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$\mathcal{H}_\infty$ Output Feedback Control for Linear, Discrete Time-Varying Systems via the Bounded Real Lemma

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Abstract In this paper we develop a solution to the discrete-time $\mathcal{H}_\infty$ output feedback control problem for Linear Time-Varying (LTV) systems. The solution is developed along the strategy set up in [Doyle *et. al.* 1989] and the main ingredient in its derivation is the extension of the well-known bounded real lemma to a (discrete) time-varying context, developed in [van der Veen and Verhaegen 1995]. This approach contributes to the conceptual simplicity, and hence to the accessibility, of the solution.

Apart from that, we treat the infinite-horizon case for LTV system of non-uniform state dimension, and varying input and output dimension. Both situations can easily occur in practice, e.g. in multirate sampled data control systems.

The algorithm that can be derived from the solution presented is then applied to the $\mathcal{H}_\infty$ output feedback of a dynamical system changing from one operation point in its operation envelope to another.

Keywords

Positive real lemma, discrete time-varying system, $\mathcal{H}_\infty$ control, Riccati equations.

1. INTRODUCTION

The major part of the research activities in system- and control theory still concentrate upon linear, time-invariant systems. However, in reality most physical and economical systems demonstrate a time-varying and/or non-linear behavior. Taking into account the fact that non-linear systems operating around a particular trajectory within their operation envelope can adequately be described as linear, time-varying (LTV) systems, the development of a coherent system theory for the latter class of systems as exist for linear time-invariant systems would be of great value.

It is only recently that basic textbooks have become available on LTV systems, such as [Rugh 1993], and that a number of basic system theoretical problems have been generalized from a time-invariant context to a time-varying one. Pioneering work in this area is made in [Dewilde and Dym 1992] and [Ball *et. al.* 1992]. These papers develop the transitions from functions defining Toeplitz matrices to operators that appear as input-output maps of time-varying systems, from analytic functions to triangular operators, etc. For a more detailed overview of the contributions in this area we refer to the proceedings of a colloquium dedicated to this topic [Dewilde et. al. 1993].

In this paper, we analyze the topic of $\mathcal{H}_\infty$ control of LTV systems. In a time-invariant context this topic is indicated by $\mathcal{H}_\infty$ control and in the past decade a burst of research activity has taken place in this field. An basic early contribution is given by [Yakubovich 1971]. Without giving a detailed overview of the contributions in this field, we mention two main strategies to solve the "standard" four block $\mathcal{H}_\infty$ problem. One is the approach indicated by the so-called "1984 approach" in [Doyle *et. al.* 1989] and is based on various standard factorizations, such as spectral and inner-outer factorization, of transfer functions. This approach is well documented in [Francis 1987]. The other is the "Riccati state space approach" presented in [Doyle *et. al.* 1989], which establishes a striking parallel between state space solutions to LQG and $\mathcal{H}_\infty$ control problems.

Most of the research activities in this area are for continuous time systems, however solutions exists in a discrete time context, such as [Antonov and Yakubovich 1980], [Gu *et. al.* 1989], [Stoorvogel 1992a], [Basar 1989] and [De Souza and Xie 1992].

For LTV system a restricted number of solutions have been published. The earliest contribution to this topic is the paper [Feintuch and Francis 1985], where the so-called "1984 approach" has been formulated into an operator theoretic framework covering discrete LTV systems. In that paper, it was however remarked that "at present, computation of uniformly optimal controllers for LTV systems is not feasible". With the algorithms that have recently been developed to calculate an inner-outer and spectral factorization and to solve the Nehari problem, see [van der Veen 1993a], [van der Veen and Verhaegen 1995], and [Dewilde and van der Veen 1993], we are now in a position to map the solution of [Feintuch and Francis 1985] into a computational scheme. However, as in the time-invariant case such a solution will give rise to controllers of large system order, a particular situation that needs to be avoided in practice.

Within the class of solutions following the "1984 approach", we have solved a prototype $\mathcal{H}_\infty$ control problem, namely the (weighted) sensitivity minimization problem, for discrete LTV systems [Verhaegen and Dewilde 1993]. As in the time-invariant case, this problem has been formulated as a Nevanlinna-Pick interpolation problem based on the inner-outer factorization of the given causal plant. Related contributions for periodic time-varying systems are [Feintuch et. al. 1986] and [Georgiou and Khargonekar 1987].

In the wake of the pioneering paper [Doyle *et. al.* 1989] a number of extensions have been published treating LTV systems. In the context of differential games we mention the contributions of [Limebeer *et. al.* 1989], [Limebeer *et. al.* 1992], [Basar and Bernhard 1991] and in the context of the maximum principle we mention [Tadmor 1989]. Apart from the work in [Basar and Bernhard 1991] which also treats the discrete time case, all these solutions are for the finite horizon case and for continuous time systems. The particularly more difficult infinite horizon case has only been treated in [Tadmor 1989] and [Ravi *et. al.* 1991] for continuous time systems and in [Halanay and Ionescu 1993], [Dragan . *et. al.* 1994] for discrete time systems, using the framework of the generalized Popov-Yakubovich theory.

In this paper, we treat the infinite horizon case for LTV discrete time systems. Apart from this, the merits of the paper are: (1) the simplicity of the solution only based on the discrete time Bounded Real Lemma for LTV systems [van der Veen and Verhaegen 1995], (2) the treatment of varying state dimensions (and input-output dimensions), e.g. [van der Veen and Dewilde 1991], [Gohberg *et. al.* 1992]. It has been observed that the latter situations can easily occur in practice. E.g. in [Yu and Verhaegen 1993] identification schemes revealed a change in the state order when the human joint dynamics, studied in that paper, underwent fast changes. The change of the input/output dimension occurs e.g. in multirate sampled data systems.

The solution presented follows the strategy developed in [Doyle *et. al.* 1989] and the contributions made in [De Souza and Xie 1992], [Furuta and Phoojaruenchanachai 1990], [Pappas *et. al.* 1980], and [Walker 1990], discussing related problems for LTI systems. As in [Doyle *et. al.* 1989], the three different stages along which we develop a solution are: (1) Solving the $\mathcal{H}_\infty$ static state feedback control problem and its dual variant of $\mathcal{H}_\infty$ state reconstruction, (2) Formulating the plant to be controlled as a linear fractional transformation of an "inner" operator and (3) Combining the first two stages in providing a solution to the $\mathcal{H}_\infty$ output feedback problem. Preliminary versions of these stages have appeared in [Verhaegen and van der Veen 1993b] and [Verhaegen and van der Veen 1993a]. The resulting conditions are given in terms of two coupled Riccati equations. The next stage is to find a relation between three appearing Riccati equations, in order to give conditions that exists of two uncoupled Riccati equations with a coupling condition.

The present paper is organized as follows. In Section 2, we give a brief overview of the notation and the representation of a state space model of LTV systems used throughout the paper. The variants of the bounded real lemma necessary to tackle the problems in the first stage are presented in Section 3 and applied to the $\mathcal{H}_\infty$ static state feedback problem in Section 4 and the $\mathcal{H}_\infty$ state reconstruction problem in Section 5. The equivalent representation of the given plant as a linear fractional transformation of an "inner" operator and the solution to the $\mathcal{H}_\infty$ output feedback problem are treated in Section 6. In Section 7 we look for a formulation in terms of two uncoupled Riccati equations with a coupling condition instead of two coupled Riccati equations. Section 8 then makes a number of remarks about the computational aspects involved and applies the derived solution to the $\mathcal{H}_\infty$ control of systems changing from one operation point to another in their operation envelope. Finally, Section 9 concludes this paper with some summarizing comments.

2. NOTATION AND STATE SPACE REPRESENTATION OF LTV SYSTEMS

Spaces

We consider time-varying transfer operators as bounded Hilbert space operators on ℓ_2-sequences. Such operators have an (infinite) matrix description

$$T = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & T_{-1,-1} & T_{-1,0} & T_{-1,1} & \cdots \\ & T_{0,-1} & \boxed{T_{0,0}} & T_{0,1} & \\ \cdots & T_{1,-1} & T_{1,0} & T_{1,1} & \cdots \\ & \vdots & & \vdots & \end{bmatrix}$$

(the square identifies the $(0,0)$-th entry), where we will allow, for generality, that the entries $T_{i,j}$ are matrices themselves, say $T_{i,j} \in \mathbb{C}^{M_i \times N_j}$. To describe such operators properly, let $M = [\cdots, \boxed{M_0}, M_1, \cdots]$ be a sequence of non-negative integers. The space of non-uniform sequences ('signals') u such that the i-th entry of the sequence is an M_i-dimensional vector is denoted by

$$\mathcal{M} \; = \; \cdots \times \mathcal{M}_0 \times \mathcal{M}_1 \times \cdots ,$$

where $\mathcal{M}_i = \mathbb{C}^{M_i}$. In this context, we call M an index sequence and write $\mathcal{M} = \mathbb{C}^M$ and $M = \#\mathcal{M}$. The space $\ell_2^{\mathcal{M}}$ is the space of sequences in $\mathcal{M}$ of finite 2-norm (bounded energy). This space is a Hilbert space. We denote by $\mathcal{X}(\mathcal{M},\mathcal{N})$ the space of operators $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$ that are bounded in operator norm. Furthermore, we denote by $\mathcal{X}_2(\mathcal{M},\mathcal{N})$ the space of elements of $\mathcal{X}(\mathcal{M},\mathcal{N})$ that are also bounded in Hilbert-Schmidt norm (see e.g. [van der Veen 1993b]).

To be consistent with earlier literature in which this notation was defined, e.g. [Dewilde and Dym 1992], [van der Veen and Dewilde 1991], [van der Veen and Dewilde 1995], [Dewilde and van der Veen 1993], we think of sequences as row vectors, and of operators as acting on the sequences at the left, so that we will write uT rather than Tu, which is the usual notation for time-invariant systems in the control literature.

An operator is said to be upper if its matrix representation is an upper matrix: $T_{i,j} = 0 \; (i > j)$, and we denote the space of bounded upper operators by $\mathcal{U}(\mathcal{M},\mathcal{N})$. In particular the space of 2×2 block operators with each entry in $\mathcal{U}$ is denoted by $\mathcal{U}^2$. The space $\mathcal{L}$ of bounded lower operators, and the space $\mathcal{D} = \mathcal{L} \cap \mathcal{U}$ of bounded diagonal operators is defined in a similar way. The identity operator in $\mathcal{D}(\mathcal{M},\mathcal{M})$ will sometimes be denoted by $I_{\mathcal{M}}$ for the sake of clarity. We will allow that some (or all but a finite) number of entries of index sequences are equal to zero. In this way, finite matrices are incorporated in the same framework.

