Normalized Coprime Factorizations and Balancing for Unstable Nonlinear Systems

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Abstract

In this paper we first study the normalized right and left coprime factorizations of a nonlinear system from a state space point of view. In order to do so, we introduce the notion of inner and co-inner nonlinear systems. Secondly we deal with balancing for unstable nonlinear systems. For linear systems there are several ways to approach this problem, and we generalize the ideas of balancing the normalized coprime factorization and of LQG balancing to the nonlinear case. LQG balancing can be seen as measuring the influence of a state component on a certain future and past energy function. We extend this interpretation to the nonlinear case. Furthermore we use the normalized coprime factorization to give another way of balancing the unstable nonlinear system. We give the relation between these two methods of balancing, which is similar to the relation in the linear case.

Keywords: balancing, nonlinear systems, normalized coprime factorizations, model reduction.

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1 Introduction

In the last decade significant advance has been made in the theory of model reduction for linear systems. One of the tools is to bring the system in balanced form and apply model reduction based on the Hankel singular values of the system. This method, which applies to stable linear systems, has been introduced by Moore 1981, with the motivation that the order of the system can be reduced based on the future output and the past input energy. These energies are related to the observability and controllability gramian. The same method has been extended for stable nonlinear systems in Scherpen 1993, using the observability and controllability function of a nonlinear system.

For model reduction of unstable linear systems, other forms of balancing have been proposed. One method makes use of the normalized coprime factorization of an unstable linear system and has been studied in e.g. Meyer 1988 and Ober/McFarlane 1989. Another method is LQG balancing, which is based on the standard LQG problem and the corresponding Riccati equations, see e.g. Jonckheere/Silverman 1983 and Opdenacker/Jonckheere 1985. In Ober/McFarlane 1989 the relation between these two methods has been given. Balancing for linear systems and the relation between LQG balancing and balancing of the normalized coprime factorization is treated from another point of view in Weiland 1991. We will extend these ideas to nonlinear systems.

In recent years, some authors have been developing theory about coprime factorizations for nonlinear systems. Some of them treat this concept from an input-output point of view, where the Bezout identity plays an important role (see e.g. Hammer 1984, 1989, Paice 1992). Others consider it from a state space point of view, with some nice results on the right coprime factorization (see e.g. Sontag 1989, Moore/Irlicht 1992, Verma/Hunt 1993), but as far as we know the results on the left coprime factorization are not very well-developed. Furthermore we think there are no results yet on normalized coprime factorizations. In this paper we give a characterization of inner and co-inner nonlinear systems, and based on this we give a state space characterization of a normalized right, respectively a normalized left coprime factorization.

In analogy with the linear theory we can perform balancing for an unstable nonlinear system on the basis of the normalized coprime factorization, and then apply the theory of balancing for stable nonlinear systems from Scherpen 1993. In order to extend LQG balancing to the nonlinear case we generalize the point of view given in Weiland 1991. We propose a method to balance unstable nonlinear systems based on a certain future and past energy function, which replace the input and output energy function used for balancing of a stable system. We call this HJB (Hamilton-Jacobi-Bellman) balancing and we give the relation between this method and balancing of the normalized coprime factorization. As we already said before, in the linear case this method is the same as LQG balancing, but in general this is not true in the nonlinear case. It is difficult to extend the usual stochastic formulation of the LQG problem to the nonlinear case. On the other hand, there exists a deterministic formulation of the LQG problem for linear systems which can be extended to nonlinear systems, see Hijab 1980. The difference however with the linear case is that in general we get an infinite dimensional compensator that solves the problem. In case the measurements are zero this compensator becomes finite dimensional and HJB balancing corresponds to this case.

In section 2 we give a brief review on LQG balancing and coprime balancing for linear systems and in section 3 we give a brief review on balancing for stable nonlinear systems. In section 4 we treat the normalized coprime factorizations. First we give a characterization of an inner nonlinear system and then of the corresponding normalized right coprime factorization. We do the same with a co-inner nonlinear system and the corresponding normalized left coprime factorization. In section 5 we define HJB balancing and we propose a procedure to bring a nonlinear system into HJB balanced form, which is related to the procedure to balance stable nonlinear systems. Furthermore we go into balancing for the normalized right coprime factorization and study the relation of HJB balancing with balancing of the normalized coprime factorization. Section 6 contains model reduction based on the concept of HJB balancing and on the concept of balancing the normalized coprime factorization for nonlinear systems. Finally in section 7 we give some conclusions.

Throughout this paper we will use a fairly standard notation. We denote by $x^T x$ or $\| x \|^2$ the squared norm of a vector $x \in \mathbb{R}^n$. We say that $u : (-\infty, 0) \to \mathbb{R}^m$ is in $L_2(-\infty, 0)$ if $\int_{-\infty}^0 \| u(t) \|^2 \, dt < \infty$. By $\frac{\partial L}{\partial x}(x)$ we denote the row-vector of partial derivatives of a differentiable function $L : \mathbb{R}^n \to \mathbb{R}$. Furthermore we denote by $x(t_2) = \varphi(t_2, t_1, x_1, u)$ the solution on time t_2 of the system $\dot{x} = f(x) + g(x)u$ with initial condition $x(t_1) = x_1$ and input $u : [t_1, t_2] \to \mathbb{R}^m$.

2 Review on balancing for linear systems

2.1 The LQG problem

LQG-balancing for linear systems has been introduced in Jonckheere/Silverman 1983, and in Opdenacker/Jonckheere 1985 this concept is developed further. The set of invariants defined in these two papers are treated from another point of view in Weiland 1991. First we will give a review of the formulation of Jonckheere/Silverman 1983 and Opdenacker/Jonckheere 1985.

LQG compensation is formulated for a minimal state-space system

$$\begin{aligned} \dot{x} &= Ax + Bu + Bd \\ y &= Cx + v \end{aligned} \tag{1}$$

where $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and d and v are Gaussian white noise processes with covariance functions $I\delta(t-\tau)$. The criterion

$$J(x_0, u(t)) = E\left(\lim_{T\to\infty} \frac{1}{T} \int_0^T (x^T C^T C x + u^T u) dt \right) \tag{2}$$

is required to be minimized and the corresponding optimal compensator is given by

$$\begin{aligned} \dot{z} &= Az + Bu + SC^T(y - Cz) \\ u &= -B^T P z \end{aligned} \tag{3}$$

where S is the stabilizing solution (i.e. $\sigma(A - SC^T C) \subset \mathbb{C}^-$) to the Filter Algebraic Riccati Equation (FARE)

$$AS + SA^T + BB^T - SC^T CS = 0 \tag{4}$$

and P is the stabilizing solution (i.e. $\sigma(A - BB^T P) \subset \mathbb{C}^-$) to the Control Algebraic Riccati Equation (CARE)

$$A^T P + PA + C^T C - PBB^T P = 0 \tag{5}$$

Theorem 2.1 (Jonckheere/Silverman 1983, Opdenacker/Jonckheere 1984) *The eigenvalues of PS are similarity invariants and there exists a state space representation where*

$$M := P = S = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix} \quad \textit{with} \quad \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > 0. \tag{6}$$

This is called a LQG balanced representation.

In Jonckheere/Silverman 1983 and Opdenacker/Jonckheere 1985 it is argued that if $\mu_k >> \mu_{k+1}$, then the state components x_1 up to x_k are more difficult both to control and to filter than x_{k+1} up to x_n and a synthesis based only on x_1 up to x_k probably retains the stability and sensitivity properties of the system. Corresponding to the partitioning of the state in the first k components and the last $n-k$ components, the partitioning of the matrices is

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}.$$

If we assume system (1) is LQG balanced, then the reduced order system is

$$\begin{aligned} \dot{x} &= A_{11}x + B_1 u + B_1 d \\ y &= C_1 x + v \end{aligned} \tag{7}$$

Under some mild conditions the reduced order system (A_{11}, B_1, C_1) is minimal, see Jonckheere/Silverman 1983.

Theorem 2.2 (Jonckheere/Silverman 1983, Opdenacker/Jonckheere 1985) *If (A_{11}, B_1, C_1) is minimal, the reduced order system (7) is LQG balanced again and the optimal compensator for system (7) is the reduced order optimal compensator of the full order system (1).*

The original idea of balancing for stable linear systems, introduced in Moore 1981, considers the Hankel singular values which are a measure of the importance of a state component. This is based on the input energy which is necessary to reach this state component and the output energy which is generated by this state component. A similar kind of reasoning, using a different pair of energy functions, can be used to achieve the similarity invariants μ_i, $i = 1, .., n$, as above, see Weiland 1991. For this we consider a minimal system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{8}$$

where $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$ (N.B.: no noise is entering the system). We define the following energy functions

$$K^-(x_0) := \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} (\| y(t) \|^2 + \| u(t) \|^2) dt$$

$$K^+(x_0) := \min_{\substack{u \in L_2(0,\infty) \\ x(\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{0}^{\infty} (\| y(t) \|^2 + \| u(t) \|^2) dt$$

$K^-(x_0)$ will be called the *past energy* and $K^+(x_0)$ the *future energy* of the system in the state x_0.

Theorem 2.3 (Weiland 1991) *$K^-(x_0) = \frac{1}{2}x_0^T S^{-1} x_0$ and $K^+(x_0) = \frac{1}{2}x_0^T P x_0$, where S and P are the stabilizing solutions of respectively (4) and (5).*

For the LQG balanced representation from Theorem 2.1 the past and future energy function are respectively $K^-(x_0) = \frac{1}{2}x_0^T M^{-1} x_0$ and $K^+(x_0) = \frac{1}{2}x_0^T M x_0$, where M is diagonal. Then the importance of the state $\tilde{x} = (0 \dots 0 \; x_i \; 0 \dots 0)$ in terms of past and future energy can be measured by the similarity invariant μ_i. For large μ_i the influence of the state $\tilde{x}$ on the future energy is large while the influence on the past energy is small. Hence if $\mu_k >> \mu_{k+1}$, the state components x_{k+1} to x_n are not important from this energy point of view and can be removed to reduce the number of state components of the model.

2.2 The normalized coprime representation

In Meyer 1988 and Ober/McFarlane 1989 balancing of the normalized coprime representation of a linear system is dealt with. In the first paper it is treated for the normalized *right* coprime factorization and in the second paper for the normalized *left* coprime factorization of a linear system. Here we will give a brief review on this subject. We consider system (8) and its transfer function $G(s) = C(sI - A)^{-1}B$. First we go into the normalized right coprime factorization and then into the normalized left coprime factorization.

The transfer matrix $G(s)$ can be represented as a normalized right coprime fraction over the ring of stable transfer matrices such that $G(s) = N(s)D^{-1}(s)$, where $N(s) = C(sI - \hat{A})^{-1}B$, $D(s) = I - B^T P(sI - \hat{A})^{-1}B$, $\hat{A} = A - BB^T P$ is stable and P is the stabilizing solution to the CARE (5). Moreover, a state space realization of the so called Graph operator $\begin{pmatrix} N(s) \\ D(s) \end{pmatrix}$ is

$$\dot{x} = (A - BB^T P)x + Bw, \qquad \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} C \\ -B^T P \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} w \tag{9}$$

with w a (fictitious) input variable. This Graph operator is representing the normalized right coprime factorization of system (8), see Meyer 1988.

