}(\tilde{z}_{2d} + E_o)^2 \tilde{z}_{2d} + \frac{K_2 E_o^2}{CRE}(E + R_1 \tilde{z}_1)\tilde{z}_{2d} + \frac{K_1^2}{2\gamma^2}\tilde{z}_1^2 + \frac{1}{2}\tilde{z}_2^2 + \tilde{z}_2 \tilde{z}_{2d} + \frac{1}{2}\tilde{z}_{2d}^2 \leq 0 \qquad (2.12)$$

Use $\tilde{z}_2 \tilde{z}_{2d} \leq \frac{1}{2}\tilde{z}_2^2 + \frac{1}{2}\tilde{z}_{2d}^2$, and $\tilde{z}_1 \tilde{z}_{2d} \leq \frac{1}{2}\tilde{z}_1^2 + \frac{1}{2}\tilde{z}_{2d}^2$, then inequality (2.12) is satisfied if

$$\left(-K_1 R_1 + \frac{1}{2}\frac{K_2 R_1 E_o^2}{CRE} + \frac{K_1^2}{2\gamma^2}\right)\tilde{z}_1^2 + \left(-\frac{K_1}{R} + 1\right)\tilde{z}_2^2 + \left(-\frac{K_2}{CR}(\tilde{z}_{2d} + 2E_o) + \frac{K_2 E_o^2 R_1}{2CRE} + 1\right)\tilde{z}_{2d}^2 \leq 0$$

is satisfied. Clearly, this last inequality holds if

1. $-K_1 R_1 + \frac{1}{2}\frac{K_2 R_1 E_o^2}{CRE} + \frac{K_1^2}{2\gamma^2} \leq 0$
2. $-\frac{K_1}{R} + 1 \leq 0$
3. $-\frac{K_2}{CR}(\tilde{z}_{2d} + 2E_o) + \frac{K_2 E_o^2 R_1}{2CRE} + 1 \leq 0$

Condition 3 implies that R_1, which is a design parameter, should fulfill

$$R_1 \leq \frac{2E}{E_o^2}\left(-\frac{CR}{K_2} + 2E_o + \tilde{z}_{2d}\right).$$

With $\tilde{z}_{2d} \geq -\frac{3}{2}E_o$ it follows that if

$$R_1 \leq \frac{2E}{E_o^2}\left(-\frac{CR}{K_2} + \frac{1}{2}E_o\right)$$

then condition 3 is fulfilled. Since $R_1 > 0$ we have that K_2 should fulfill

$$K_2 > \frac{2CR}{E_o}.$$

From condition 2 we obtain

$$K_1 \geq R.$$

Finally, from condition 1 we obtain that

$$\gamma^2 \geq \frac{1}{R_1}\frac{K_1^2 CRE}{(2K_1 CRE - K_2 E_o^2)} \quad \text{and} \quad 2K_1 CRE - K_2 E_o^2 > 0.$$

Now, to optimize the disturbance attenuation property we have to find K_1 and K_2 such that γ is as small as possible. This leads us to the constrained optimization problem summarized in the following proposition.

Proposition 2.3 Consider the system (2.1) in closed loop with the controller (2.2), (2.3) with the design parameter $R_1 < \frac{E}{E_o}$. Then, for all disturbances ω such that $z_{2d}(t) > -\frac{1}{2}E_o$, we have the bound

$$\|z_2\|_2 \leq \tilde{\gamma}\|\omega\|_2$$

where $\tilde{\gamma}$ is the solution to the following optimization problem

$$\begin{aligned} \tilde{\gamma}^2 \quad &:= \quad \min_{K_3,K_4} \frac{1}{2R_1} \cdot \frac{K_3 K_4^2 E_o}{K_4 - 1} \\ \text{with} \qquad & R_1 \leq \frac{E}{E_o}\left(-\frac{1}{K_3} + 1\right) \\ & K_3 K_4 \geq \frac{RE}{E_o} \\ & K_3 > 1 \\ & K_4 > 1 \end{aligned}$$

Proof

The proof is given in the steps towards this proposition, where for convenience we have defined

$$K_3 := K_2 \cdot \frac{E_o}{2CR}, \quad K_4 := \frac{K_1}{K_2} \cdot \frac{2CRE}{E_o^2}$$

□□□

The following remarks are in order:

- Eventhough we can not solve analytically the optimization problem posed above, standard software can be used to find $\tilde{\gamma}$ for a given system and a damping gain satisfying $R_1 < \frac{E}{E_o}$. It is interesting to note that the latter bound exhibits again the tradeoff between robust stability and robust performance to external disturbances. This stems from the fact that R_1, which relates with the convergence rate as explained in Section 2.3.1, cannot be chosen larger than $\frac{E}{E_o}$ to insure the disturbance attenuation $\tilde{\gamma}$. Furthermore, the expressions above provide some useful guidelines for the selection of the system parameters to enhance the disturbance attenuation properties of the amplifier.
- Notice from (2.3) that to avoid singularities the controller state z_{2d} should be always positive and bounded away from zero. As discussed in [13] this requirement, which is consistent with the domain of validity of the averaged model, is needed even in the absence of external disturbances. Hence, the assumption made above on the disturbances is by no means restrictive in the present context.