Furthermore, upper, lower and diagonal Hilbert-Schmidt spaces are given by $\mathcal{U}_2 = \mathcal{X}_2 \cap \mathcal{U}$, $\mathcal{L}_2 = \mathcal{X}_2 \cap \mathcal{L}$ and $\mathcal{D}_2 = \mathcal{X}_2 \cap \mathcal{D}$, respectively.

An operator $T \in \mathcal{X}$ can be viewed as a time-varying transfer operator: its i-th rows contains the impulse response of the system for an impulse at time i. An operator $T \in \mathcal{U}$ is said to be *causal*, because when $y = uT$ is the response of an input u which is zero up till time instant k, then y is also zero up till this point. If T is a time-invariant system, then its matrix representation has a Toeplitz structure.

The causal bilateral shift-operator on sequences is denoted by Z: it is such that

$$[\cdots \quad \boxed{u_0} \quad u_1 \quad \cdots]Z \; = \; [\cdots \quad \boxed{u_{-1}} \quad u_0 \quad \cdots].$$

If $u \in \mathcal{M}$, then we write $uZ \in \mathcal{M}^{(1)}$, where $\mathcal{M}^{(1)}$ is equal to the space sequence $\mathcal{M}$, shifted over one position to the left. We will also need a diagonal shift operator: the k-th diagonal shift of $A \in \mathcal{X}$ is $A^{(k)} = Z^{*k} A Z^k$, and will shift the entries of A over k positions into the South-East direction: $(A^{(k)})_{i,j} = A_{i-k,j-k}$.

We will say that a hermitian operator X is uniformly positive, $X \gg 0$, if

$$\exists \varepsilon > 0 : \quad uXu^* \; > \varepsilon \, uu^* \;, \text{for all } u \in \ell_2 \,.$$

If X is uniformly positive, then it is boundedly invertible.

We also use the concept of an outer operator. We say that an operator $W \in \mathcal{U}$ is outer if $\mathcal{U}_2 W$ is dense in $\mathcal{U}_2$. If W is invertible this implies that $W^{-1} \in \mathcal{U}$.

Realizations

A linear, discrete, causal time-varying state realization is given by a collection $\{A_k, B_k, C_k, D_k\}$ of matrices, and consists of the recursion

$$\begin{aligned} x_{i+1} &= x_i A_i + u_i B_i \\ y_i &= x_i C_i + u_i D_i \,. \end{aligned}$$

We allow the dimensions of all matrices to be time-varying. We collect the state matrices into diagonals:

$$A = \operatorname{diag}[\,\cdots\; A_0\;\; A_1\;\; \cdots\,]\,,$$

and likewise for B, C, D, so that the above realization equation can be written as

$$\begin{aligned} xZ^{-1} &= xA + uB \\ y &= xC + uD \end{aligned} \qquad \mathbf{T} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \tag{1}$$

where

$$\begin{aligned} A &\in \mathcal{D}(\mathcal{B}, \mathcal{B}^{(-1)})\,, & C &\in \mathcal{D}(\mathcal{B}, \mathcal{N})\,, \\ B &\in \mathcal{D}(\mathcal{M}, \mathcal{B}^{(-1)})\,, & D &\in \mathcal{D}(\mathcal{M}, \mathcal{N})\,. \end{aligned} \tag{2}$$

The space sequence $\mathcal{B}$ is called the system order of the realization; if $A_k \in \mathbb{C}^{d_k \times d_{k+1}}$, then $\mathcal{B}_k = \mathbb{C}^{d_k}$. $\mathbf{T}$ is a realization of T if its entries T_{ij} or diagonals $T_{[i]}$ are given by

$$T_{ij} = \begin{cases} 0\,, & i > j \\ D_i\,, & i = j \\ B_i A_{i+1} \cdots A_{j-1} C_j\,, & i < j \end{cases} \qquad \Leftrightarrow \qquad T_{[i]} = \begin{cases} 0, & i < 0 \\ D, & i = 0 \\ B^{(i)} A^{(i-1)} \cdots A^{(1)} C, & i > 0\,. \end{cases} \tag{3}$$

Let ℓ_A denote the spectral radius of the operator AZ: $\ell_A = \lim_{n\to\infty} \|\,(AZ)^n\,\|^{1/n}$. If $\ell_A < 1$ then the realization is strictly or asymptotically stable, and $(I - AZ)$ is invertible so that $T = D + BZ(I - AZ)^{-1}C$. For time-varying systems exponential stability is equivalent with uniform asymptotic stability (e.g. [Rugh 1993]).

Finally, operators representing input-output maps are sometimes indexed. In this way, the input-output map T_{wz} relates the input sequence w to the output sequence z,

$$z = wT_{wz}.$$

3. THE BOUNDED REAL LEMMA AND ITS EXTENSION

In this section, we consider a causal system T with state realization $\mathbf{T} = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$ such that A, B, C, D have the dimensions as indicated in equation (2).

3.1. Spectral factorization of the operator $\Gamma_o I - T^* T$

Let us recall Theorem 12 and Proposition 14 of [van der Veen and Verhaegen 1995].

Theorem 1. (Theorem 12, [van der Veen and Verhaegen 1995], Time-varying bounded real lemma) *Let $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a locally finite operator with state realization $\{A, B, C, D\}$ such that $\ell_A < 1$. Let $\Gamma_o \in \mathcal{D}(\mathcal{N}, \mathcal{N})$ such that $\Gamma_o \gg 0$. Then $\Gamma_o I - T^* T \gg 0$ if and only if there exists a solution $M_o \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ of*

$$M_o^{(-1)} \;=\; A^* M_o A \;+\; \left[A^* M_o C + B^* D\right] (\Gamma_o I - D^* D - C^* M_o C)^{-1} \left[D^* B + C^* M_o A\right] + B^* B\,. \tag{4}$$

such that $\Gamma_o I - D^*D - C^*M_oC \gg 0$ *and* $M_o \geq 0$. *If in addition the realization of* T *is observable and [uniformly] controllable, then* M_o *is [uniformly] positive.*

If $\Gamma_o I - T^*T \gg 0$, *let* $W \in \mathcal{U}(\mathcal{N},\mathcal{N})$ *be a factor of* $\Gamma_o I - T^*T = W^*W$. *A realization* $\{A, B_W, C, D_W\}$ *for* W *such that* W *is outer is then given by the smallest solution* M_o *of the above equation, and solutions* D_W, B_W *of*

$$\left\{ \begin{array}{rcl} D_W^* D_W & = & \Gamma_o I - D^*D - C^*M_oC \\ B_W & = & -D_W^{-*}\left[D^*B + C^*M_oA\right]. \end{array} \right. \tag{5}$$

$\square$

Note that $\Gamma_o I - T^*T \gg 0$ for $\Gamma_o = \gamma^2 I$ in case of a time-invariant system means that the $\mathcal{H}_\infty$ norm of T (the transfer function from u to y) is bounded from above by γ. The following proposition and corollary show the relation between the smallest solution and the stabilizing solution to the Riccati equation.

Proposition 2. **(Proposition 14, [van der Veen and Verhaegen 1995])** *Let* $W \in \mathcal{U}(\mathcal{M},\mathcal{M})$ *be an outer invertible operator, with state realization* $\mathbf{W} = \{A, B, C, D\}$. *Then* $S = W^{-1} \in \mathcal{U}(\mathcal{M},\mathcal{M})$ *has a state realization given by*

$$\mathbf{S} = \left[\begin{array}{cc} A - CD^{-1}B & -CD^{-1} \\ D^{-1}B & D^{-1} \end{array} \right] \tag{6}$$

Moreover, $\mathbf{W}$ *is (uniformly) controllable (e.g. [van der Veen 1993b]) if and only if* $\mathbf{S}$ *is (uniformly) controllable,* $\mathbf{W}$ *is (uniformly) observable if and only if* $\mathbf{S}$ *is (uniformly) observable. Let* $A^\times = A - CD^{-1}B$. *If* $\ell_A < 1$ *and* $\mathbf{W}$ *is controllable or observable, then* $\ell_{A^\times} < 1$. $\square$

Corollary 3. *Let the conditions of Theorem 1 hold, and let the same quantities as in this theorem be defined, with* M_o *the smallest solution of (4). Assume that* $\mathbf{T}$ *is observable. Then the operator* $A_o^\times$, *defined as:*

$$A_o^\times = A + C(\Gamma_o I - D^*D - C^*M_oC)^{-1}\left[D^*B + C^*M_oA\right] \tag{7}$$

satisfies

$$\ell_{A_o^\times} < 1 \tag{8}$$

PROOF Since $\{A, B_W, C, D_W\}$ is an observable realization of an outer factor W of $\Gamma_o I - T^*T$, and $A_o^\times = A - CD_W^{-1}B_W$, Proposition 2 immediately shows that condition (8) is satisfied. $\square$

Corollary 4. *The solution* M_o *of (4) that stabilizes* $A_o^\times$ *as in equation (7) is unique.*

PROOF This follows immediately from Lemma 4 of [van der Veen and Verhaegen 1995]. $\square$

In order to address the $\mathcal{H}_\infty$ control problems of this paper, we need the following extension of the version of the Bounded Real Lemma in Theorem 1. In the proof of this extension, we make use of the notion of uniform stabilizability. Uniform stabilizability and detectability are studied in more detail in e.g. [Anderson and Moore 1991].