In a similar way the transfer matrix $G(s)$ can be represented as a normalized left coprime fraction over the ring of stable transfer matrices such that $G(s) = \tilde{D}(s)^{-1}\tilde{N}(s)$, where $\tilde{N}(s) = C(sI - \tilde{A})^{-1}B$, $\tilde{D}(s) = C(sI - \tilde{A})^{-1}SC^T - I$, $\tilde{A} = A - SC^T C$ is stable and S is the stabilizing solution to the FARE (4). Obviously $\hat{y}(s) = G(s)\hat{u}(s)$ is equivalent with $0 = \tilde{N}(s)\hat{u}(s) - \tilde{D}(s)\hat{y}(s)$. Moreover, a state space realization of $(\tilde{N}(s) \quad \tilde{D}(s))$ is

$$\dot{x} = (A - SC^T C)x + (B \quad SC^T)\tilde{w}, \qquad z = Cx + (0 \quad -I)\tilde{w} \tag{10}$$

If we take $\tilde{w} = \begin{pmatrix} u \\ y \end{pmatrix}$ as the input variable, then the zero-dynamics of (10) for $z = 0$ is a state space representation of $G(s)$. This represents the normalized left coprime factorization of system (8), see Ober/McFarlane 1989.

For balancing we first consider system (9). The Hankel singular values of this system (for a definition of Hankel singular values see e.g. Glover 1984) are called the *Graph Hankel singular values* of system (8). It is not difficult to obtain that they are also the Hankel singular values of system (10). These singular values have the following property

Theorem 2.4 (Meyer 1988, Ober/McFarlane 1989) *The Graph Hankel singular values of system (8) are strictly less than one.*

Denote the Graph Hankel singular values by τ_i, $i = 1, .., n$, and assume $\tau_1 \geq \ldots \geq \tau_n$. The relation between τ_i, $i = 1, .., n$, and the similarity invariants μ_i, $i = 1, .., n$, of Theorem 2.1 is given by the following theorem:

Theorem 2.5 (Ober/McFarlane 1989, Weiland 1991) $\mu_i = \tau_i(1 - \tau_i^2)^{-\frac{1}{2}}$ *for* $i = 1, .., n$.

In particular, this means that the reduced model that is obtained by model reduction based on the concept of balancing the normalized coprime factorization is the same as the reduced model that is obtained by model reduction based on the concept of LQG balancing.

3 Review on balancing for stable nonlinear systems

Balancing for stable nonlinear systems is dealt with in Scherpen 1993. As in the linear case, this is a method based on the input energy that is necessary to reach a state and the output energy that is generated by this state. We will give a brief review on this subject in this section.

Consider a smooth, i.e., C^∞, nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{11}$$

where $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$, $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$ and $x = (x_1, \ldots, x_n)$ are local coordinates for a smooth state space manifold denoted by M. Throughout we assume that the system has an equilibrium. Without loss of generality we take this equilibrium in 0, i.e. $f(0) = 0$ and we also take $h(0) = 0$.

Definition 3.1 The controllability and observability function of a nonlinear system are defined as

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty) = 0, x(0) = x_0}} \frac{1}{2} \int_{-\infty}^{0} \| u(t) \|^2 \, dt \tag{12}$$

respectively

$$L_o(x_0) = \frac{1}{2} \int_0^\infty \| y(t) \|^2 \, dt, \qquad x(0) = x_0, \qquad u(t) \equiv 0, \quad 0 \leq t < \infty \tag{13}$$

The value of the controllability function at x_0 is the minimum amount of control energy required to reach the state x_0 and the value of the observability function at x_0 is the amount of output energy generated by x_0. These functions do not necessarily exist (i.e. are finite), in particular, L_o can be infinite if the system is unstable and if x_0 can not be reached from 0, then by convention $L_c(x_0)$ will be infinite. We throughout assume L_c and L_o are *finite*. Also, for the rest of this paper we assume L_c and L_o are *smooth* functions of x.

Theorem 3.2 (Scherpen 1993) *If $f(x)$ is asymptotically stable on a neighborhood W of 0, then for all $x \in W$, $L_o(x)$ is the unique smooth solution of the following Lyapunov type of equation:*

$$\frac{\partial L_o}{\partial x}(x)f(x) + \frac{1}{2}h^T(x)h(x) = 0, \qquad L_o(0) = 0 \tag{14}$$

Furthermore for all $x \in W$, $L_c(x)$ is the unique smooth solution of the following Hamilton-Jacobi equation:

$$\frac{\partial L_c}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial L_c}{\partial x}(x)g(x)g^T(x)\frac{\partial^T L_c}{\partial x}(x) = 0, \qquad L_c(0) = 0 \tag{15}$$

satisfying $-(f(x) + g(x)g^T(x)\frac{\partial^T L_c}{\partial x}(x))$ *is asymptotically stable on W.*

Remark 3.3 (Scherpen 1993) L_c and L_o are *non-negative.*

Remark 3.4 If we assume that $f(x)$ is asymptotically stable and that (14) has a smooth solution, it follows that L_o, as in (13), exists, i.e. is finite. Furthermore, if we assume that (15) has a smooth solution L and that $-(f(x) + g(x)g(x)^T \frac{\partial^T L}{\partial x}(x))$ is asymptotically stable, it follows that L_c, as in (12), exist, i.e. is finite.

Theorem 3.5 (Scherpen 1993) *Assume f is asymptotically stable on W and (15) has a smooth solution $\bar{L}_c$ on W. Then $\bar{L}_c(x_0) > 0$ for $x_0 \in W$, $x_0 \neq 0$, if and only if $-(f(x) + g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ is asymptotically stable on W.*

For the following definition see e.g. Hill/Moylan 1976 and van der Schaft 1992.

Definition 3.6 The system (11) is reachable from x_0 if for any $\bar{x} \in M$ there exists a $\bar{t} \geq 0$ and input u such that $\bar{x} = \varphi(\bar{t}, 0, x_0, u)$.
The system (11) is zero-state observable if any trajectory where $u(t) \equiv 0, y(t) \equiv 0$ implies $x(t) \equiv 0$, i.e., for all $x \in M$, $h(\varphi(t,0,x,0)) = 0, t \geq 0 \Rightarrow \varphi(t,0,x,0) = 0, t \geq 0$.

The following theorem is closely related to some results in Hill/Moylan 1976 and van der Schaft 1992. For the proof, see Scherpen 1993.

Theorem 3.7 *Assume $f(x)$ is asymptotically stable on a neighborhood W of 0. If the system (11) is zero-state observable on W, then $L_o(x_0) > 0$, $\forall x_0 \in W$, $x_0 \neq 0$.*

Now we consider nonlinear systems of the form (11) with controllability and observability function L_c respectively L_o as in definition 3.1, and with the following additional assumptions:

1. $f(x)$ is asymptotically stable on some neighborhood Y of 0
2. $L_c(x)$ and $L_o(x)$ are smooth and finite functions of x on Y
3. the system is zero-state observable on Y
4. $\frac{\partial^2 L_c}{\partial x^2}(0) > 0$ and $\frac{\partial^2 L_o}{\partial x^2}(0) > 0$

Lemma 3.8 (Scherpen 1993) *There exists a coordinate transformation $x = \phi(\bar{x})$, $\phi(0) = 0$, such that $L_c(x)$ in the new coordinates $\bar{x} = \phi^{-1}(x)$ is of the following form:*

$$L_c(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T\bar{x} \tag{16}$$

Furthermore we can write $L_o(x)$ in the new coordinates $\bar{x} = \phi^{-1}(x)$ in the following form:

$$L_o(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T M(\bar{x})\bar{x} \quad \text{where} \quad M(0) = \frac{\partial^2 L_o}{\partial x^2}(0) \tag{17}$$

with $M(\bar{x})$ a $n \times n$ symmetric matrix with entries which are smooth functions of $\bar{x}$.

Theorem 3.9 (Scherpen 1993) *Consider system (11) and assume there exists a neighborhood V of 0 where the number of distinct eigenvalues of $M(\bar{x})$ is constant for $\bar{x} \in V$. Then there exists a neighborhood U of zero and a coordinate transformation $x = \psi(z)$, $\psi(0) = 0$, such that $L_c(x)$ in the new coordinates $z \in W := \psi^{-1}(U)$ is of the following form:*

$$\tilde{L}_c(z) := L_c(\psi(z)) = \frac{1}{2}z^T z \tag{18}$$

while in the new coordinates $L_o(x)$ is of the following form:

$$\tilde{L}_o(z) := L_o(\psi(z)) = \frac{1}{2}z^T \begin{pmatrix} \tau_1(z) & & 0 \\ & \ddots & \\ 0 & & \tau_n(z) \end{pmatrix} z \tag{19}$$

where $\tau_1(z) \geq ... \geq \tau_n(z)$ are smooth functions of z, called the singular value functions.

Remark 3.10 For a linear system the singular value functions τ_i, $i = 1, .., n$ are constant and are equal to the squared Hankel singular values.

The form of the controllability and observability function in (18) and (19) is not yet entirely balanced. For that we need another additional coordinate transformation. We take as smooth transformation $\bar{z}_i = \eta_i(z_i) := \tau_i(0, .., 0, z_i, 0, ..0)^{\frac{1}{4}} z_i$, $i = 1, .., n$ and hence $\bar{z} = \eta(z) := (\; \eta_1(z_1) \quad \ldots \quad \eta_n(z_n)\;)$ on $\bar{z} \in \bar{W} := \eta(W)$. Define $\hat{L}_c(\bar{z}) := \tilde{L}_c(\eta^{-1}(\bar{z}))$ and $\hat{L}_o(\bar{z}) := \tilde{L}_o(\eta^{-1}(\bar{z}))$. Then (18) and (19) become respectively:

$$\hat{L}_c(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} \sigma_1(\bar{z}_1)^{-1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n(\bar{z}_n)^{-1} \end{pmatrix} \bar{z} \tag{20}$$

$$\hat{L}_o(\bar{z}) = \frac{1}{2}\bar{z}^T \begin{pmatrix} \sigma_1(\bar{z}_1)^{-1}\tau_1(\eta^{-1}(\bar{z})) & & 0 \\ & \ddots & \\ 0 & & \sigma_n(\bar{z}_n)^{-1}\tau_n(\eta^{-1}(\bar{z})) \end{pmatrix} \bar{z} \tag{21}$$

where $\sigma_i(\bar{z}_i) = \tau_i(0, .., 0, \eta_i^{-1}(\bar{z}_i), 0, .., 0)^{\frac{1}{2}}$ for $i = 1, .., n$. It follows that $\hat{L}_c(0, .., 0, \bar{z}_i, 0, .., 0) = \frac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i)^{-1}$ and $\hat{L}_o(0, .., 0, \bar{z}_i, 0, .., 0) = \frac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i)$ for $i = 1, .., n$. We call a nonlinear system *balanced* if it has a controllability and observability function of the form of respectively (20) and (21). This means that we can balance system (11) by a coordinate transformation of the form $x = \chi(\bar{z}) := \psi(\eta^{-1}(\bar{z}))$ for $\bar{z} \in \bar{W}$, where ψ is as in Theorem 3.9.