3 Rigid Robots

In this section we consider the problem of attenuation of input disturbances in rigid robots performing a trajectory tracking task[2]. In this case we will show that both, passivity-based and feedback linearization schemes, yield closed loops with arbitrarily good disturbance attenuation properties without compromising the convergence rate.

[2]This problem has been considered before by [11], [1], and we refer the reader to those papers for further motivation.

It is well known (e.g. [15]) that the free dynamics of rigid robots (with rotational joints) are described by

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u + \omega \tag{3.1}$$

where $q \in \mathcal{R}^n$ denotes the joint angular positions, $d_1 I \geq D(q) \geq d_2 I > 0$ the inertia matrix, $C(q,\dot{q})\dot{q}$ the centrifugal and Coriolis forces, $G(q)$ the gravitational forces, and u the input torques, which we assume are perturbed by some external disturbance ω.

The kjth element of $C(q,\dot{q})$ is univocally defined from the elements of $D(q)$ via the Christoffel symbols of the first kind [10]

$$c_{kj} := \frac{1}{2}\sum_{i=1}^{n}(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k})\dot{q}_i$$

such that

$$\dot{D}(q) = C(q,\dot{q}) + C^T(q,\dot{q}) \tag{3.2}$$

In the absence of external disturbances, a globally exponentially stable controller that preserves passivity in closed loop is given by

$$u = -D(q)(\ddot{q}_d - \lambda\dot{\tilde{q}}) - C(q,\dot{q})(\dot{q}_d - \lambda\tilde{q}) - k_1(\dot{\tilde{q}} + \lambda\tilde{q}) - k_2\tilde{q} + G(q) \tag{3.3}$$

where $q_d(t)$ is the desired angular trajectory, $\tilde{q} = q - q_d$, and $\lambda, k_1, k_2 > 0$ are design parameters, see e.g. [3]. We are interested here in the choice of these parameters for optimal attenuation of the torque disturbance on the position and speed tracking errors. The solution to this problem is summarized in the proposition below. A similar result, as well as its extension to flexible joint robots, may be found in [1], [2].

Proposition 3.1 Consider (3.1) in closed loop with (3.3) with the output signal $z := [\tilde{q}, \dot{\tilde{q}}]^T$.
For a fixed $\gamma > 0$, assume

$$\begin{aligned} k_1 &> \frac{1}{2}(\frac{1}{\gamma^2} + 1 + \lambda) \\ k_2 &> \frac{1}{2\lambda}(\lambda^2 + \lambda + 1) \end{aligned}$$

Under these conditions, the $\mathcal{L}_2$ gain of the operator $T_{\omega z} : \omega \rightarrow z$ satisfies the bound $\|T_{\omega z}\|_2 \leq \gamma$. Consequently, arbitrarily good disturbance attenuation is achievable by increasing the gain k_1.

Proof
The proof follows immediately by plugging in the Hamilton-Jacobi inequality (2.9) the quadratic function

$$V(s,\tilde{q}) := \frac{1}{2}s^T D(q)s + \frac{k_2}{2}|\tilde{q}|^2$$

where, for convenience, we have defined $s := \dot{\tilde{q}} + \lambda\tilde{q}$, to obtain, after some basic bounding and the use of the skew-symmetry property (3.2), the inequality

$$-[k_1 - \frac{1}{2}(\frac{1}{\gamma^2} + 1 + \lambda)]|s|^2 - [\lambda k_2 - \frac{1}{2}(\lambda^2 + \lambda + 1)]|\tilde{q}|^2 \leq 0$$

□□□

Evaluating the derivative of V along the trajectories of the unforced closed loop system we get

$$\dot{V} = -k_1|s|^2 - k_2\lambda|\tilde{q}|^2$$

Henceforth, in contrast with the converter problem, in this case there is no tradeoff between converge rate and disturbance attenuation to be made. This seems to stem from the fact that rigid robots are fully actuated systems, that is, the number of degees of freedom is equal to number of controls.

Its easy to see that a similar property is enjoyed by the feedback linearization (computed torque) controller

$$u = D(q)[\ddot{q}_d - k_1\dot{\tilde{q}} - k_2\tilde{q}] + C(q,\dot{q})\dot{q} + G(q)$$

which yields the closed loop system

$$\ddot{\tilde{q}} + k_1\dot{\tilde{q}} + k_2\tilde{q} = D^{-1}(q)\omega$$

Notice that, in view of the uniform boundedness of $D(q)$, we have $|D^{-1}(q)\omega| \leq \frac{1}{d_1}|\omega|$.

4 Concluding Remarks

The long term motivation of the present study is to provide a framework to compare, from a perspective different from stabilization, various existing controllers proposed for Euler-Lagrange systems. A similar research, albeit specialized to robots with flexible joints, was reported in [4], where the comparison was based on continuity properties and adaptivity. As alternative performance indicator we propose here to adopt the robustness to external disturbances, which is measured via the $\mathcal{L}_2$ norm of the corresponding closed loop operator. Although this indicator may be extremely conservative[3], we believe it is a natural way to set up the basic tradeoff between robust stability and robust performance.

In this brief note we have studied these questions for passivity-based and feedback linearizing controllers as applied to DC-to-DC converters and rigid robots. Current research is under way to extend this study to other systems and controllers. In particular, we are interested in carrying out the analysis for backstepping-based controllers and induction motors.

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