Definition 5. *The pair* $(A\,,\,B)$ *is uniformly stabilizable if and only if there exists a bounded operator* $F \in \mathcal{D}(\mathcal{B},\mathcal{M})$ *such that*

$$\ell_{A+FB} < 1$$

$\square$

Lemma 6. (Extended Lyapunov lemma, [Nicolao 1992]) *Suppose the pair* $(A\,,\,B)$ *is uniformly stabilizable. If there exists a solution* $X \in \mathcal{D}(\mathcal{B},\mathcal{B})$ *and* $X \geq 0$ *of:*

$$X^{(-1)} = A^*XA + B^*B \tag{9}$$

then $\ell_A < 1$. *Conversely, if* A *has* $\ell_A < 1$, *then there exists a unique bounded solution* $X \geq 0$ *of equation (9).* ☐

Theorem 7. *Let* $T \in \mathcal{U}(\mathcal{M},\mathcal{N})$ *be a locally finite operator with realization* $\{A,B,C,D\}$. *Let* $\Gamma_o \in \mathcal{D}(\mathcal{N},\mathcal{N})$ *such that* $\Gamma_o \gg 0$. *Then* $\Gamma_o I - T^*T \gg 0$ *and* $\ell_A < 1$ *if and only if there exists a unique solution* $M_o \in \mathcal{D}(\mathcal{B},\mathcal{B})$ *of equation (4) such that* $\Gamma_o I - D^*D - C^*M_oC \gg 0$, $M_o \geq 0$, *and* $A_o^\times$ *as in equation (7) is such that* $\ell_{A_o^\times} < 1$.

PROOF (⟹) Without loss of generality, we assume the state realization of T to be in the following observer canonical form:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad B = [B_1 \quad B_2], \qquad C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix},$$

with the pair $(A_{11}\,,\,C_1)$ observable and $\ell_{A_{11}} < 1$, $\ell_{A_{22}} < 1$. Let also M_o be partitioned conformably, i.e.

$$M_o = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix}$$

Then it follows that equation (4) is equivalent to the following three equations:

$$\begin{aligned}
M_{11}^{(-1)} &= A_{11}^*M_{11}A_{11} + [B_1^*D + A_{11}^*M_{11}C_1](\Gamma_o I - D^*D - C_1^*M_{11}C_1)^{-1}[D^*B_1 + C_1^*M_{11}A_{11}] + \\
&\quad B_1^*B_1 && (10) \\
M_{12}^{(-1)} &= A_{11}^*M_{11}A_{12} + \left(A_{11}^* + [B_1^*D + A_{11}^*M_{11}C_1](\Gamma_o I - D^*D - C_1^*M_{11}C_1)^{-1}C_1^*\right) M_{12}A_{22} + \\
&\quad (B_1^*D + A_{11}^*M_{11}C_1)(\Gamma I - D^*D - C_1^*M_{11}C_1)^{-1}(D^*B_2 + C_1^*M_{11}A_{12}) + B_1^*B_2 && (11) \\
M_{22}^{(-1)} &= A_{22}^*M_{22}A_{22} + (A_{12}^*M_{11} + A_{22}^*M_{12}^*)A_{12} + A_{12}^*M_{12}A_{22} + B_2^*B_2 + \\
&\quad [B_2^*D + A_{12}^*M_{11}C_1 + A_{22}^*M_{12}^*C_1](\Gamma_o I - D^*D - C_1^*M_{11}C_1)^{-1} && (12) \\
&\quad [D^*B_2 + C_1^*M_{11}A_{12} + C_1^*M_{12}A_{22}]
\end{aligned}$$

Since the pair $(A_{11}\,,\,C_1)$ is observable, Theorem 1 shows that the solution to equation (10) exists such that $M_{11} \geq 0$ and $(\Gamma_o I - D^*D - C_1^*M_{11}C_1) \gg 0$. Furthermore, this solution defines the operator $A_{11}^\times$ as:

$$A_{11}^\times = A_{11} + C_1(\Gamma_o I - D^*D - C_1^*M_{11}C_1)^{-1}[D^*B_1 + C_1^*M_{11}A_{11}]$$

such that by Corollary 3, $\ell_{A_{11}^\times} < 1$. Hence, since $\ell_{A_{22}} < 1$, equation (11), which is a Stein equation in M_{12}, has a bounded unique solution. For the same reason, equation (12) has bounded unique solution. As a consequence, the condition $\Gamma_o I - T^*T \gg 0$ assures that a unique solution M_o to equation (4) exists, such that $\Gamma_o I - D^*D - C^*M_oC = \Gamma_o I - D^*D - C_1^*M_{11}C_1 \gg 0$.

The operator $A_o^\times$, defined as in equation (7) now has the following structure,

$$A_o^\times = \begin{bmatrix} A_{11}^\times & \star \\ 0 & A_{22} \end{bmatrix}$$

where the $\star$ operator is irrelevant. Since $\ell_{A_{11}^\times} < 1$ and $\ell_{A_{22}} < 1$, the same result holds for $\ell_{A_o^\times}$. Finally, since $\ell_A < 1$ and $\Gamma_o I - D^*D - C^*M_oC \gg 0$, it follows from the Riccati equation (4) that $M_o \geq 0$ and bounded.

(⟸) Based on Theorem 1, we only have to proof that $\ell_A < 1$.

Since $\Gamma_o I - D^* D - C^* M_o C \gg 0$, we can define the operator $\hat{B}$ as:

$$\hat{B} = \begin{bmatrix} (\Gamma_o I - D^* D - C^* M_o C)^{-\frac{1}{2}} [D^* B + C^* M_o A] \\ B \end{bmatrix}$$

Since $\ell_{A_o^\times} < 1$, the pair $(A\,,\,\hat{B})$ is uniformly stabilizable, and hence by Lemma 6 and equation (4) we conclude that $\ell_A < 1$. ☐

3.2. Spectral factorization of the operator $\Gamma_c I - TT^*$

The dual variants of Theorem 1, Definition 5, Lemma 6 and Theorem 7 of the previous subsection will be stated in this section for notational convenience and without proof as Theorem 8, Definition 9, Lemma 10 and Theorem 11 respectively.

Theorem 8. *Let $T \in \mathcal{U}(\mathcal{M},\mathcal{N})$ be a locally finite operator with a controllable state realization $\{A,B,C,D\}$ such that $\ell_A < 1$. Let $\Gamma_c \in \mathcal{D}(\mathcal{M},\mathcal{M})$ such that $\Gamma_c \gg 0$. Then $\Gamma_c I - TT^* \gg 0$ if and only if there exists a solution $M_c \in \mathcal{D}(\mathcal{B},\mathcal{B})$ of*

$$M_c = A M_c^{(-1)} A^* + \left[CD^* + A M_c^{(-1)} B^*\right] (\Gamma_c I - DD^* - B M_c^{(-1)} B^*)^{-1} \left[DC^* + B M_c^{(-1)} A^*\right] + CC^* \tag{13}$$

such that $\Gamma_c I - DD^* - B M_c^{(-1)} B^* \gg 0$ and $M_c \geq 0$. If in addition the realization of T is [uniformly] controllable, then M_c is [uniformly] positive.

If $\Gamma_c I - TT^* \gg 0$, let $V \in \mathcal{U}(\mathcal{M},\mathcal{M})$ be a factor of $\Gamma_c I - TT^* = VV^*$. A realization $\{A,B,C_V,D_V\}$ for V such that V is outer is then given by the smallest solution M_c of the above equation, and solutions D_V, C_V of

$$\begin{cases} D_V D_V^* &= \Gamma_c I - DD^* - B M_c^{(-1)} B^* \\ C_V &= -\left[CD^* + A M_c^{(-1)} B^*\right] D_V^{-*}\,. \end{cases} \tag{14}$$

The realization of V^{-1} derived from this realization of V has the following A-operator:

$$A_c^\times = A + \left[CD^* + A M_c^{(-1)} B^*\right] (\Gamma_c I - DD^* - B M_c^{(-1)} B^*)^{-1} B \tag{15}$$

and the latter satisfies,

$$\ell_{A_c^\times} < 1 \tag{16}$$

The solution $M_c \geq 0$ to (13), such that $A_c^\times$ is stable is unique. ☐

Definition 9. *The pair $(A\,,\,C)$ is uniformly detectable if and only if there exists a bounded operator $K \in \mathcal{D}(\mathcal{N},\mathcal{B}^{(-1)})$ such that,*

$$\ell_{A+CK} < 1$$

☐

Lemma 10. *Suppose the pair $(A\,,\,C)$ is uniformly detectable. Then if there exists a solution $X \in \mathcal{D}(\mathcal{B},\mathcal{B})$ and $X \geq 0$ of:*

$$X = A X^{(-1)} A^* + CC^* \tag{17}$$

then $\ell_A < 1$. Conversely, if A has $\ell_A < 1$, then there exists a unique bounded solution $X \geq 0$ of equation (17) ☐

Theorem 11. *Let* $T \in \mathcal{U}(\mathcal{M},\mathcal{N})$ *be a locally finite operator with realization* $\{A,B,C,D\}$*. Let* $\Gamma_c \in \mathcal{D}(\mathcal{N},\mathcal{N})$ *such that* $\Gamma_c \gg 0$*. Then* $\Gamma_c I - TT^* \gg 0$ *and* $\ell_A < 1$ *if and only if there exists a unique solution* $M_c \in \mathcal{D}(\mathcal{B},\mathcal{B})$ *of equation (13) such that* $\Gamma_c I - DD^* - BM_c^{(-1)}B^* \gg 0$, $M_c \geq 0$, *and the operator* $A_c^{\times}$ *as in equation (15) is such that* $\ell_{A_c^{\times}} < 1$. ☐

4. $\mathcal{H}_\infty$ STATIC STATE FEEDBACK

Using Theorem 11, we are now in a position to generalize the solution to the static $\mathcal{H}_\infty$ state feedback control problem for time-invariant systems to the time-varying case.

Consider the linear time-varying system that has state space realization **T**:

$$\begin{array}{rcllll} xZ^{-1} & = & xA & + \; wB_1 & + \; uB_2 & \\ z & = & xC_1 & + \; wD_{11} & + \; uD_{21} & \\ y & = & x & & & \end{array} \qquad (18)$$

where

$$\begin{array}{rcl@{\qquad}rcl} A & \in & \mathcal{D}(\mathcal{B},\mathcal{B}^{(-1)}), & C_1 & \in & \mathcal{D}(\mathcal{B},\mathcal{N}_1), \\ B_1 & \in & \mathcal{D}(\mathcal{M}_1,\mathcal{B}^{(-1)}), & D_{11} & \in & \mathcal{D}(\mathcal{M}_1,\mathcal{N}_1), \\ B_2 & \in & \mathcal{D}(\mathcal{M}_2,\mathcal{B}^{(-1)}), & D_{21} & \in & \mathcal{D}(\mathcal{M}_2,\mathcal{N}_1), \end{array}$$

and x is the state sequence, w the exogenous input sequence (disturbances), u the control input sequence, y the measured output sequence, and z the to-be-controlled output sequence. Note that we do not assume that the A-operator of (18) has $\ell_A < 1$ or in other words, that we allow the system to be unstable (i.e., the state sequence x may be unbounded). In that case the operator $(I - AZ)^{-1}$ is not bounded, which implies that T does not exists, i.e., that $T \in \mathcal{U}(\mathcal{M}_1 + \mathcal{M}_2, \mathcal{N}_1 + \mathcal{B})$ does not hold. We make the following assumptions, which are standard for the $\mathcal{H}_\infty$ problem for time-invariant systems:

Assumptions 12.