4 Coprime representations

4.1 The normalized right coprime factorization

In this section we will give the characterization of a normalized right coprime factorization of a nonlinear system. Before we do this we discuss the concept of inner nonlinear systems, which is an extension of inner linear systems. For linear systems we have the following:

Definition 4.1 A stable transfer matrix $G(s) = D + C(sI - A)^{-1}B$ of a linear system (A, B, C, D) is called *inner* if $G(-s)^T G(s) = I$ for all $s \in \mathbb{C}$.

Consider the following nonlinear system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x) + Du \end{aligned} \tag{22}$$

with the same properties as system (11), D a $p \times m$ constant matrix and $f(x)$ asymptotically stable. Let the observability function of this system be smooth and finite. The Hamiltonian extension of this system is (see Crouch/van der Schaft 1987):

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ \dot{p} = -\left(\frac{\partial f}{\partial x}(x) + \frac{\partial g}{\partial x}(x)u\right)^T p - \left(\frac{\partial h}{\partial x}(x)\right)^T u_a \end{cases}$$
$$\begin{cases} y = h(x) + Du \\ y_a = g(x)^T p + D^T u_a \end{cases} \tag{23}$$

Now we take $u_a = y = h(x) + Du$ which results in:

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ \dot{p} = -\left(\frac{\partial f}{\partial x}(x) + \frac{\partial g}{\partial x}(x)u\right)^T p - \frac{\partial^T h}{\partial x}(x)(h(x) + Du) \end{cases}$$
$$y_a = g(x)^T p + D^T h(x) + D^T D u \tag{24}$$

For a linear system (A, B, C, D), the transfer matrix of (24) is given as $G(-s)^T G(s)$, with $G(s) = D + C(sI - A)^{-1}B$. Now we can give a characterization of an inner nonlinear system:

Definition 4.2 System (22) is called *inner* if the input-output map of system (24) from u to y_a, with $x(0) = 0$, is the identity.

The next theorem coincides with the linear case, see e.g. Ober/McFarlane 1989.

Theorem 4.3 *System (22) is inner if*

- $D^T D = I$ *and*
- $\frac{\partial L_o}{\partial x}(x) g(x) + h(x)^T D = 0$

Furthermore, if system (22) is reachable from 0*, these conditions are also necessary.*

Proof System (24) is a Hamiltonian system with Hamiltonian:

$$\hat{H}(x,p,u) = p^T(f(x)+g(x)u) + \frac{1}{2}h(x)^T h(x) + u^T D^T h(x) + \frac{1}{2}u^T D^T D u \tag{25}$$

Write $p = \bar{p} + \frac{\partial L_o}{\partial x}(x)$, then the Hamiltonian $\hat{H}$ transforms into $\bar{H}$:

$$\begin{aligned} \bar{H}(x,\bar{p},u) = \;& \bar{p}^T(f(x)+g(x)u) + \tfrac{1}{2}h(x)^T h(x) + u^T D^T h(x) + \tfrac{1}{2}u^T D^T D u \\ & + \tfrac{\partial L_o}{\partial x}(x) f(x) + \tfrac{\partial L_o}{\partial x}(x) g(x) u) + \tfrac{1}{2}u^T D^T D u = \bar{p}^T f(x) + \bar{p}^T g(x) u \\ & + u^T g(x)^T \tfrac{\partial^T L_o}{\partial x}(x) + u^T D^T h(x) + \tfrac{1}{2}u^T D^T D u \end{aligned}$$

From this we conclude: $(D^T D = I$ and $\frac{\partial L_o}{\partial x}(x)g(x) + h(x)^T D = 0) \iff \bar{H}(x,0,u) = \hat{H}(x, \frac{\partial^T L_o}{\partial x}(x), u) = \frac{1}{2}u^T u$, and since $p(t) = \frac{\partial^T L_o}{\partial x}(x(t))$ is the solution of $\dot{p}(t) = -\frac{\partial^T f}{\partial x}(x(t))p(t) - u(t)^T \frac{\partial^T g}{\partial x}(x(t))p(t) - \frac{\partial^T h}{\partial x}(x(t))\;(h(x(t)) + Du(t))$, where $p(0) = 0$ and $x(t)$ is the solution of $\dot{x}(t) = f(x(t)) + g(x(t))u(t)$, for $x(0) = 0$, we have proven the if part of the theorem. If we assume system (22) is reachable from 0 (e.g. the case that $g(x) = 0$ is excluded), it immediately follows that the reverse implication is also true. ∎

This characterization of inner nonlinear systems is of importance for the characterization of a normalized right coprime factorization of a nonlinear system. To make a right coprime factorization we consider a system of the form (11), i.e.

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x) \tag{26}$$

with the same properties as (11) and let the system be zero-state observable. Contrary to what is assumed in section 3 we do not assume the asymptotic stability of the system. Now consider the following two systems:

$$\begin{aligned} \dot{x} &= \tilde{f}(x) + g(x)w, \\ y &= h(x) \end{aligned} \tag{27}$$

$$\begin{aligned} \dot{x} &= \tilde{f}(x) + g(x)w, \\ u &= \tilde{h}(x) + w \end{aligned} \tag{28}$$

and the inverse system of (28):

$$\begin{aligned} \dot{x} &= \tilde{f}(x) - g(x)\tilde{h}(x) + g(x)u, \\ w &= u - \tilde{h}(x) \end{aligned} \tag{29}$$

If we can choose $\tilde{f}(x)$ and $\tilde{h}(x)$ such that $\tilde{f}(x) - g(x)\tilde{h}(x) = f(x)$ and $\tilde{f}(x)$ asymptotically stable, then taking the output of system (29) as an input of system (27) exactly results in system (26). This is a *right factorization* of system (26). Comparing this with the theory for linear systems at the end of section 2, we should take for system (26) the linear system (8) with transfer matrix $G(s) = N(s)D^{-1}(s)$, where $N(s)$ is the transfer matrix of system (27), $D(s)$ is the transfer matrix of system (28) and hence $D^{-1}(s)$ is the transfer matrix of the inverse system (29). Both $N(s)$ and $D(s)$ are stable transfer matrices and they have no zeros in the right half plane in common, which implies that the factorization is coprime. Furthermore the Graph operator is inner and therefore this is a normalized right coprime factorization.

As a nonlinear equivalent of coprimeness we say that, if $\tilde{f}(x)$ is asymptotically stable, the factorization given by (27) and (28) is coprime, if the zero dynamics of the combination of both systems is trivial. Together with the notion of an inner nonlinear system, this results in the following definition for a normalized right coprime representation:

Definition 4.4 A *normalized right coprime factorization* of system (26) is given by two systems of the form (27) and (28) where $\tilde{f}$ is asymptotically stable and the combination of both systems, i.e.

$$\begin{aligned} \dot{x} &= \tilde{f}(x) + g(x)w, \\ \begin{pmatrix} y \\ u \end{pmatrix} &= \begin{pmatrix} h(x) \\ \tilde{h}(x) \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} w \end{aligned} \tag{30}$$

is inner and has the property that $y \equiv 0$ and $u \equiv 0$ implies that $x \equiv 0$.

Consider the following Hamilton-Jacobi-Bellman equation, which is known from optimal control theory (see e.g. Lee/Markus 1967):

$$\frac{\partial V}{\partial x}(x)f(x) - \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x)g(x)^T\frac{\partial^T V}{\partial x}(x) + \frac{1}{2}h(x)^T h(x) = 0, \qquad V(0) = 0 \tag{31}$$

Now assume (31) has a smooth non-negative solution $V(x)$ and consider the following system:

$$\begin{aligned} \dot{x} &= \left(f(x) - g(x)g(x)^T\tfrac{\partial^T V}{\partial x}(x)\right) + g(x)w, \\ &\begin{cases} y = h(x) \\ u = -g(x)^T\frac{\partial^T V}{\partial x}(x) + w \end{cases} \end{aligned} \tag{32}$$

The following lemmas are related to results from e.g. Lee/Markus 1967:

Lemma 4.5 *System (32) is zero-state observable.*

Proof For (32) with $w \equiv 0$ and x a solution to $\dot{x} = f(x) - g(x)g(x)^T\frac{\partial^T V}{\partial x}(x)$ we have

$$-g(x)^T\frac{\partial^T V}{\partial x}(x) = 0 \Rightarrow \dot{x} = f(x)$$

Together with the zero-state observability of system (26) this results in

$$\begin{pmatrix} h(x) \\ -g(x)^T\frac{\partial^T V}{\partial x}(x) \end{pmatrix} = 0 \Rightarrow x = 0$$

This proves the lemma. ∎

Lemma 4.6 *Let V be a smooth positive definite solution to the Hamilton-Jacobi-Bellman equation (31), then $\dot{x} = f(x) - g(x)g(x)^T\frac{\partial^T V}{\partial x}(x)$ is locally asymptotically stable and globally asymptotically stable if V is proper.*

Proof

$$\frac{\partial V}{\partial x}(x)f(x) - \frac{\partial V}{\partial x}(x)g(x)g(x)^T\frac{\partial^T V}{\partial x}(x) = -\frac{1}{2}\frac{\partial V}{\partial x}(x)g(x)g(x)^T\frac{\partial^T V}{\partial x}(x) - \frac{1}{2}h(x)^T h(x) \le 0$$

By the zero-state observability of system (32) this implies for $\dot{x} = (f(x) - g(x)g(x)^T \frac{\partial^T V}{\partial x}(x)$:

$$-\frac{1}{2}\frac{\partial V}{\partial x}(x)g(x)g(x)^T \frac{\partial^T V}{\partial x}(x) - \frac{1}{2}h(x)^T h(x) = 0 \Rightarrow \left\{ g(x)^T \frac{\partial^T V}{\partial x}(x) = 0,\ h(x) = 0 \right\} \Rightarrow x = 0$$

Asymptotic stability now follows by LaSalle's invariance principle. ■

We can use this lemma to show that (32) represents the normalized right coprime factorization:

Theorem 4.7 *Let V be a smooth proper positive definite solution to the Hamilton-Jacobi-Bellman equation (31), then a normalized right coprime factorization of system (26) is given by the systems (27) and (28) where $\tilde{f}(x) = f(x) - g(x)g(x)^T \frac{\partial^T V}{\partial x}(x)$ and $\tilde{h}(x) = -g(x)^T \frac{\partial^T V}{\partial x}(x)$.*

Proof By lemma 4.6 we know $\tilde{f}(x) = f(x) - g(x)g(x)^T \frac{\partial^T V}{\partial x}(x)$ is globally asymptotically stable. Therefore we only have to prove that (32) is an inner system and that for (32) $y \equiv 0$ and $u \equiv 0$ implies that $x \equiv 0$. By section 3 the observability function of (32) is given by the smooth solution $\bar{L}_o > 0$ of the following Hamilton-Jacobi equation:

$$\frac{\partial \bar{L}_o}{\partial x}(x)(f(x) - g(x)g(x)^T \frac{\partial^T V}{\partial x}(x)) + \frac{1}{2}h(x)^T h(x) + \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x)g(x)^T \frac{\partial^T V}{\partial x}(x) = 0 \quad (33)$$

where $\bar{L}_o(0) = 0$. Obviously this equation has as the solution $\bar{L}_o = V$. Therefore we have

$$\frac{\partial \bar{L}_o}{\partial x}(x)g(x) + \begin{pmatrix} h(x)^T & -\frac{\partial V}{\partial x}(x)g(x) \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} = 0, \qquad \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} = I$$

and thus system (32) is inner. Furthermore, ($y = 0$ and $u = 0$) $\Rightarrow$ $\big($ $h(x) = 0$ and $w = g(x)^T \frac{\partial^T V}{\partial x}(x)$ $\big)$ $\Rightarrow$ ($\dot{x} = f(x)$ and $h(x) = 0$) and by zero-state observability of the original system (26) this implies that $x = 0$. This proves the theorem. ■

If we consider the linearization of all systems in this section, we find that this gives us the theory for normalized right coprime factorizations for linear systems, see section 2.