1. *The pair* (A, B_2) *is uniformly stabilizable, and*
2. *The operator* $D_{21}D_{21}^*$ *is uniformly positive.*

Remark 13. *Assumption 2 has been removed in the time-invariant case in [Stoorvogel 1992b]. It is remarked that this removal gives rise to a substantial increase in the amount of intricacies in the proof.*

The $\mathcal{H}_\infty$ static state feedback control problem can be stated as follows (Fig. 1): For a given level of disturbance attenuation $\Gamma_c \gg 0$, $\Gamma_c \in \mathcal{D}(\mathcal{M}_1,\mathcal{M}_1)$, find (if it exists) a bounded static state feedback control law $u = yF = xF$, with $F \in \mathcal{D}(\mathcal{B},\mathcal{M}_2)$, such that:

1. The A-operator of the closed-loop system in Figure 1, that is $A + FB_2$, has $\ell_{A+FB_2} < 1$ (i.e., the closed loop operator exists),
2. The closed-loop operator T_{wz} between w and z with realization $\{A+FB_2, B_1, C_1+FD_{21}, D_{11}\}$ satisfies:

$$\Gamma_c I - T_{wz}T_{wz}^* \gg 0$$

A solution to the $\mathcal{H}_\infty$ static state feedback problem is provided in the next theorem. The proof of this theorem

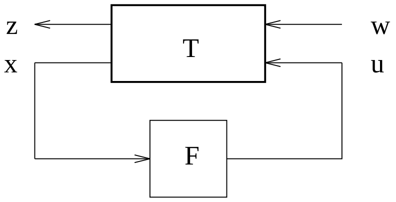

Figure 1. Block-schematic representation of the $\mathcal{H}_\infty$ static state feedback problem.

is stated in the Appendix, since it is quite long and technical. In the proof some strategies of time-invariant results are followed, e.g. [De Souza and Xie 1992], [Ran and Vreugdenhil 1988].

Theorem 14. *Let T be a locally finite operator with state space realization (18) and satisfying the Assumptions 12. Furthermore, let $\Gamma_c \in \mathcal{D}(\mathcal{M}_1,\mathcal{M}_1)$ be a prescribed level of disturbance attenuation, such that $\Gamma_c \gg 0$. Then an operator $F \in \mathcal{D}(\mathcal{B},\mathcal{M}_2)$ solves the $\mathcal{H}_\infty$ static state feedback control problem if and only if there exists a unique solution $M_c \in \mathcal{D}(\mathcal{B},\mathcal{B})$ of,*

$$\begin{aligned} M_c &= \left[C_1 D_E^* + A M_c^{(-1)} B_E^*\right] \left(\begin{bmatrix} \Gamma_c I & 0 \\ 0 & 0 \end{bmatrix} - D_E D_E^* - B_E M_c^{(-1)} B_E^* \right)^{-1} \left[D_E C_1^* + B_E M_c^{(-1)} A^*\right] \\ &\quad + A M_c^{(-1)} A^* + C_1 C_1^* \end{aligned} \tag{19}$$

with $D_E = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}$ and $B_E = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, such that $\Gamma_c I - D_{11} D_{11}^* - B_1 M_c^{(-1)} B_1^* \gg 0$, $M_c \geq 0$ and the operator $A_c^\times$, defined as:

$$A_c^\times = A + \left[C_1 D_E^* + A M_c^{(-1)} B_E^*\right] \left(\begin{bmatrix} \Gamma_c I & 0 \\ 0 & 0 \end{bmatrix} - D_E D_E^* - B_E M_c^{(-1)} B_E^* \right)^{-1} B_E$$

satisfies $\ell_{A_c^\times} < 1$. With this solution M_c of equation (19), the static state feedback law is given as,

$$F = \left[C_1 D_E^* + A M_c^{(-1)} B_E^*\right] \left(\begin{bmatrix} \Gamma_c I & 0 \\ 0 & 0 \end{bmatrix} - D_E D_E^* - B_E M_c^{(-1)} B_E^* \right)^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{20}$$

$\square$

5. STATE RECONSTRUCTION

Consider the time-varying system T with state space realization:

$$\begin{array}{rclclcl} xZ^{-1} &=& xA &+& wB_1 &+& uB_2 \\ z &=& xC_1 &+& &+& uD_{21} \\ y &=& xC_2 &+& wD_{12} &+& uD_{22} \end{array} \tag{21}$$

where

$$\begin{array}{lllllllll} A & \in & \mathcal{D}(\mathcal{B},\mathcal{B}^{(-1)}), & C_1 & \in & \mathcal{D}(\mathcal{B},\mathcal{N}_1), & C_2 & \in & \mathcal{D}(\mathcal{B},\mathcal{N}_2), \\ B_1 & \in & \mathcal{D}(\mathcal{M}_1,\mathcal{B}^{(-1)}), & & & & D_{12} & \in & \mathcal{D}(\mathcal{M}_1,\mathcal{N}_2), \\ B_2 & \in & \mathcal{D}(\mathcal{M}_2,\mathcal{B}^{(-1)}), & D_{21} & \in & \mathcal{D}(\mathcal{M}_2,\mathcal{N}_1), & D_{22} & \in & \mathcal{D}(\mathcal{M}_2,\mathcal{N}_2) \end{array}$$

Again we do not assume $\ell_A < 1$, and thus the input-output operator T does not exist. We don't have full state information available, only the output measurements y. Based on y we want to build a reconstructor of the state such that the resulting error asymptotically tends to zero, and such that the influence of the disturbance w on the error is attenuated.

We make the following assumptions, which are standard for time-invariant systems (see also Remark 13):

Assumptions 15.

1. *The pair* $(A\,,\,C_2)$ *is uniformly detectable, and*
2. *The operator* $D_{12}^* D_{12}$ *is uniformly positive.*

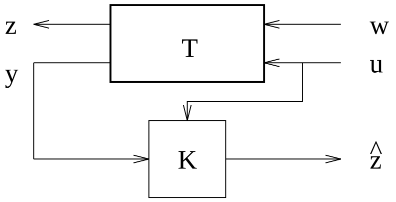

Figure 2. Block-schematic representation of the $\mathcal{H}_\infty$ state reconstruction problem.

The reconstruction of the output z is done with the filter K, see Figure 2, with structure,

$$\begin{array}{rcccccc} \hat{x}Z^{-1} & = & \hat{x}A & + & (y-\hat{x}C_2-uD_{22})L & + & uB_2 \\ \hat{z} & = & \hat{x}C_1 & + & & + & uD_{21} \end{array} \tag{22}$$

where $L \in \mathcal{D}(\mathcal{N}_2, \mathcal{B}^{(-1)})$ represents the 'Kalman' gain, that has to be determined. Then the error on the reconstructed state quantities $\tilde{x} = x - \hat{x}$ and the error on the reconstructed output $e = z - \hat{z}$ satisfy,

$$\begin{array}{rcl} \tilde{x}Z^{-1} & = & \tilde{x}(A-C_2L) + w(B_1 - D_{12}L) \\ e & = & \tilde{x}C_1 \end{array} \tag{23}$$

The $\mathcal{H}_\infty$ state reconstruction problem can now be stated as follows (Figure 2): For a given level of disturbance attenuation $\Gamma_o \gg 0$, $\Gamma_o \in \mathcal{D}(\mathcal{N}_1, \mathcal{N}_1)$, find (if it exists) a bounded operator L, with $L \in \mathcal{D}(\mathcal{N}_2, \mathcal{B}^{(-1)})$, such that:

1. The A-operator of the filter K in Figure 2, that is $A - C_2L$, has $\ell_{A-C_2L} < 1$, and
2. The operator T_{we} between w and e $(= z - \hat{z})$ with realization $\{A - C_2L, B_1 - D_{12}L, C_1, 0\}$ satisfies:

$$\Gamma_o I - T_{we}^* T_{we} \gg 0$$

A solution to the $\mathcal{H}_\infty$ state reconstruction problem is provided in the next theorem. Since the proof of this theorem follows from Theorem 7 in a similar way to how the proof of its dual variant, namely Theorem 14, followed from Theorem 11, it is stated without proof.

Theorem 16. *Let T be a locally finite operator with state realization (21) and satisfying Assumptions 15. Furthermore, let $\Gamma_o \in \mathcal{D}(\mathcal{N}_1, \mathcal{N}_1)$, be a prescribed disturbance attenuation level such that $\Gamma_o \gg 0$. Then an operator $L \in \mathcal{D}(\mathcal{N}_2, \mathcal{B}^{(-1)})$ solves the $\mathcal{H}_\infty$ state reconstruction problem if and only if there exists a unique solution $M_o \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ of*

$$M_o^{(-1)} \;=\; [B_1^* D_E + A^* M_o C_E] \left(\begin{bmatrix} \Gamma_o I & 0 \\ 0 & 0 \end{bmatrix} - D_E^* D_E - C_E^* M_o C_E \right)^{-1} [D_E^* B_1 + C_E^* M_o A]$$

$$+A^*M_oA+B_1^*B_1 \tag{24}$$

with $D_E = \begin{bmatrix} 0 & D_{12} \end{bmatrix}$ *and* $C_E = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$*, such that* $\Gamma_o I - C_1^* M_o C_1 \gg 0,\ M_o \geq 0,$ *and the operator* $A_o^\times$*, defined as:*

$$A_o^\times = A + C_E \left(\begin{bmatrix} \Gamma_o I & 0 \\ 0 & 0 \end{bmatrix} - D_E^* D_E - C_E^* M_o C_E \right)^{-1} [D_E^* B_1 + C_E^* M_o A]$$

satisfies $\ell_{A_o^\times} < 1$. *With this solution* M_o *of equation (24), the observer gain operator L is given as,*

$$L = -\begin{bmatrix} 0 & I \end{bmatrix} \left(\begin{bmatrix} \Gamma_o I & 0 \\ 0 & 0 \end{bmatrix} - D_E^* D_E - C_E^* M_o C_E \right)^{-1} [D_E^* B_1 + C_E^* M_o A] \tag{25}$$

□

6. $\mathcal{H}_\infty$ OUTPUT FEEDBACK

In this section we consider the $\mathcal{H}_\infty$ output feedback problem which we solve by combination of the problems stated in the previous two sections, and an intermediate problem. Let the time-varying system T be given with state space realization (21). Consider the time-varying controller K with state space realization:

$$\begin{aligned} \xi Z^{-1} &= \xi\Phi + y\Psi_1 \\ u &= \xi\Psi_2 + y\Psi_3 \end{aligned} \tag{26}$$

where Φ, Ψ_1, Ψ_2 and Ψ_3 are bounded diagonal operators and where the state dimensions still has to be determined. Both systems are connected as displayed in Figure 3. The following assumptions are made (see also Remark 13):

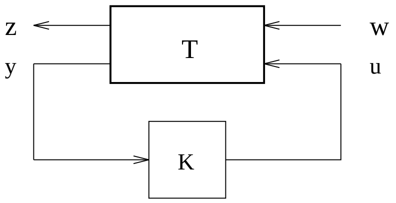

Figure 3. Block-schematic representation of the $\mathcal{H}_\infty$ output feedback problem.

Assumptions 17.

1. *The pair* $(A\ ,\ B_2)$ *is uniformly stabilizable, the operator* $D_{21}D_{21}^* \gg 0$ *and* $\Gamma_c = \gamma I_{\mathcal{M}_1}$ *with* $\gamma > 0$ *is chosen such that a solution exists to the* $\mathcal{H}_\infty$ *static state feedback problem treated in Section 4 and solved in Theorem 14.*

2. *The pair* $(\overline{A}\ ,\ \overline{C}_2) = (A + \tilde{B}_1 U_1^{-1} B_1\ ,\ C_2 + \tilde{B}_1 U_1^{-1} D_{12})$*, with the quantities* $\tilde{B}_1$ *and* U_1 *defined in equation (69) and (66) respectively, is uniformly detectable. Furthermore, the operator* $D_{12}^* D_{12}$ *is uniformly positive.*

We can state the $\mathcal{H}_\infty$ feedback problem as follows (Figure 3): For a given level of disturbance attenuation $\Gamma_c = \gamma I_{\mathcal{M}_1}$ with $\gamma > 0$, find a state space realization $\{\Phi, \Psi_1, \Psi_2, \Psi_3\}$ of the controller K in equation (26), such that:

1. The A-operator of the closed-loop system in Figure 3, which has the following form:

   $$A_{\text{cl}} = \left[\begin{array}{cc} A + C_2\Psi_3(I - D_{22}\Psi_3)^{-1}B_2 & C_2\Psi_1 + C_2\Psi_3(I - D_{22}\Psi_3)^{-1}D_{22} \\ \Psi_2(I - D_{22}\Psi_3)^{-1}B_2 & \Phi + \Psi_2(I - D_{22}\Psi_3)^{-1}D_{22}\Psi_1 \end{array} \right],$$

   satisfies $\ell_{A_{\text{cl}}} < 1$. When this is the case, the closed-loop system depicted in Figure 3 is internally stable, as defined in Definition 21.

2. The operator T_{wz} between w and z in Figure 3 satisfies,

   $$\Gamma_c I - T_{wz} T_{wz}^* \gg 0$$

As outlined in the introduction, a solution to this problem will be developed in three different stages. The first stage is the solution to the $\mathcal{H}_\infty$ static state feedback control problem discussed in Section 4. Theorem 14 provides the static feedback gain operator F that solves this problem. Continuing with this solution we will now subsequently treat the next two stages, namely:

Stage 2 Using the solution of the first stage we formulate and solve an intermediate problem that falls within the class of $\mathcal{H}_\infty$ state reconstruction problems. The quantity that will be reconstructed in this intermediate problem is xF.

Stage 3 Relate the solution derived in Stage 2 to the original $\mathcal{H}_\infty$ output feedback problem.

Stage 2 relies on a representation of the LTV plant T by two other LTV plants (see Subsection 6.1), and for one of these plants the intermediate problem can be solved.

6.1. Equivalent representation of the given LTV plant *T*

As for the solution given to the time-invariant $\mathcal{H}_\infty$ output feedback problem in [Doyle *et. al.* 1989], and [Furuta and Phoojaruenchanachai 1990] we derive based on the solution to the $\mathcal{H}_\infty$ static state feedback problem two LTV systems from the given plant T making use of the following identity:

$$xZ^{-1}M_c^{(-1)}Zx^* - xM_cx^* = 0$$

Making use of the state space representation in equation (21) and the expression for M_c in equation (75) this identity can be written as,

$$\begin{array}{rcl} 0 & = & x\left[U_3U_2^{-1}U_3^* - \tilde{B}_1U_1^{-1}\tilde{B}_1^*\right]x^* - xC_1C_1^*x^* + wB_1M_c^{(-1)}B_1^*w^* + uB_2M_c^{(-1)}B_2^*u^* \\ & & +2xAM_c^{(-1)}B_1^*w^* + 2uB_2M_c^{(-1)}B_1^*w^* + 2xAM_c^{(-1)}B_2^*u^* \end{array}$$

Adding and subtracting the term $w\Gamma_cw^*$, $uD_{21}D_{21}^*u^*$, $2xC_1D_{21}^*u^*$ and using the expressions in equation (68-72) for $D_{11} \equiv 0$, yields:

$$\begin{array}{rcl} 0 & = & w\Gamma_cw^* - zz^* + x\left[U_3U_2^{-1}U_3^* - \tilde{B}_1U_1^{-1}\tilde{B}_1^*\right]x^* - \underline{wU_1w^*} + uU_2u^* - u\tilde{B}_3U_1^{-1}\tilde{B}_3^*u^* \\ & & +2x\tilde{B}_2u^* + \underline{2(x\tilde{B}_1 + u\tilde{B}_3)w^*} \end{array}$$

Completing the squares with the underlined terms, using the expression for U_3 in equation (71), F in equation (74), and using v, and r defined as

$$\begin{array}{rcl} v & = & (u - xF)U_2^{\frac{1}{2}} \qquad (27) \\ r & = & (w - (x\tilde{B}_1 + u\tilde{B}_3)U_1^{-1})U_1^{\frac{1}{2}}\Gamma_c^{-\frac{1}{2}} \qquad (28) \end{array}$$

then the above identity can be written compactly as,

$$w\Gamma_c w^* - zz^* = r\Gamma_c r^* - vv^* \qquad (29)$$

Using equation (27), a first new LTV system P that can be derived from the LTV system T has the following input-output relationship,

$$\left[\begin{array}{cc} z & r \end{array}\right] = \left[\begin{array}{cc} w & v \end{array}\right] P \qquad (30)$$

The latter system P has the state space representation:

$$\begin{array}{rclcccc} xZ^{-1} & = & x(A + FB_2) & + & wB_1 & + & vU_2^{-\frac{1}{2}}B_2 \\ z & = & x(C_1 + FD_{21}) & + & & + & vU_2^{-\frac{1}{2}}D_{21} \\ r & = & -x(\tilde{B}_1 U_1^{-\frac{1}{2}}\Gamma_c^{-\frac{1}{2}} + F\tilde{B}_3 U_1^{-\frac{1}{2}}\Gamma_c^{-\frac{1}{2}}) & + & wU_1^{\frac{1}{2}}\Gamma_c^{-\frac{1}{2}} & - & vU_2^{-\frac{1}{2}}\tilde{B}_3 U_1^{-\frac{1}{2}}\Gamma_c^{-\frac{1}{2}} \end{array} \qquad (31)$$

In the same way, we can define a second LTV system $\overline{T}$, such that,

$$\left[\begin{array}{cc} v & y \end{array}\right] = \left[\begin{array}{cc} r & u \end{array}\right] \overline{T} \qquad (32)$$

$\overline{T}$ has the state space representation,

$$\begin{array}{rclcccc} xZ^{-1} & = & x(A + \tilde{B}_1 U_1^{-1} B_1) & + & r\Gamma_c^{\frac{1}{2}} U_1^{-\frac{1}{2}} B_1 & + & u(\tilde{B}_3 U_1^{-1} B_1 + B_2) \\ v & = & -xFU_2^{\frac{1}{2}} & + & & + & uU_2^{\frac{1}{2}} \\ y & = & x(C_2 + \tilde{B}_1 U_1^{-1} D_{12}) & + & r\Gamma_c^{\frac{1}{2}} U_1^{-\frac{1}{2}} D_{12} & + & u(\tilde{B}_3 U_1^{-1} D_{12} + D_{22}) \end{array}$$

denoted compactly as,

$$\begin{array}{rclcccc} xZ^{-1} & = & x\overline{A} & + & r\overline{B}_1 & + & u\overline{B}_2 \\ v & = & x\overline{C}_1 & + & & + & u\overline{D}_{21} \\ y & = & x\overline{C}_2 & + & r\overline{D}_{12} & + & u\overline{D}_{22} \end{array} \qquad (33)$$

When we organize the two LTV systems P and $\overline{T}$ as in the right hand side block-scheme of Figure 4, we have an equivalent input-output representation of the given LTV system T, depicted on the left hand side of Figure 4.

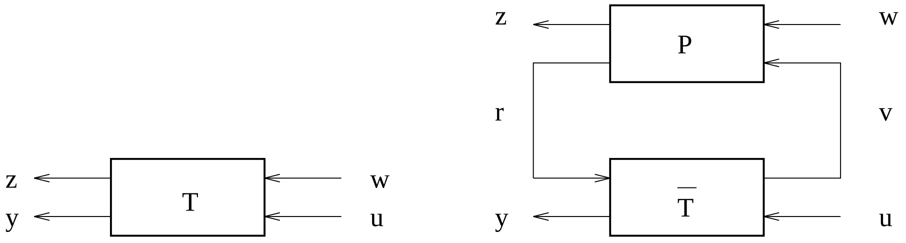

Figure 4. Equivalent input-output representation of the given LTV system T.