Theorem 4.8 *Consider the linearization of system (26) in $x = 0$ and $u = 0$ and assume it is minimal. Then it is of the form (8) of section 2, where $A = \frac{\partial f}{\partial x}(0)$, $B = g(0)$ and $C = \frac{\partial h}{\partial x}(0)$. The linearization of system (32) gives exactly the state space realization, given by (9), of the Graph operator of system (8), where $P = \frac{\partial^2 \bar{L}_o}{\partial x^2}(0)$.*

Proof Follows directly from linearization, see also Scherpen 1993. ■

In this section we developed a representation of the normalized right coprime factorization, which is stable and zero-state observable. With some extra assumptions we can apply the theory of balancing for stable nonlinear systems to this representation. This is done in section 5.2.

4.2 The normalized left coprime factorization

Similar to the foregoing section, where we discussed the concept of an inner nonlinear system, we discuss here co-inner nonlinear systems, which again is an extension of the theory for linear systems. We will use this to define a normalized left coprime factorization of a nonlinear system. For linear systems we have the following:

Definition 4.9 A stable transfer matrix $G(s) = D + C(sI - A)^{-1}B$ of a linear system (A, B, C, D) is called *co-inner* if $G(s)G(-s)^T = I$ for all $s \in \mathbb{C}$.

Consider system (22) and its Hamiltonian extension (23). Let the controllability function of (22) be smooth and finite. Now we take $u = y_a = g(x)^T p + D^T u_a$, which results in:

$$\begin{cases} \dot{x} = f(x) + g(x)g(x)^T p + g(x)D^T u_a \\ \dot{p} = -\left(\frac{\partial f}{\partial x}(x) + \frac{\partial g}{\partial x}(x)g(x)^T p + \frac{\partial g}{\partial x}(x)D^T u_a\right)^T p - \frac{\partial^T h}{\partial x}(x)u_a \\ y = h(x) + Dg(x)^T p + DD^T u_a \end{cases} \tag{34}$$

For a linear system (A, B, C, D) the transfer matrix of (34) is given as $G(s)G(-s)^T$, where $G(s) = D + C(sI - A)^{-1}B$. Now we can give a characterization of a co-inner nonlinear system:

Definition 4.10 System (22) is called *co-inner* if the input-output map of system (34) from u_a to y , with $x(0) = 0$, is the identity.

Theorem 4.11 *System (22) is co-inner if*

- $DD^T = I$ *and*
- $h(x) + Dg(x)^T \dfrac{\partial^T L_c}{\partial x}(x) = 0$

Furthermore, if (22) is reachable from 0*, these conditions are also necessary.*

Proof System (34) is a Hamiltonian system with Hamiltonian:

$$\hat{H}(x, p, u_a) = p^T f(x) + \frac{1}{2}p^T g(x)g(x)^T p + p^T g(x)D^T u_a + h(x)^T u_a + \frac{1}{2}u_a^T DD^T u_a \tag{35}$$

Write $p = \bar{p} + \frac{\partial L_c}{\partial x}(x)$, then the Hamiltonian $\hat{H}$ transforms into $\bar{H}$:

$$\begin{aligned} \bar{H}(x, \bar{p}, u_a) = \; & \bar{p}^T f(x) + \tfrac{\partial L_c}{\partial x}(x)f(x) + \tfrac{1}{2}\bar{p}^T g(x)g(x)^T p + \bar{p}^T g(x)g(x)^T \tfrac{\partial^T L_c}{\partial x}(x) + \\ & \tfrac{1}{2}\tfrac{\partial L_c}{\partial x}(x)g(x)g(x)^T \tfrac{\partial^T L_c}{\partial x}(x) + \bar{p}^T g(x)D^T u_a + \tfrac{\partial L_c}{\partial x}(x)g(x)D^T u_a + \\ & h(x)^T u_a + \tfrac{1}{2}u_a^T D^T D u_a = \bar{p}^T f(x) + \tfrac{1}{2}\bar{p}^T g(x)g(x)^T \bar{p} + \\ & \bar{p}^T g(x)g(x)^T \tfrac{\partial^T L_c}{\partial x}(x) + \bar{p}^T g(x)D^T u_a + \tfrac{\partial L_c}{\partial x}(x)g(x)D^T u_a + h(x)^T u_a + \\ & \tfrac{1}{2}u_a^T D^T D u_a \end{aligned}$$

From this we conclude: $(DD^T = I$ and $h(x) + Dg(x)^T \frac{\partial^T L_c}{\partial x}(x) = 0) \iff \bar{H}(x, 0, u_a) = \hat{H}(x, \frac{\partial^T L_c}{\partial x}(x), u_a) = \frac{1}{2}u_a^T u_a$ and since $p(t) = \frac{\partial^T L_c}{\partial x}(x(t))$ is the solution of $\dot{p} = -\frac{\partial^T f}{\partial x}(x)p - p^T g(x)\frac{\partial^T g}{\partial x}(x)p - u_a^T D\frac{\partial^T g}{\partial x}(x)p - \frac{\partial^T h}{\partial x}(x)u_a$ where $p(0) = 0$ and $x(t)$ is the solution of $\dot{x} = f(x) + g(x)g(x)^T p + g(x)D^T u_a$, for $x(0) = 0$, we have proven the if part of the theorem. If we assume system (22) is reachable from 0, it immediately follows that the reverse implication is also true. ∎

As we said before, this characterization of co-inner nonlinear systems is of importance for the characterization of a normalized left coprime factorization of a nonlinear system. To make a normalized left coprime factorization we consider a system of the form (11) and also (26), i.e.

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x) \end{aligned} \tag{36}$$

with the same properties as (11) and let the system be zero-state observable. Again, contrary to what is assumed in section 3 we do not assume the asymptotic stability of the system. Now we consider the equation $z = h(x) - y$ to make a left coprime factorization. In the linear case, see section 2, the realization (10) of the transfer matrix $(\,\tilde{N}(s) \quad \tilde{D}(s)\,)$ is such that the transfer from y and u to z gives the left factorization of the system. This formulation does not hold for nonlinear systems, since we can not give two independent input-output maps from y to z and from u to z, but if we consider $z = h(x) - y$ and its kernel, we can extend the linear theory to come up with the following definition:

Definition 4.12 A *normalized left coprime factorization* of system (36) is represented by a system of the form

$$\begin{aligned} \dot{x} &= \tilde{f}(x) + (\, g(x) \quad \tilde{g}(x)\,)\,\tilde{w} \\ z &= h(x) + (\,0 \quad -I\,)\,\tilde{w} \end{aligned} \tag{37}$$

where $\tilde{f}$ is asymptotically stable. The system is co-inner and if we take $\tilde{w} = \begin{pmatrix} u \\ y \end{pmatrix}$ as the input variable, then the zero-dynamics of (37) for $z = 0$ gives the dynamics of system (36). Additionally, the system has a stable right inverse, i.e. there exists a stable system

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x}) + \bar{g}(\bar{x})s \\ \bar{z} &= \bar{h}(\bar{x}) + \bar{D}s \end{aligned} \tag{38}$$

such that for $x(0) = \bar{x}(0)$ and $\tilde{w} = \bar{z}$, the output z of system (37) is $z = s$.

Remark 4.13 The above definition is not strictly analogous to the definition of the normalized right coprime factorization (Definition 4.4), since here the coprimeness is stated in terms of a stable right inverse. It seems complicated to give a satisfying definition for the left coprimeness in terms of the input, output and the state, while this is easier done for the right coprimeness. The existence of a stable right inverse means that the Bezout identity (see e.g. Paice 1992) is fulfilled, which implies the left coprimeness of the system (37). Using stable left inverses, we can give a similar definition for right coprimeness, but this is a little more restricted than the definiton we used, see Remark 4.17.

It is easily checked that the linear system (10) of section 2 is a normalized left coprime factorization according to Definition 4.12. In section 2 we saw that the linear theory makes use of the FARE. Similarly, here we introduce the following Hamilton-Jacobi-Bellman equation:

$$\frac{\partial W}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial W}{\partial x}(x)g(x)g(x)^T\frac{\partial^T W}{\partial x}(x) - \frac{1}{2}h(x)^T h(x) = 0, \quad W(0) = 0 \tag{39}$$

Assume this equation has a smooth non-negative solution $W(x)$ on a neighborhood Y of 0. The following result is known from optimal control theory, see e.g. Lee/Markus 1967.