6.2. An intermediate $\mathcal{H}_\infty$ state reconstruction problem for the LTV system $\overline{T}$

For the LTV system $\overline{T}$, given by the state space realization in equation (33), our aim of this section is to design an observer $\overline{K}$ to reconstruct the quantity xF. Following the outline of Section 5, for notational convenience we state the equivalent of Theorem 16 for $\overline{T}$ in the next theorem. This yields a solution to the intermediate problem.

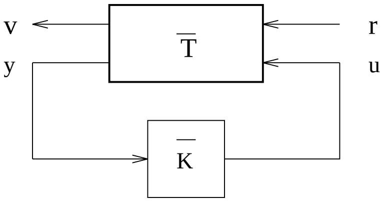

Figure 5. Block-schematic representation of the intermediate $\mathcal{H}_\infty$ state reconstruction problem.

Theorem 18. *Let $\overline{T}$ be a locally finite operator with state realization (33) and satisfying Assumptions 17(2-3). Furthermore, let $\overline{\Gamma}_o \in \mathcal{D}(\mathcal{M}_2, \mathcal{M}_2)$, be a prescribed disturbance attenuation level such that $\overline{\Gamma}_o \gg 0$. Then an operator $\overline{L} \in \mathcal{D}(\mathcal{N}_2, \mathcal{B}^{(-1)})$ solves the <u>intermediate</u> $\mathcal{H}_\infty$ state reconstruction problem if and only if there exists a unique solution $\overline{M}_o \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ of*

$$\begin{aligned} \overline{M}_o^{(-1)} &= [\overline{B}_1^*\overline{D}_E + \overline{A}^*\overline{M}_o\overline{C}_E]\left(\begin{bmatrix} \overline{\Gamma}_o I & 0 \\ 0 & 0 \end{bmatrix} - \overline{D}_E^*\overline{D}_E - \overline{C}_E^*\overline{M}_o\overline{C}_E\right)^{-1}[\overline{D}_E^*\overline{B}_1 + \overline{C}_E^*\overline{M}_o\overline{A}] \\ &\quad + \overline{A}^*\overline{M}_o\overline{A} + \overline{B}_1^*\overline{B}_1 \end{aligned} \tag{34}$$

with $\overline{D}_E = \begin{bmatrix} 0 & \overline{D}_{12} \end{bmatrix}$ and $\overline{C}_E = \begin{bmatrix} \overline{C}_1 & \overline{C}_2 \end{bmatrix}$, such that $\overline{\Gamma}_o I - \overline{C}_1^*\overline{M}_o\overline{C}_1 \gg 0$, $\overline{M}_o \geq 0$ and the operator $\overline{A}_o^\times$, defined as:

$$\overline{A}_o^\times = \overline{A} + \overline{C}_E\left(\begin{bmatrix} \overline{\Gamma}_o I & 0 \\ 0 & 0 \end{bmatrix} - \overline{D}_E^*\overline{D}_E - \overline{C}_E^*\overline{M}_o\overline{C}_E\right)^{-1}[\overline{D}_E^*\overline{B}_1 + \overline{C}_E^*\overline{M}_o\overline{A}]$$

satisfies $\ell_{\overline{A}_o^\times} < 1$. With this solution $\overline{M}_o$ of equation (24), the observer gain operator $\overline{L}$ is given as,

$$\overline{L} = -\begin{bmatrix} 0 & I \end{bmatrix}\left(\begin{bmatrix} \overline{\Gamma}_o I & 0 \\ 0 & 0 \end{bmatrix} - \overline{D}_E^*\overline{D}_E - \overline{C}_E^*\overline{M}_o\overline{C}_E\right)^{-1}[\overline{D}_E^*\overline{B}_1 + \overline{C}_E^*\overline{M}_o\overline{A}] \tag{35}$$

□

Remark 19. *The above Theorem shows that when a solution $\overline{M}_o \geq 0$ exists to the Riccati equation (34) then Assumption 17 (2) is satisfied.* □

6.3. A solution to the $\mathcal{H}_\infty$ output feedback problem

As suggested by the intermediate problem formulation, we use the LTV system $\overline{\mathbf{T}}$ to solve the $\mathcal{H}_\infty$ output feedback problem. To be able to do so, the LTV system **P** needs some further investigation. **P** is defined in the previous subsection and has some interesting properties that are highlighted in the following Lemma.

Lemma 20. *Let the LTV system* $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ *be defined as in equation (30), with state space realization given in equation (31), then the following conditions hold:*

1. $P \in \mathcal{U}^2$.
2. $P \begin{bmatrix} I & \\ & \Gamma_c \end{bmatrix} P^* = \begin{bmatrix} \Gamma_c & \\ & I \end{bmatrix}$.
3. $P_{12}^{-1} \in \mathcal{U}$.

Proof **1.** Let the state space realization in equation (31) be denoted more compactly as,

$$\mathbf{P} = \begin{bmatrix} A+FB_2 & C_1+FD_{21} & -(\tilde{B}_1+F\tilde{B}_3)U_1^{-\frac{1}{2}}\Gamma_c^{-\frac{1}{2}} \\ B_1 & 0 & U_1^{\frac{1}{2}}\Gamma_c^{-\frac{1}{2}} \\ U_2^{-\frac{1}{2}}B_2 & U_2^{-\frac{1}{2}}D_{21} & -U_2^{-\frac{1}{2}}\tilde{B}_3U_1^{-\frac{1}{2}}\Gamma_c^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} A' & C_1' & C_2' \\ B_1' & D_{11}' & D_{12}' \\ B_2' & D_{21}' & D_{22}' \end{bmatrix} \tag{36}$$

then we see by Theorem 14 that the operator $A+FB_2$ has $\ell_{A+FB_2} < 1$, and therefore $P \in \mathcal{U}^2$.

2. Using the expression for M_c in equation (75) and for F in equation (74), we derive the following relationship,

$$\mathbf{P} \begin{bmatrix} M_c^{(-1)} & & \\ & I & \\ & & \Gamma_c \end{bmatrix} \mathbf{P}^* = \begin{bmatrix} M_c & & \\ & \Gamma_c & \\ & & I \end{bmatrix} \tag{37}$$

Using the compact notation for the state space representation of P as given in the right hand side of equation (36), we can express $P \begin{bmatrix} I & \\ & \Gamma_c \end{bmatrix} P^*$ as,

$$\begin{aligned} P \begin{bmatrix} I & \\ & \Gamma_c \end{bmatrix} P^* = & \begin{bmatrix} D_{11}'D_{11}'^* + D_{12}'\Gamma_c D_{12}'^* & D_{11}'D_{21}'^* + D_{21}'\Gamma_c D_{22}'^* \\ & D_{21}'D_{21}'^* + D_{22}'\Gamma_c D_{22}'^* \end{bmatrix} \\ & + \begin{bmatrix} D_{11}'C_1'^* + D_{12}'\Gamma_c C_2'^* \\ D_{21}'C_1'^* + D_{22}'\Gamma_c C_2'^* \end{bmatrix} (I - Z^*A'^*)^{-1} Z^* \begin{bmatrix} B_1'^* & B_2'^* \end{bmatrix} \\ & + \begin{bmatrix} B_1' \\ B_2' \end{bmatrix} Z(I-A'Z)^{-1} \begin{bmatrix} C_1'D_{11}'^* + C_2'\Gamma_c D_{12}'^* & C_1'D_{21}'^* + C_2'\Gamma_c D_{22}'^* \end{bmatrix} \\ & + \begin{bmatrix} B_1' \\ B_2' \end{bmatrix} Z(I-A'Z)^{-1} \left[C_1'C_1'^* + C_2'\Gamma_c C_2'^*\right] (I-Z^*A'^*)^{-1} Z^* \begin{bmatrix} B_1'^* & B_2'^* \end{bmatrix}. \end{aligned}$$

Together with equation (37) it is easy to verify that $P \begin{bmatrix} I & \\ & \Gamma_c \end{bmatrix} P^*$ is equal to $\begin{bmatrix} \Gamma_c & \\ & I \end{bmatrix}$.

3. The state representation of P_{12} is,

$$\mathbf{P}_{12} = \begin{bmatrix} A+FB_2 & -(\tilde{B}_1+F\tilde{B}_3)U_1^{-\frac{1}{2}}\Gamma_c^{-\frac{1}{2}} \\ B_1 & U_1^{\frac{1}{2}}\Gamma_c^{-\frac{1}{2}} \end{bmatrix}$$

Since $U_1^{\frac{1}{2}}\Gamma_c^{-\frac{1}{2}}$ is invertible, the A-operator of the state space representation of P_{12}^{-1} equals:

$$A_{P_{12}^{-1}} = A + FB_2 + (\tilde{B}_1 + F\tilde{B}_3)U_1^{-1}B_1$$

and this operator is equal to the operator $A_c^\times$ defined in Theorem 14. The latter operator satisfies $\ell_{A_c^\times} < 1$, and item 3 of the Lemma is proved. $\square$

In the following lemma, we consider LTV systems P satisfying the conditions 1 to 3 of Lemma 20 operating in closed-loop with a LTV system Q as depicted in Figure 6,

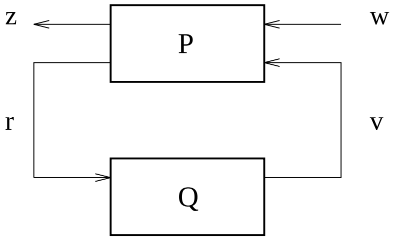

Figure 6. Closed-loop configuration of a LTV system P satisfying the conditions of Lemma 18 with a LTV system Q.

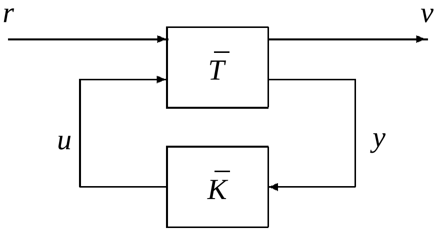

Figure 7. Block diagram of the closed-loop LTV system Q.

In this lemma, we make use of the following definition of internal stability.