Lemma 4.14 *Let W be a smooth positive definite solution on Y to the Hamilton-Jacobi-Bellman equation (39), then $\dot{x} = -(f(x) + g(x)g(x)^T\frac{\partial^T W}{\partial x}(x))$ is locally asymptotically stable and asymptotically stable on Y if W is proper on Y.*

Proof From (39) we obtain

$$\begin{aligned}\frac{\partial W}{\partial x}(x)\left(-\left(f(x)+g(x)g(x)^T\frac{\partial^T W}{\partial x}(x)\right)\right) = & -\tfrac{1}{2}\frac{\partial W}{\partial x}(x)g(x)g(x)^T\frac{\partial^T W}{\partial x}(x) \\ & -\tfrac{1}{2}h(x)^T h(x) \leq 0\end{aligned}$$

Together with the zero-state observability of system (36) we have

$$\begin{pmatrix} h(x) \\ g(x)^T\frac{\partial^T W}{\partial x}(x)\end{pmatrix} = 0 \Rightarrow x = 0$$

Asymptotic stability now follows by LaSalle's invariance principle. ■

This lemma explains why sometimes W is called the anti-stabilizing solution of (39). To construct a normalized left coprime factorization we need a stable vectorfield $\tilde{f}$, see definition 4.12. It follows from (39) that $\frac{\partial W}{\partial x}(0) = 0$ and thus we can write $\frac{\partial W}{\partial x}(x) = x^T M(x)$, where $M(x)$ an $n \times n$ matrix is, with all entries smooth functions of x and $M(0) = \frac{\partial^2 W}{\partial x^2}(0)$, see Milnor 1963. We assume $\frac{\partial^2 W}{\partial x^2}(0) > 0$ and therefore there exists a neighborhood U of 0 for which $M(x)$ is nonsingular and thus is invertible on U. Furthermore, since $h(0) = 0$, we can write $h(x) = C(x)x$ where $C(x)$ is an $p \times n$ matrix with entries that are smooth functions of x and $C(0) = \frac{\partial h}{\partial x}(0)$. Now consider for $x \in U$

$$\begin{aligned}\dot{x} &= \left(f(x) - (M(x))^{-1}C(x)^T C(x)x\right) + \begin{pmatrix} g(x) & (M(x))^{-1}C(x)^T\end{pmatrix}\tilde{w} \\ z &= h(x) + \begin{pmatrix} 0 & -I\end{pmatrix}\tilde{w}\end{aligned} \tag{40}$$

Lemma 4.15 *Let W be a smooth positive definite solution to the Hamilton-Jacobi-Bellman equation (39), then $\dot{x} = f(x) - (M(x))^{-1}C(x)^T C(x)x$ is locally asymptotically stable and asymptotically stable on U if W is proper on U.*

Proof Writing (39) with $\frac{\partial W}{\partial x}(x) = x^T M(x)$ and $h(x) = C(x)x$, we get

$$\begin{aligned}& x^T M(x)f(x) + \tfrac{1}{2}x^T M(x)g(x)g(x)^T M(x)^T x - \tfrac{1}{2}x^T C(x)^T C(x)x = \\ & x^T M(x)\left(f(x) - (M(x))^{-1}C(x)^T C(x)x\right) + \tfrac{1}{2}x^T M(x)g(x)g(x)^T M(x)^T x + \\ & \tfrac{1}{2}x^T C(x)^T C(x)x = 0\end{aligned}$$

From this we obtain

$$\begin{aligned}x^T M(x)\left(f(x) - (M(x))^{-1}C(x)^T C(x)x\right) = & -\tfrac{1}{2}x^T M(x)g(x)g(x)^T M(x)^T x \\ & -\tfrac{1}{2}x^T C(x)^T C(x)x \leq 0\end{aligned}$$

By the zero-state observability of system (36) this implies for $\dot{x} = f(x) - (M(x))^{-1}C(x)^T C(x)x$:

$$\begin{aligned}& -\tfrac{1}{2}x^T M(x)g(x)g(x)^T M(x)^T x - \tfrac{1}{2}x^T C(x)^T C(x)x = 0 \Rightarrow \\ & \left\{g(x)^T M(x)^T x = 0, \quad h(x) = C(x)x = 0\right\} \Rightarrow x = 0\end{aligned}$$

Asymptotic stability now follows by LaSalle's invariance principle. ■

Theorem 4.16 *Let W and V be a smooth proper positive definite solutions to the Hamilton-Jacobi-Bellman equation (39) respectively (31), then a representation of the normalized left coprime factorization of system (36) on U is given by the system (40).*

Proof By Lemma 4.15 we know $\dot{x} = f(x) - (M(x))^{-1} C(x)^T C(x)x$ is asymptotically stable. Obviously for $\tilde{w} = \begin{pmatrix} u \\ y \end{pmatrix}$ and $z = 0$ we get the dynamics of the original system (36). Therefore we only have to prove that (40) is an co-inner system and that there exist a stable right inverse of the form (38) for (40). By section 3 the controllability function of (40) is given by the smooth solution $\bar{L}_c \geq 0$ of the following Hamilton-Jacobi equation:

$$\begin{array}{l} \frac{\partial \bar{L}_c}{\partial x}(x)\left(f(x) - (M(x))^{-1} C(x)^T C(x)x\right) + \frac{1}{2}\frac{\partial \bar{L}_c}{\partial x}(x)g(x)g(x)^T \frac{\partial^T \bar{L}_c}{\partial x}(x) + \\ \frac{1}{2}\frac{\partial \bar{L}_c}{\partial x}(x)\,(M(x))^{-1} C(x)^T C(x)\,(M(x))^{-T}\, \frac{\partial^T \bar{L}_c}{\partial x}(x) = 0 \end{array} \tag{41}$$

where $\bar{L}_c(0) = 0$. Obviously this equation has as solution $\bar{L}_c = W$ and thus the controllability function exists on U. Therefore $\frac{\partial^T \bar{L}_c}{\partial x}(x) = M(x)^T x$ and we have

$$C(x)x + \begin{pmatrix} 0 & -I \end{pmatrix} \begin{pmatrix} g(x)^T \\ C(x)(M(x))^{-T} \end{pmatrix} \frac{\partial^T \bar{L}_c}{\partial x}(x) = 0, \quad \begin{pmatrix} 0 & -I \end{pmatrix} \begin{pmatrix} 0 \\ -I \end{pmatrix} = I$$

and thus by Theorem 4.11 system (40) is co-inner. Furthermore, consider the system

$$\begin{array}{l} \dot{\bar{x}} = f(\bar{x}) - g(\bar{x})g(\bar{x})^T \frac{\partial^T V}{\partial x}(\bar{x}) - (M(\bar{x}))^{-1} C(\bar{x})^T s \\ \bar{z} = \begin{pmatrix} -g(\bar{x})^T \frac{\partial^T V}{\partial x}(\bar{x}) \\ h(\bar{x}) \end{pmatrix} + \begin{pmatrix} 0 \\ -I \end{pmatrix} s \end{array} \tag{42}$$

then by Lemma 4.6 we know that $f(\bar{x}) - g(\bar{x})g(\bar{x})^T \frac{\partial^T V}{\partial x}(\bar{x})$ is asymptotically stable. Take $\tilde{w} = \bar{z}$ as input for system (40) and $x(0) = \bar{x}(0)$, then the output for system (40) is $z = s$ and hence the system is left coprime. This proves the theorem. ∎

Remark 4.17 In Remark 4.13 we mentioned that right coprimeness can be defined in a similiar way as left coprimeness. Then we need to construct a stable left inverse of system (32), such that the serial connection of the two systems is the identity mapping. This can be done by considering the smooth proper positive definite solution to the Hamilton-Jacobi equation (39) and by constructing the left inverse in a similar way as (42) is constructed. Such a construction is analogous to the linear case which is treated in Nett/Jacobson/Balas 1984. This paper considers a doubly coprime fractional representation for a linear system, and if we follow this approach, we can construct a similar representation for the nonlinear case. For the normalized right coprime factorization this approach is a little more restrictive than the approach of section 4.1, since the inverse of the matrix $M(x)$ is needed, which might not exist globally.

Similar to the right coprime factorization, we have the following connection with the linearization.

Theorem 4.18 *Consider the linearization of system (36) in $x = 0$ and $u = 0$, which has the form of system (8) with transfer matrix $G(s)$, see section 2. The linearization of system (40) gives exactly the state space realization (10) of $\begin{pmatrix} \tilde{N}(s) & \tilde{D}(s) \end{pmatrix}$, where $\tilde{N}(s)$ and $\tilde{D}(s)$ are the left factors of $G(s)$. Moreover, $S = \left(\frac{\partial^2 \bar{L}_c}{\partial x^2}(0)\right)^{-1} = (M(0))^{-1}$.*

Proof Follows directly from linearization. ∎

5 Balancing for unstable nonlinear systems

5.1 The HJB balanced form

For closed loop balancing we first follow the idea of Weiland 1991 which has been treated briefly in section 2.1. Consider the system (11)

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{43}$$

We assume the system is zero-state observable. First we define the following energy functions (see also section 2.1):

Definition 5.1 The past and future energy function of a nonlinear system are defined as

$$K^-(x_0) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} (\| y(t) \|^2 + \| u(t) \|^2) dt \tag{44}$$

respectively

$$K^+(x_0) = \min_{\substack{u \in L_2(0,\infty) \\ x(\infty)=0,\, x(0)=x_0}} \frac{1}{2}\int_{0}^{\infty} (\| y(t) \|^2 + \| u(t) \|^2) dt \tag{45}$$

We assume that $K^-(x_0)$ and $K^+(x_0)$ are *smooth* and *finite*. From optimal control theory we know that K^+ is the smooth non-negative solution to the Hamilton-Jacobi-Bellman equation (31) from section 4.1. Furthermore, K^- is the smooth non-negative solution to the Hamilton-Jacobi-Bellman equation (39) from section 4.2, see e.g. Lee/Markus 1967. K^+ is minimized by an input $u = -g(x)^T \frac{\partial^T K^+}{\partial x}(x)$ and K^- by an input $u = g(x)^T \frac{\partial^T K^-}{\partial x}(x)$. Summarizing we get the following theorem:

Theorem 5.2 *K^+ is the smooth non-negative solution to the following Hamilton-Jacobi-Bellman equation:*

$$\frac{\partial K^+}{\partial x}(x)f(x) - \frac{1}{2}\frac{\partial K^+}{\partial x}(x)g(x)g(x)^T\frac{\partial^T K^+}{\partial x}(x) + \frac{1}{2}h^T(x)h(x) = 0, \qquad K^+(0) = 0 \tag{46}$$

satisfying $f(x) - g(x)g(x)^T\frac{\partial^T K^+}{\partial x}(x)$ *is asymptotically stable. Furthermore,* K^- *is the smooth non-negative solution to the following Hamilton-Jacobi-Bellman equation:*

$$\frac{\partial K^-}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial K^-}{\partial x}(x)g(x)g^T(x)\frac{\partial^T K^-}{\partial x}(x) - \frac{1}{2}h(x)^T h(x) = 0, \qquad K^-(0) = 0 \tag{47}$$

satisfying $-(f(x) + g(x)g(x)^T\frac{\partial^T K^-}{\partial x}(x))$ *is asymptotically stable.*

Note that equation (46) is the same as equation (31) and that equation (47) is the same as equation (39).

Remark 5.3 If we assume there exists a smooth solution K of (46), such that $f(x) - g(x)g(x)^T\frac{\partial^T K}{\partial x}(x)$ is asymptotically stable, then K^+ as in (45) exists, i.e. is finite. Similarly, it is also valid for (47) and K^-, see cf. Lee/Markus 1967.

Theorem 5.4 *Assume (46) has a smooth proper solution K on W. Then $K(x_0) > 0$ for $x_0 \in W$, $x_0 \neq 0$, if and only if $f(x) - g(x)g(x)^T \frac{\partial^T K}{\partial x}(x)$ is asymptotically stable on W. Similarly, assume (47) has a smooth proper solution $\bar{K}$ on W, then $\bar{K}(x_0) > 0$ for $x_0 \in W$, $x_0 \neq 0$, if and only if $-(f(x) + g(x)g(x)^T \frac{\partial^T \bar{K}}{\partial x}(x))$ is asymptotically stable on W.*

Proof Assume $K > 0$, then by Lemma 4.6 $f(x) - g(x)g(x)^T \frac{\partial^T K}{\partial x}(x)$ is asymptotically stable on W.
Now assume $f(x) - g(x)g(x)^T \frac{\partial^T K}{\partial x}(x)$ is asymptotically stable on W. By Theorem 5.2 we know that $K = K^+$ on W, where K^+ is the future energy function of system (43).

$$K^+(x_0) = \min_{\substack{u \in L_2(0,\infty) \\ x(\infty)=0, x(0)=x_0}} \frac{1}{2}\int_0^\infty (\| y(t) \|^2 + \| u(t) \|^2) dt$$

Furthermore we know from optimal control theory that the minimum is reached for $u = -g(x)^T \frac{\partial^T K^+}{\partial x}(x)$. Hence

$$K^+(x_0) = \frac{1}{2}\int_0^\infty \frac{\partial K^+}{\partial x}(x(t))g(x(t))g(x(t))^T \frac{\partial^T K^+}{\partial x}(x(t)) + h(x(t))^T h(x(t)) dt$$

Now let $x_0 \neq 0$. If $\frac{\partial K^+}{\partial x}(x(t))g(x(t))g(x(t))^T \frac{\partial^T K^+}{\partial x}(x(t)) + h(x(t))^T h(x(t)) = 0$ for $0 \leq t < \infty$ then $u(t) = 0$ and $h(x(t)) = 0$, for all t, $0 \leq t < \infty$. But by the zero-state observability of system (43) this means that $x(t) = 0$ for all $0 \leq t < \infty$ and this contradicts $x_0 \neq 0$. Hence $K^+(x_0) > 0, \forall x_0 \in W$, $x_0 \neq 0$.
The second part of the theorem can be proven by using the same type of arguments. ∎

Now we consider nonlinear systems of the form (43) with future and past energy function respectively K^+ and K^- as in definition 5.1 and with the following additional assumptions:

1. K^+ and K^- are smooth and finite on a neighborhood Y of 0
2. $\frac{\partial^2 K^+}{\partial x^2}(0) > 0$ and $\frac{\partial^2 K^-}{\partial x^2}(0) > 0$
3. the system is zero-state observable on Y

Similar to Lemma 3.8 and Theorem 3.9 we can bring K^+ and K^- into a special form:

Lemma 5.5 *There exists a coordinate transformation $x = \varphi(\bar{x})$, $\varphi(0) = 0$, such that $K^-(x)$ in the new coordinates $\bar{x} = \varphi^{-1}(x)$ is of the following form:*

$$K^-(\varphi(\bar{x})) = \frac{1}{2}\bar{x}^T \bar{x} \tag{48}$$

Furthermore we can write $K^+(x)$ in the new coordinates $\bar{x} = \varphi^{-1}(x)$ in the following form:

$$K^+(\varphi(\bar{x})) = \frac{1}{2}\bar{x}^T N(\bar{x})\bar{x} \quad \textit{where} \quad N(0) = \frac{\partial^2 K^+}{\partial x^2}(0) \tag{49}$$

where $N(\bar{x})$ is a $n \times n$ symmetric matrix with entries which are smooth functions of $\bar{x}$.

Proof This follows the proof of Lemma 3.8. ∎

Theorem 5.6 *Consider system (43) and assume there exists a neighborhood V of 0 where the number of distinct eigenvalues of $N(\bar{x})$ is constant for $\bar{x} \in V$. On a neighborhood U of zero there exists a coordinate transformation $x = \gamma(z)$, $\gamma(0) = 0$, such that $K^-(x)$ in the new coordinates $z \in W := \gamma^{-1}(U)$ is of the following form:*

$$\tilde{K}^-(z) := K^-(\gamma(z)) = \frac{1}{2} z^T z \tag{50}$$

while in the new coordinates $K^+(x)$ is of the following form:

$$\tilde{K}^+(z) := K^+(\gamma(z)) = \frac{1}{2} z^T \begin{pmatrix} v_1(z) & & 0 \\ & \ddots & \\ 0 & & v_n(z) \end{pmatrix} z \tag{51}$$

where $v_1(z) \geq ... \geq v_n(z)$ are smooth functions of z, called the HJB singular value functions (HJB stands for Hamilton-Jacobi-Bellman).

Proof This follows the proof of Theorem 3.9. ■

Remark 5.7 For linear systems the HJB singular value functions v_i, $i = 1, .., n$ are constant and are equal to the squared similarity invariants of Theorem 2.1.

Like in section 3.1 the form of the past and future energy function in (50) and (51) is not yet entirely the form we want. For that we need an additional coordinate transformation. We take as transformation $\bar{z}_i = \xi_i(z_i) := v_i(0, .., 0, z_i, 0, ..0)^{\frac{1}{4}} z_i$, $i = 1, .., n$ and hence $\bar{z} = \xi(z) := (\, \xi_1(z_1) \;\; \ldots \;\; \xi_n(z_n)\,)$ on $\bar{z} \in \bar{W} := \xi(W)$. Define $\bar{K}^-(\bar{z}) := \tilde{K}^-(\xi^{-1}(\bar{z}))$ and $\bar{K}^+(\bar{z}) := \tilde{K}^+(\xi^{-1}(\bar{z}))$. Then (50) and (51) become respectively:

$$\bar{K}^-(\bar{z}) = \frac{1}{2} \bar{z}^T \begin{pmatrix} \mu_1(\bar{z}_1)^{-1} & & 0 \\ & \ddots & \\ 0 & & \mu_n(\bar{z}_n)^{-1} \end{pmatrix} \bar{z} \tag{52}$$

$$\bar{K}^+(\bar{z}) = \frac{1}{2} \bar{z}^T \begin{pmatrix} \mu_1(\bar{z}_1)^{-1} v_1(\xi^{-1}(\bar{z})) & & 0 \\ & \ddots & \\ 0 & & \mu_n(\bar{z}_n)^{-1} v_n(\xi^{-1}(\bar{z})) \end{pmatrix} \bar{z} \tag{53}$$

where $\mu_i(\bar{z}_i) = v_i(0, .., 0, \xi_i^{-1}(\bar{z}_i), 0, .., 0)^{\frac{1}{2}}$ for $i = 1, .., n$. It follows that $\bar{K}^-(0, .., 0, \bar{z}_i, 0, .., 0) = \frac{1}{2}\bar{z}_i^2 \mu_i(\bar{z}_i)^{-1}$ and $\bar{K}^+(0, .., 0, \bar{z}_i, 0, .., 0) = \frac{1}{2}\bar{z}_i^2 \mu_i(\bar{z}_i)$ for $i = 1, .., n$. In terms of the past and future energy we can infer from $v_i(\xi^{-1}(\bar{z})) > v_{i+1}(\xi^{-1}(\bar{z}))$ that the state component $\bar{z}_i$ is more important than the state component $\bar{z}_{i+1}$ on $\bar{W}$. We call a nonlinear system *HJB balanced* if it has a past and future energy function respectively of the form (52) and (53). This means that we can bring system (43) in a HJB balanced form by a coordinate transformation of the form $x = \alpha(\bar{z}) := \gamma(\xi^{-1}(\bar{z}))$ where γ is as in Theorem 5.6. For a linear system this means that the system is in the LQG balanced form, since then $\bar{K}^-(\bar{z}) = \frac{1}{2}\bar{z}^T S^{-1} \bar{z}$ and $\bar{K}^+(\bar{z}) = \frac{1}{2}\bar{z}^T P \bar{z}$ with $P = S = M$ as in Theorem 2.1.

For linear systems HJB balancing is the same as LQG balancing. However, the formulation of LQG balancing for linear systems can not be extended easily to nonlinear systems. This is nevertheless an interesting problem to consider. The usual stochastic formulation of the

LQG problem seems not to be the right formulation for nonlinear systems. However, there exists a *deterministic* formulation of the LQG problem, which is equivalent to the stochastic formulation, and which has been extended to nonlinear systems, see Mortensen 1968 and Hijab 1980. Consider the following system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \quad x(0) = x_0 \\ y &= h(x) + v \end{aligned} \tag{54}$$

where v is the (deterministic) noise that enters the system. We want to minimize the following energy functional:

$$J(x_0, u(t), v(t)) = W_0(x_0) + \frac{1}{2}\int_0^t \| u(\tau) \|^2 + \| v(\tau) \|^2 \, d\tau \tag{55}$$

over the input triple $(x_0, u(\cdot), v(\cdot))$, which corresponds uniquely with an input pair $(x_0, u(\cdot))$, if the observations $y(\cdot)$ are given, see Hijab 1980. $W_0(x_0)$ is a real-valued function representing the initial costs, with $W_0(0) = 0$. Let $\hat{x}(t) \in \mathbb{R}^n$ be a deterministic estimate of the state at time t, $t \geq 0$, given the observations $y(\tau)$, $0 \leq \tau \leq t$ (i.e. $\hat{x}(t)$ is the endpoint of the state trajectory of an input pair $(\hat{x}_0, \hat{u}(\cdot))$ that is minimizing the energy functional (55) based on the observations). Let W_0 generate a non-degenerate estimate $\hat{x}_0$, i.e. $\det\left(\frac{\partial^2 W_0}{\partial x^2}(\hat{x}(0))\right) \neq 0$, then for small $t \geq 0$ and for x near $\hat{x}(0)$ we will have $\det\left(\frac{\partial^2 W}{\partial x^2}(t, x)\right) \neq 0$. Now the dynamics of the deterministic etimate $\hat{x}(t)$ is given by (see Hijab 1980):

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u + \left(\frac{\partial^2 W}{\partial x^2}(t, \hat{x})\right)^{-1} \left(\frac{\partial h}{\partial x}(\hat{x})\right)^T (y(t) - h(\hat{x})) \tag{56}$$

where $W(t, x)$ is a smooth solution of the Mortensen equation:

$$\frac{\partial W}{\partial t} + h(x)y(t) + \frac{\partial W}{\partial x} f(x) + \frac{1}{2}\frac{\partial W}{\partial x} g(x)g(x)^T \frac{\partial^T W}{\partial x} - \frac{1}{2}h(x)^T h(x) = 0, \tag{57}$$

$W(0, \cdot) = W_0(\cdot)$, and $V(x)$ is the smooth positive solution of the Hamilton-Jacobi-Bellman equation (46):

$$\frac{\partial V}{\partial x}(x)f(x) - \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x)g(x)^T \frac{\partial^T V}{\partial x}(x) + \frac{1}{2}h^T(x)h(x) = 0, \quad V(0) = 0 \tag{58}$$

Based on the separation principles, this motivates to consider the nonlinear compensator (56) together with

$$u = -g(\hat{x})^T \frac{\partial^T V}{\partial x}(\hat{x}) \tag{59}$$

If system (54) is linear, then (56) is equal to the compensator (3) of section 2.1. In that case $\left(\frac{\partial^2 W}{\partial x^2}(t, x)\right)^{-1}$ is constant, i.e. is equal to the matrix S that is the stabilizing solution of equation (4) (the FARE). This results directly from (57), see e.g. Hijab 1980 and van der Schaft 1993. For general nonlinear systems equation (57) does not have a solution such that $\frac{\partial^2 W}{\partial x^2}(t, x)$ is independent of the time t. Nevertheless we observe that equation (47) is part of the equation (57). Indeed for $y(t) \equiv 0$ for all $-\infty < t \leq 0$, $W(t, x) = K^-(x)$ is a smooth solution of (57). In this case we can obtain the HJB singular value functions from the solutions W and V of the equations (57) and (58) as is done above for respectively K^+

and K^-, and then the HJB singular value functions are a measure for the difficulties both to control and filter the corresponding state component.