Definition 21. *The closed-loop configuration depicted in Figure 6 is internally stable if and only if,*

$$(I-P_{22}Q)^{-1} \in \mathcal{U},\ (I-P_{22}Q)^{-1}P_{22} \in \mathcal{U},\ (I-QP_{22})^{-1} \in \mathcal{U} \text{ and } (I-QP_{22})^{-1}Q \in \mathcal{U}. \qquad \square$$

since the proof of the lemma is long and technical; it can be found in the Appendix.

Lemma 22. *Let* $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ *be a given LTV system satisfying the conditions 1 to 3 of Lemma 20. Then, the system of Figure 6 is internally stable, well-posed and* $\Gamma_c I - T_{wz}T_{wz}^* \gg 0$ *if and only if* $\Gamma_c I - QQ^* \gg 0$, *Q in Figure 7 is internally stable and well-posed.* $\square$

The above Lemma is the key towards the solution of the $\mathcal{H}_\infty$ output feedback problem. In order to apply this Lemma, we consider the feedback configuration in Figure 6 with the LTV system Q replaced by the LTV system of Figure 5. This is depicted in our final Figure 8. The solution to $\mathcal{H}_\infty$ output feedback problem is summarized in the next theorem.

Theorem 23. *Let T be a locally finite operator with state space realization in equation. (21) and satisfying the Assumptions 17. Furthermore, let $\Gamma_c = \gamma I_{\mathcal{M}_1}$ be a prescribed disturbance attenuation level with $\gamma > 0$. For this Γ_c, let M_c be a solution to the Riccati equation (19) satisfying the conditions of Theorem 14. Let this M_c define the state space representation of the LTV system $\overline{T}$ as in equation (33). Let $\overline{\Gamma}_o = \gamma I_{\mathcal{M}_2}$ and let $\overline{M}_o$*

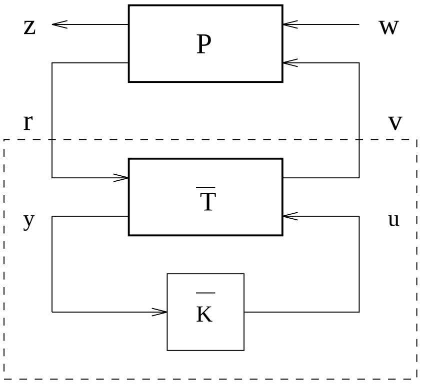

Figure 8. Block-schematic representation of the solution to $\mathcal{H}_\infty$ output feedback problem.

be a solution to the Riccati equation (34) satisfying the conditions stated in Theorem 18, then the controller $\overline{K}$ *defined by the observer gain operator* $\overline{L}$ *of equation (35) solves the* $\mathcal{H}_\infty$ *output feedback problem.*

PROOF In addition to the system $\overline{T}$, the operator M_c defines the LTV system P in Figure 8 with state space representation as in equation (31). Since this system P satisfies conditions 1 to 3 of Lemma 20, we only have to show that the LTV system within the dashed box of Figure 8 satisfies the conditions stipulated on the LTV system Q in Lemma 22, in order to apply this Lemma. These are:

1. $\underline{\Gamma_c I - \overline{T}_{rv}\overline{T}_{rv}^* \gg 0}$: Since the solution $\overline{M}_o$ of Theorem 18 guarantees that $\overline{\Gamma}_o I - \overline{T}_{rv}^*\overline{T}_{rv} \gg 0$ and since $\Gamma_c = \gamma I_{\mathcal{M}_1}$ and $\overline{\Gamma}_o = \gamma I_{\mathcal{M}_2}$, the identity,

   $$\left[\Gamma_c I - \overline{T}_{rv}\overline{T}_{rv}^*\right]^{-1} = \Gamma_c^{-1}\left[I + \overline{T}_{rv}(\overline{\Gamma}_o I - \overline{T}_{rv}^*\overline{T}_{rv})^{-1}\overline{T}_{rv}^*\right]$$

   shows that this condition holds.

2. $\underline{\overline{T}_{rv} \in \mathcal{U}}$: To show that the operator $\overline{T}_{rv}$ belongs to $\mathcal{U}$, we derive a state space representation of this operator. Recall the state space realization for $\overline{T}$ in equation (33) with $u = \hat{x}F$:

   $$\begin{array}{rcccccc} xZ^{-1} & = & x\overline{A} & + & r\overline{B}_1 & + & \hat{x}F\overline{B}_2 \\ v & = & x\overline{C}_1 & + & & + & \hat{x}F\overline{D}_{21} \\ y & = & x\overline{C}_2 & + & r\overline{D}_{12} & + & \hat{x}F\overline{D}_{22} \end{array}$$

   Substituting the last output equation in the state equation for the observer $\overline{K}$ yields:

   $$\hat{x}Z^{-1} = \hat{x}\overline{A} + \hat{x}F\overline{B}_2 + (x\overline{C}_2 - \hat{x}\overline{C}_2 + r\overline{D}_{12})\overline{L}$$

   And therefore, the state space representation for $\overline{T}_{rv}$ becomes:

   $$\begin{array}{rcl} \begin{bmatrix} xZ^{-1} & \hat{x}Z^{-1} \end{bmatrix} & = & \begin{bmatrix} x & \hat{x} \end{bmatrix} \begin{bmatrix} \overline{A} & \overline{C}_2\overline{L} \\ F\overline{B}_2 & \overline{A} + F\overline{B}_2 - \overline{C}_2\overline{L} \end{bmatrix} + r\begin{bmatrix} \overline{B}_1 & \overline{D}_{12}\overline{L} \end{bmatrix} \\ v & = & \begin{bmatrix} x & \hat{x} \end{bmatrix} \begin{bmatrix} \overline{C}_1 \\ F\overline{D}_{21} \end{bmatrix} \end{array}$$

   We now perform the following constant similarity transformation to this state realization:

   $$\begin{bmatrix} I & I & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} \overline{A} & \overline{C}_2\overline{L} & \overline{C}_1 \\ F\overline{B}_2 & \overline{A} + F\overline{B}_2 - \overline{C}_2\overline{L} & F\overline{D}_{21} \\ \overline{B}_1 & \overline{D}_{12}\overline{L} & \end{bmatrix} \begin{bmatrix} I & -I & \\ & I & \\ & & I \end{bmatrix} =$$

$$\left[\begin{array}{ccc} \overline{A}+F\overline{B}_2 & 0 & \overline{C}_1+F\overline{D}_{21} \\ F\overline{B}_2 & \overline{A}-\overline{C}_2\overline{L} & F\overline{D}_{21} \\ \overline{B}_1 & -\overline{B}_1+\overline{D}_{12}\overline{L} & \end{array}\right]$$

From this state representation we conclude that $\overline{T}_{rv}$ is in $\mathcal{U}$ if $\ell_{\overline{A}+F\overline{B}_2} < 1$ and $\ell_{\overline{A}-\overline{C}_2\overline{L}} < 1$. The latter condition is guaranteed by the solution $\overline{M}_o$ in Theorem 18. The first condition holds by Theorem 14, since by the definition of the quantities $\overline{A}$ and $\overline{B}_2$ in equation (33):

$$\overline{A}+F\overline{B}_2 = A+\tilde{B}_1U_1^{-1}B_1+F\tilde{B}_3U_1^{-1}B_1+FB_2$$

and the right hand side equals the operator $A_c^{\times}$ defined in Theorem 14.

Hence, we conclude by Lemma 22, that with the controller $\overline{K}$, the closed-loop system in Figure 8 is well-posed, internally stable and satisfies, $\Gamma_c I - T_{wz}T_{wz}^* \gg 0$. □

6.4. Necessary conditions

Theorem 23 provides sufficient conditions on the solution to the robust output feedback problem. Till now, we did not consider the necessity of these conditions. Based on our knowledge of the discrete time-invariant robust output feedback problem (i.e., [Stoorvogel 1992a]), it is expected that the conditions of Theorem 23 are also necessary. We proof the reverse implication of Theorem 23 by using Lemma 22, and further generalizing the proof of the time-invariant case.

Theorem 24. *Let T be a locally finite operator with state space realization as in equation (21). Let $((A,B_2)$ be uniformly stabilizable, $D_{21}^*D_{21} \gg 0$, and $\Gamma_c = \gamma^2 I_{\mathcal{M}_1}$ be a prescribed disturbance attenuation level with $\gamma > 0$. Assume that there exists a controller that solves the robust output feedback problem for T. Then there exists a solution to (19) fulfilling the conditions of Theorem 14. Now, let also $(\overline{C}_2,\overline{A})$ be uniformly detectable, $\overline{D}_{12}\overline{D}_{12}^* \gg 0$, and $\overline{\Gamma}_o = \gamma^2 I_{\mathcal{M}_2}$. Then there exists a solution to (34) that fulfills the conditions of Theorem 18.*

PROOF Since the robust output feedback problem for T is solvable, there exists a causal operator $R \in \mathcal{U}(\mathcal{M}_1,\mathcal{M}_2)$ such that for all $w \in \ell_2^{\mathcal{M}_1}$ with $u = wR$ we have $x \in \ell_2^{\mathcal{B}}$, and there exists a $\varepsilon > 0$ such that $\| \ \gamma^2 \, w\,w^* \ - z\,z^* \ \| > \varepsilon \,\|\, ww^* \,\|$. This means that the robust static state feedback control problem is solvable, and thus by Theorem 14 it follows that there exists a solution to (19) that fulfills the corresponding conditions. This proves the first part of the theorem.
In order to prove the second part of the theorem we make use of Lemma 22. Since the robust output feedback problem for T is solvable, by Lemma 22 we know that we only have to investigate Q. Now the proof follows straightforwardly by duality. $(\overline{C}_2,\overline{A})$ is uniformly detectable means that $(\overline{A}^*,\overline{C}_2^*)$ is uniformly stabilizable. Therefore, for the dual system of $\overline{T}$ we can follow the same arguments as in the first part of this theorem, and then we obtain that there exists a solution to (34) that fulfills the conditions of Theorem 18. □

7. TWO UNCOUPLED RICCATI EQUATIONS AND A COUPLING CONDITION

The introduction of the intermediate robust state reconstruction problem has the disadvantage that the assumptions which are made in Theorem 23 are not in the original diagonal operators that appear in the state-space description (21 of the system, but already involve the diagonal operators of the intermediate system $\overline{T}$. These assumptions are in terms of the original diagonal operators in combination with the solution M_c to the Riccati equation (19) A similar problem rises in the theory for discrete time-invariant systems, and is tackled in [Walker 1990] by using results of [Pappas *et. al.* 1980]. Therefore, we first generalize some of the results of [Pappas *et. al.* 1980] to time-varying systems, and then we use this to obtain a relation with a discrete time-varying Riccati equation that is given in the original system operators [Scherpen and Verhaegen 1995].