Like in the linear case, we can use HJB-balancing for model reduction. The HJB singular value functions are a measure for the importance of a state component in terms of the past and future energy functions and, as we argued above, in a weaker sense they are a measure for the difficulties both to control and filter a state component.

5.2 Balancing of the coprime representation

In section 4 we discussed the normalized coprime factorization of a nonlinear system. We explain here how to balance the representation of the normalized right coprime factorization of a nonlinear system. We don't go into balancing of the representation of the normalized left coprime factorization, since we saw that in the linear case this comes down to the same singular values as for balancing the normalized right coprime factorization and thus the same reduced model. In the nonlinear case we obtain something similar. Furthermore, the normalized left coprime factorization is more restricted, since we need that the matrix $M(x)$ in equation (40) is nonsingular.

Consider section 4.1 and the representation of the normalized coprime factorization of system (26) given by system (32). From the proof of Theorem 4.7 we conclude that the observability function $\bar{L}_o$, of system (32) is well defined and is the positive definite solution of (31). Additionally, let us assume that the controllability function $\bar{L}_c$ of (32) is smooth and exist. Then it fulfills

$$\frac{\partial \bar{L}_c}{\partial x}(x)(f - g(x)g(x)^T \frac{\partial^T \bar{L}_o}{\partial x}(x)) + \frac{1}{2}\frac{\partial \bar{L}_c}{\partial x}(x)g(x)g(x)^T \frac{\partial^T \bar{L}_c}{\partial x} = 0 \tag{60}$$

satisfying $\bar{L}_c > 0$. To apply the theory of section 3 we also assume that

- $\frac{\partial^2 \bar{L}_o}{\partial x^2}(0) > 0$ and $\frac{\partial^2 \bar{L}_c}{\partial x^2}(0) > 0$

We can apply Theorem 3.9 to the system (32) and we call the singular value functions of this system the *Graph singular value functions* of the original system (26). Analogously to Theorem 2.4 for linear systems we obtain in the nonlinear case the slightly weaker result:

Theorem 5.8 $\bar{L}_o \leq \bar{L}_c$.

Proof After subtracting (60) from (33) and rewriting the resulting equation, we get:

$$\frac{\partial(\bar{L}_c - \bar{L}_o)}{\partial x}(x)(-(f(x) + g(x)g(x)^T \frac{\partial^T(\bar{L}_c - \bar{L}_o)}{\partial x}(x))) + \frac{1}{2}\frac{\partial(\bar{L}_c - \bar{L}_o)}{\partial x}(x)g(x)g(x)^T \frac{\partial^T(\bar{L}_c - \bar{L}_o)}{\partial x}(x)$$
$$+\frac{1}{2}h(x)^T h(x) = 0$$

By the asymptotic stability of $-(f(x) + g(x)g(x)^T \frac{\partial^T(\bar{L}_c - \bar{L}_o)}{\partial x}(x))$ we have that $\bar{L}_c - \bar{L}_o \geq 0$ and thus $\bar{L}_o \leq \bar{L}_c$. ■

In particular, this implies that the Graph singular value functions $\tilde{\tau}_i(x)$, $i = 1, .., n$, of a nonlinear system satisfy $\tilde{\tau}_i(0, .., 0, x_i, 0, .., 0) \leq 1$ for $i = 1, .., n$.

In section 2.2 we discussed balancing of the coprime representation of a linear system. We also gave the relation between the Graph Hankel singular values, τ_i, $i = 1,..,n$, and the similarity invariants μ_i, $i = 1,..,n$, which gives the relation between LQG-balancing and balancing of the normalized right coprime representation. For nonlinear systems we can find a similar relation.

Consider a system of the form (43) and assume the assumptions 1 to 3 from section 5.1 are fulfilled. Now we consider the Hamilton-Jacobi-Bellman equations (46) and (47), which have as smooth solutions the future and past energy function K^+ and K^-. Furthermore consider the equations (31) and (60) which have as solution the observability respectively controllability function, $\bar{L}_o$ and $\bar{L}_c$ of system (32), which is representing the normalized right coprime factorization of the original system (43).

Theorem 5.9 *The solutions of (31) and (60) are related to the solutions of (46) and (47) by $K^+ = \bar{L}_o$ and $K^- = \bar{L}_c - \bar{L}_o$.*

Proof Obviously (31) and (46) are the same equations and hence $K^+ = \bar{L}_o$. From (60) we obtain:

$$\frac{\partial \bar{L}_c}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial(\bar{L}_o - \bar{L}_c)}{\partial x}(x)g(x)g(x)^T\frac{\partial^T(\bar{L}_o - \bar{L}_c)}{\partial x}(x) - \frac{1}{2}\frac{\partial \bar{L}_o}{\partial x}(x)g(x)g(x)^T\frac{\partial^T \bar{L}_o}{\partial x}(x) = 0$$

If we subtract (31) from this, we obtain:

$$\frac{\partial(\bar{L}_c - \bar{L}_o)}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial(\bar{L}_c - \bar{L}_o)}{\partial x}(x)g(x)g(x)^T\frac{\partial^T(\bar{L}_c - \bar{L}_o)}{\partial x}(x) - \frac{1}{2}h(x)^T h(x) = 0$$

and hence $K^- = \bar{L}_c - \bar{L}_o$. ∎

Corollary 5.10 *$\bar{L}_o(x) < \bar{L}_c(x)$ for $x \neq 0$.*

To find the relation between the HJB singular value functions and the Graph singular value functions, we assume that the representation of the normalized right coprime factorization (32) of system (43) has the form such that the observability and controllability function for $x \in U$ are of the following form:

$$\bar{L}_c(x) = \frac{1}{2}x^T x$$

$$\bar{L}_o(x) = \frac{1}{2}x^T \begin{pmatrix} \tilde{\tau}_1(x) & & 0 \\ & \ddots & \\ 0 & & \tilde{\tau}_n(x) \end{pmatrix} x$$

where the $\tilde{\tau}_i(x)$'s are the Graph singular value functions as defined in section 3.2. From Corollary 5.10 we obtain that $\tilde{\tau}_i(0,..,0,x_i,0,..,0) < 1$ for $i = 1,..,n$. Furthermore we assume that for $x \in U$, with U a neighborhood of 0, the transformation which is necessary to bring the system in the form of Theorem 5.6, is $z = \gamma(x)$, $\gamma(0) = 0$, for $z \in W := \gamma(U)$. Hence $\tilde{K}^-(z) := K^-(\gamma^{-1}(z))$ is of the form (50) and $\tilde{K}^-(z) := K^+(\gamma^{-1}(z))$ is of the form (51) where the $v_i(z)$'s are the HJB singular value functions.

Theorem 5.11 *There exists a neighborhood U of 0 such that by the coordinate transformation $z = \gamma(x)$ for all $z \in W = \gamma(U)$:*

$$v_i(z) = \frac{\tilde{\tau}_i(\gamma^{-1}(z))}{1 - \tilde{\tau}_i(\gamma^{-1}(z))}, \qquad i = 1,..,n$$

.

Proof Since $\tilde{\tau}_i(0,..,0,x_i,0,..,0) < 1$, $i = 1,..,n$, and by continuity, there exists a neighborhood U of 0 such that $\tilde{\tau}_i(x) < 1$ for all $x \in U$. By the forms of the observability and controllability functions we infer that K^+ and K^- are of the following form:

$$K^+(x) = \frac{1}{2}x^T \begin{pmatrix} \tilde{\tau}_1(x) & & 0 \\ & \ddots & \\ 0 & & \tilde{\tau}_n(x) \end{pmatrix} x$$

$$K^-(x) = \frac{1}{2}x^T \begin{pmatrix} 1-\tilde{\tau}_1(x) & & 0 \\ & \ddots & \\ 0 & & 1-\tilde{\tau}_n(x) \end{pmatrix} x$$

It follows that the coordinate transformation necessary to bring K^- and K^+ in the form of respectively (50) and (51) is given by $z_i = \gamma_i(x) := (1-\tilde{\tau}_i(x))^{\frac{1}{2}} x_i$, $i = 1,..,n$ and $z = \gamma(x) = (\gamma_1(x) \cdots \gamma_n(x))$ for $z \in \gamma(U) = W$. After this transformation it follows that the HJB singular value functions are $v_i(z) = \dfrac{\tilde{\tau}_i(\gamma^{-1}(z))}{1-\tilde{\tau}_i(\gamma^{-1}(z))}$, $i = 1,..,n$ for $z \in W$. ∎

From this theorem it is clear that $\tilde{\tau}_1(\gamma^{-1}(z)) \geq \ldots \geq \tilde{\tau}_n(\gamma^{-1}(z))$ is equivalent with $v_1(z) \geq \ldots \geq v_n(z)$. Furthermore the form of the transformation $z = \gamma(x)$ in the proof of Theorem 5.11 is such that $x_k = 0$ is equivalent with z_k=0 and hence if we want to reduce the order of the original model based on these singular value functions, HJB-balancing and balancing of the coprime representation for $x \in U$ both result in the same reduced order model.

6 Model reduction

6.1 Model reduction of the coprime representation

Model reduction of linear systems based on the Graph Hankel singular values is treated in Meyer 1988, see section 2.2. It turned out that model reduction applied to the normalized right coprime representation (9) of the original linear system (8), results in a representation of the normalized right coprime factorization of the reduced order original system. Here we will study the same problem for nonlinear systems.

Consider section 4.1, where system (32) is a representation of the normalized right coprime factorization of system (26). Assume system (32) is in balanced form. Then the controllability and observability function, $\bar{L}_c$ and $\bar{L}_o$, are of a form as is described in section 3, i.e. are of the form of equation (20) respectively (21) with the τ_i's replaced by the $\tilde{\tau}_i$'s. Reducing the order of system (32) can be done in a way as is prescribed in Scherpen 1993. Hence if for $k < n$ $\tilde{\tau}_k(x) > \tilde{\tau}_{k+1}(x)$, $x \in U$, then for system (32) the state component x_k is more important than x_{k+1} in terms of input and output energy, see Scherpen 1993. To reduce the order of the system we put $x_{k+1} = ... = x_n = 0$. Corresponding to the partitioning of the state in the first k components and the last $n-k$ components, the partitioning is as follows:

$$f(x) = \begin{pmatrix} f_a(x^a,x^b) \\ f_b(x^a,x^b) \end{pmatrix}, \quad g(x) = \begin{pmatrix} g_a(x^a,x^b) \\ g_b(x^a,x^b) \end{pmatrix}, \quad h(x) = h(x^a,x^b)$$

where $x^a = (x_1, \dots, x_k)$, $x^b = (x_{k+1}, \dots, x_n)$ and thus $x = (x^a, x^b)$. This partitioning for system (32) results in:

$$\begin{cases} \dot{x}^a = f_a(x) - g_a(x)g_a(x)^T \frac{\partial^T \bar{L}_o}{\partial x^a}(x) - g_a(x)g_b(x)^T \frac{\partial^T \bar{L}_o}{\partial x^b}(x) + g_a(x)w \\ \dot{x}^b = f_b(x) - g_b(x)g_a(x)^T \frac{\partial^T \bar{L}_o}{\partial x^a}(x) - g_b(x)g_b(x)^T \frac{\partial^T \bar{L}_o}{\partial x^b}(x) + g_b(x)w \end{cases}$$
$$\begin{cases} y = h(x) \\ u = -g_a(x)^T \frac{\partial \bar{L}_o}{\partial x^a}(x) - g_b(x)^T \frac{\partial \bar{L}_o}{\partial x^b}(x) + w \end{cases} \tag{61}$$

We define the reduced system of system (61) as follows:

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})w \\ \tilde{y} &= \tilde{h}_1(\tilde{x}) \\ \tilde{u} &= \tilde{h}_2(\tilde{x}) + w \end{aligned} \tag{62}$$

Also we define the reduced system of the original system (26) as:

$$\begin{aligned} \dot{\tilde{x}} &= \hat{f}(\tilde{x}) + \hat{g}(\tilde{x})u \\ \tilde{y} &= \hat{h}(\tilde{x}) \end{aligned} \tag{63}$$

If we reduce the order of the system (61) by putting $x_b = 0$, in general we will not get a representation of the normalized right coprime representation of the reduced original system (63). The next theorem will give a condition under which this does hold:

Theorem 6.1 *If $\frac{\partial \bar{L}_o}{\partial x^b}(x^a, 0) = 0$, then the reduced system (62) is a representation of the normalized right coprime factorization of the system (63).*

Proof If $\frac{\partial \bar{L}_o}{\partial x^b}(x^a, 0) = 0$, the form of the Hamilton-Jacobi-Bellman equation (31), which has as a smooth solution $\bar{L}_o$, implies that $\bar{L}_o(x_a, 0)$ is the solution of:

$$\begin{aligned} &\frac{\partial \bar{L}_o}{\partial x^a}(x^a, 0) \left(f_a(x^a, 0) - g_a(x^a, 0)g_a(x^a, 0)^T \frac{\partial^T \bar{L}_o}{\partial x^a}(x^a, 0) \right) + \frac{1}{2} h(x^a, 0)^T h(x^a, 0) \\ &+ \frac{1}{2} \frac{\partial \bar{L}_o}{\partial x^a}(x^a, 0) g_a(x^a, 0) g_a(x^a, 0)^T \frac{\partial^T \bar{L}_o}{\partial x^a}(x^a, 0) = 0 \end{aligned}$$

Define $\tilde{f}(\tilde{x}) := \left(f_a(x^a, 0) - g_a(x^a, 0)g_a(x^a, 0)^T \frac{\partial^T \bar{L}_o}{\partial x^a}(x^a, 0) \right)$, $\tilde{g}(\tilde{x}) := g_a(x^a, 0)$, $\tilde{h}_1(\tilde{x}) := h(x^a, 0)$ and $\tilde{h}_2(\tilde{x}) := -g_a(x^a, 0)^T \frac{\partial^T \bar{L}_o}{\partial x^a}(x^a, 0)$, then $\tilde{L}_o(\tilde{x}) := \bar{L}_o(x^a, 0)$ is the observability function of system (62). Furthermore, define $\hat{f}(\tilde{x}) := f_a(x^a, 0)$, $\hat{g}(\tilde{x}) := g_a(x^a, 0)$ and $\hat{h}(\tilde{x}) := h(x^a, 0)$. Then obviously (62) has a form that matches with the concept of a representation of a normalized right coprime factorization of system (63). ■

Remark 6.2 For linear systems the condition $\frac{\partial \bar{L}_o}{\partial x^b}(x^a, 0) = 0$ is always fulfilled.

For system (61) and the reduced order system (62) we can find some more properties of model reduction based on balancing in Scherpen 1993.

6.2 Model reduction of the HJB balanced representation

A brief review on LQG balancing and model reduction based on that can be found in section 2.1, see Jonckheere/Silverman 1983 and Opdenacker/Jonckheere 1985. In section 5.1 we treated HJB balancing for nonlinear systems and defined the HJB singular value functions. HJB balancing makes use of the past and future energy functions and a state component for which the corresponding HJB singular value function is small, can be removed from the original model in order to obtain a reduced-order model.

Consider section 5.1 where we introduced HJB-balancing and consider the system (43) after the transformation which brings it in HJB balanced form:

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}) + g(\bar{z})u \\ y &= \bar{h}(\bar{z}) \end{aligned} \tag{64}$$

for $\bar{z} \in \bar{W}$. Hence the past and future energy function, $\bar{K}^-$ and $\bar{K}^+$, are respectively of the form (52) and (53). Again, reducing the system (64) based on HJB balancing can be done in the same way as is described in Scherpen 1993. If for $k < n$ the singular value functions fulfill $v_k(z) < v_{k+1}(z)$, $z \in W$, then we can state that for system (64) the state component x_k is more important in terms of the past and future energy than the state component x_{k+1}. To reduce the model we put $\bar{z}_k = \ldots = \bar{z}_n = 0$ and again the partitioning of the system is done in a corresponding way, with $\bar{z}^a = (\bar{z}_1, \ldots, \bar{z}_k)$, $\bar{z}^b = (\bar{z}_{k+1}, \ldots, \bar{z}_n)$ and $\bar{z} = (\bar{z}^a, \bar{z}^b)$. Define the reduced system (64) as follows:

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}(\tilde{z}) + g(\tilde{z})u \\ \tilde{y} &= \tilde{h}(\tilde{z}) \end{aligned} \tag{65}$$

Assume the past and future energy function of this system are respectively $\tilde{K}^-$ and $\tilde{K}^+$. Obviously by the form of $\bar{K}^-$ and the form of the corresponding Hamilton-Jacobi-Bellman equation (47) we conclude that $\tilde{K}^-(\tilde{z}) = \bar{K}^-(\bar{z}^a, 0)$, but in general we can not conclude the same for $\tilde{K}^+$.

Theorem 6.3 *If* $\frac{\partial \bar{K}^+}{\partial \bar{z}^b}(\bar{z}^a, 0)\bar{f}_b(\bar{z}^a, 0) = 0$ *and* $\frac{\partial \bar{K}^+}{\partial \bar{z}^b}(\bar{z}^a, 0)\bar{g}_b(\bar{z}^a, 0) = 0$ *then the reduced system (65) is in HJB balanced form again and the future energy function of (65) is* $\tilde{K}^+(\tilde{z}) = \bar{K}^+(\bar{z}^a, 0)$.

Proof This follows directly from the form of the Hamilton-Jacobi-Bellman equation (46). ∎

If we consider the system (54) and the compensator (56), we can also consider model reduction based on the HJB singular value functions. Hence assume we transformed (54) to the form of (64) with additionally a noise working on the output y.

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}) + g(\bar{z})u \\ y &= \bar{h}(\bar{z}) + v \end{aligned} \tag{66}$$

for $\bar{z} \in \bar{W}$. Furthermore, assume $\bar{W}(t, \bar{z})$ is the solution to Mortensen equation (57) of system (66). Obviously, if $\bar{W}(t, \bar{z}) = \bar{K}^-(\bar{z})$ and thus $\bar{W}$ is independent of t, then as above Theorem 6.3 holds. Furthermore, by the form of $\bar{K}^-$ we get:

$$\frac{\partial^2 \bar{K}^-}{\partial \bar{z}^2}(\bar{z}) = \begin{pmatrix} \beta_1(\bar{z}_1) & & 0 \\ & \ddots & \\ 0 & & \beta_n(\bar{z}_n) \end{pmatrix}, \quad \text{where}$$

$$\beta_i(\bar{z}_i) = \mu_i^{-1}(\bar{z}_i) + \frac{3}{2}\frac{\partial \mu_i^{-1}}{\partial \bar{z}_i}(\bar{z}_i)z_i + \frac{1}{2}\frac{\partial^2 \mu_i^{-1}}{\partial \bar{z}_i^2}(\bar{z}_i)\bar{z}_i^2, \quad \text{for} \quad i = 1,..,n$$

This form ensures that the state equation of the compensator of the reduced system is the reduced order state equation of the compensator of the original system (66). Otherwise, if the solution to the Mortensen equation (57), $\bar{W}$, is not independent of t, we can not conclude this. Obviously, reducing the system based on the HJB singular value functions, results in a reduced order system with a past energy function $\tilde{K}^-(\tilde{z}) = \bar{K}^-(\bar{z}^a, 0)$, as above and hence under the conditions of Theorem 6.3 the reduced system is HJB balanced again. Nevertheless we can not state in general that the reduced order compensator is a compensator of the reduced order system. Therefore we need another condition (which is automatically satisfied in the linear case):

Theorem 6.4 *If the conditions of Theorem 6.3 are fulfilled and* $\frac{\partial^2 \bar{W}}{\partial \bar{z}^a \bar{z}^b}(t, \bar{z}^a, 0) = 0$*, then the reduced order compensator of the reduced order system has the form of the compensator (56) together with (59).*

Proof This follows from the form of (56). ∎

7 Conclusions

We have given a characterization of normalized left and right coprime factorizations for nonlinear systems. This has been done by first giving a characterization of respectively an inner and a co-inner nonlinear system.

Furthermore we have introduced two procedures to balance unstable nonlinear systems. One of them is balancing of the normalized right coprime factorization and the other method, called HJB balancing, is making use of the past and future energy of the system. We gave the relation between both methods, which is in fact the same as for linear systems. Both methods are extensions of the methods for unstable linear systems, with the difference that in the linear case LQG balancing and HJB balancing are equivalent, but in the nonlinear case the nonlinear version of the LQG problem is in general not of a form such that we can interpret HJB balancing as a measure of the difficulty to control and to filter a state component.

Finally, we applied model reduction based on the concept of HJB balancing and balancing of the normalized right coprime factorization. We gave some sufficient conditions under which the reduced order system has some nice properties again, i.e. when it is a coprime factorization or is in balanced form.

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