7.1. An equivalent representation of the time-varying Riccati equation

We are concerned with the discrete time-varying (forward) algebraic Riccati equation of the form (we adopt the notation of [Pappas *et. al.* 1980])

$$X = FX^{(-1)}F^* - FX^{(-1)}G_1^*(G_2 + G_1X^{(-1)}G_1^*)^{-1}G_1X^{(-1)}F^* + H. \tag{38}$$

Here H, $X \in \mathcal{D}(\mathcal{B},\mathcal{B})$, $F \in \mathcal{D}(\mathcal{B},\mathcal{B}^{(-1)})$, $G_1 \in \mathcal{D}(\mathcal{M},\mathcal{B}^{(-1)})$, $G_2 \in \mathcal{D}(\mathcal{M},\mathcal{M})$, and $G_2 = G_2^*$ is invertible, $H = H^*$. We assume that (F, G_1) is a stabilizable pair, and that (C,F) is a detectable pair, where $CC^* = H$. Finally, we define $G := G_1^*G_2^{-1}G_1$.

From the discrete maximum principle (see e.g. Whittle [Whittle 1990]) we obtain (similar to the time-invariant case) the Hamiltonian difference equations

$$\begin{pmatrix} x_{k+1} & \beta_{k+1} \end{pmatrix} \begin{pmatrix} I & 0 \\ G_k & F_k^* \end{pmatrix} = \begin{pmatrix} x_k & \beta_k \end{pmatrix} \begin{pmatrix} F_k & -H_k \\ 0 & I \end{pmatrix} \tag{39}$$

for every time-step, where x_k denotes the state at time t_k and β_k denotes the corresponding adjoint vector. We can rewrite this as

$$\begin{pmatrix} xZ^{-1} & \beta Z^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ G & F^* \end{pmatrix} = \begin{pmatrix} x & \beta \end{pmatrix} \begin{pmatrix} F & -H \\ 0 & I \end{pmatrix}. \tag{40}$$

Remark 25. *The assumption that G_2 is invertible is sufficient for our purposes (as can be seen in the next subsection). However, it is a restriction that can be dropped by working with the time-varying extension of the matrix pencil given in [Van Dooren 1981] (see also [Ionescu and Weiss 1992]) for the time-invariant case.*

Now we can state the following theorem

Theorem 26. *Assume that there exists a stabilizing solution of the algebraic Riccati equation (38), i.e., such that*

$$F^\times = F - FX^{(-1)}G_1^*(G_2 + G_1X^{(-1)}G_1^*)^{-1}G_1 \tag{41}$$

fulfills $\ell_{F^\times} < 1$. Then such solution X to equation (38) can be written as $X = Q^{-1}P$ for any Q non-singular, and Q and P that fulfill

$$S\begin{pmatrix} Q^{(-1)} & P^{(-1)} \end{pmatrix} \begin{pmatrix} I & 0 \\ G & F^* \end{pmatrix} = \begin{pmatrix} Q & P \end{pmatrix} \begin{pmatrix} F & -H \\ 0 & I \end{pmatrix} \tag{42}$$

where $S = QF^\times Q^{-(-1)}$.

PROOF We can rewrite (38) as

$$X - H = F^\times X^{(-1)}F^* \tag{43}$$

Take an arbitrary non-singular $Q \in \mathcal{D}(\mathcal{B},\mathcal{B})$, define $S := QF^\times Q^{-(-1)}$, and set $P := QX$ (thus $X = Q^{-1}P$). Substitute this in (43), then

$$P - QH = QF^\times Q^{-(-1)}P^{(-1)}F^* = SP^{(-1)}F^*. \tag{44}$$

Furthermore, (41) yields

$$\begin{aligned} Q^{-1}SQ^{(-1)}X^{(-1)}G_1^* &= FX^{(-1)}G_1^* - FX^{(-1)}G_1^*(G_2 + G_1X^{(-1)}G_1^*)^{-1}G_1X^{(-1)}G_1^* \\ &= FX^{(-1)}G_1^*(G_2 + G_1X^{(-1)}G_1^*)^{-1}G_2 \end{aligned}$$

Multiplication from the left by $G_2^{-1}G_1$, using (41), and using $X = Q^{-1}P$ yields (see also [Pappas *et. al.* 1980])

$$SP^{(-1)}G = QF - SQ^{(-1)} \tag{45}$$

Now we obtain (42) from (44) and (45). □

The next theorem states the reverse implication.

Theorem 27. *Assume that there exists a pair $Q, P \in \mathcal{D}(\mathcal{B},\mathcal{B})$ and a $S \in \mathcal{D}(\mathcal{B},\mathcal{B}^{(-1)})$, with Q non-singular, such that (42) holds. Then $X = Q^{-1}P$ solves the Riccati equation (38), assuming that $(G_2 + G_1^*X^{(-1)}G_1)$ is invertible.*

PROOF From (42) it follows that

$$F = Q^{-1}SP^{(-1)}G + Q^{-1}SQ^{(-1)} \tag{46}$$

and

$$Q^{-1}P = H + Q^{-1}SP^{(-1)}F^* \tag{47}$$

Substitute both (46) and (47) in (38), then, after some straightforward calculations (following [Pappas *et. al.* 1980]), this indeed yields a solution to the algebraic Riccati equation (38). □

For the backward algebraic Riccati equation we can easily obtain a similar result. We state this result without proof.

Theorem 28. *Assume that there exists a solution to the algebraic Riccati equation*

$$X^{(-1)} = F^*XF - F^*XG_1(G_2 + G_1^*XG_1)^{-1}G_1^*XF + H \tag{48}$$

where $H, X \in \mathcal{D}(\mathcal{B},\mathcal{B})$, $F \in \mathcal{D}(\mathcal{B},\mathcal{B}^{(-1)})$, $G_1 \in \mathcal{D}(\mathcal{B},\mathcal{N})$, $G_2 \in \mathcal{D}(\mathcal{N},\mathcal{N})$ and $G_2 = G_2^*$, $H = H^*$, and $G := G_1G_2^{-1}G_1^*$, such that

$$F^\times := F - G_1(G_2 + G_1^*XG_1)^{-1}G_1^*XF$$

is asymptotically stable (i.e., $\ell_{F^\times} < 1$). Then such solution X to equation (48) can be written as $X = Q^{-1}P$ for any Q non-singular, and Q and P that fulfill

$$S\begin{pmatrix} Q & P \end{pmatrix}\begin{pmatrix} I & 0 \\ G & F \end{pmatrix} = \begin{pmatrix} Q^{(-1)} & P^{(-1)} \end{pmatrix}\begin{pmatrix} F^* & -H \\ 0 & I \end{pmatrix} \tag{49}$$

where $S = Q^{(-1)}(F^\times)^*Q^{-1}$. □

7.2. A relation between three Riccati equations

It is well known that for continuous time-invariant systems the Riccati equations that occur in the solution of the robust output feedback problem are given in the original realization matrices, together with a coupling condition. For time-invariant discrete time systems this has been investigated in [Walker 1990], and a relationship between three Riccati equations has been found. Clearly, in the discrete time-varying case the conditions in Theorem 23 are not given in the original system operators. Therefore, in this section we generalize the result of [Walker 1990] to LTV systems.

Consider the stabilizing solution M_c to the algebraic Riccati equation (19), the stabilizing solution M_o to the algebraic Riccati equation (24), and the stabilizing solution $\overline{M}_o$ to the algebraic Riccati equation (34). Then we can give the following relation

Theorem 29. *If the prescribed disturbance levels are* $\Gamma_c = \gamma^2 I_{\mathcal{M}_1}$, $\Gamma_o = \gamma^2 I_{\mathcal{N}_1}$, *and* $\overline{\Gamma}_o = \gamma^2 I_{\mathcal{M}_2}$, *then*

$$\overline{M}_o = (I - \gamma^{-2} M_o M_c)^{-1} M_o \tag{50}$$

PROOF First we define

$$\begin{array}{lcl} E_o & := & D_o^* D_o - \left[\begin{array}{cc} \Gamma_o I & 0 \\ 0 & 0 \end{array}\right] \\ G_o & := & C_o E_o^{-1} C_o^* \\ \overline{E}_o & := & \overline{D}_o^* \overline{D}_o - \left[\begin{array}{cc} \overline{\Gamma}_o I & 0 \\ 0 & 0 \end{array}\right] \\ \overline{G}_o & := & \overline{C}_o \overline{E}_o^{-1} \overline{C}_o^* \end{array} \qquad \begin{array}{lcl} F_o & := & A - C_o E_o^{-1} D_o^* B_1 \\ H_o & := & B_1^* B_1 - B_1^* D_o E_o^{-1} D_o^* B_1 \\ \overline{F}_o & := & \overline{A} - \overline{C}_o \overline{E}_o^{-1} \overline{D}_o^* \overline{B}_1 \\ \overline{H}_o & := & \overline{B}_1^* \overline{B}_1 - \overline{B}_1^* \overline{D}_o \overline{E}_o^{-1} \overline{D}_o^* \overline{B}_1 \end{array} \tag{51}$$

By Theorem 28 we know that M_o, and $\overline{M}_o$ can be written as $M_o = Q^{-1} P$ and $\overline{M}_o = \overline{Q}^{-1} \overline{P}$, respectively, where Q, and $\overline{Q}$ are non-singular, and Q, P and $\overline{Q}$, $\overline{P}$, respectively, fulfill

$$S(\, Q \quad P\,) L = (\, Q^{(-1)} \quad P^{(-1)}\,) N, \;\; L := \begin{pmatrix} I & 0 \\ G_o & F_o \end{pmatrix} \text{ and } N := \begin{pmatrix} F_o^* & -H_o \\ 0 & I \end{pmatrix} \tag{52}$$

where $S = Q^{(-1)} (F_o^{\times})^* Q^{-1}$, and

$$\overline{S}(\, \overline{Q} \quad \overline{P}\,) \overline{L} = (\, \overline{Q}^{(-1)} \quad \overline{P}^{(-1)}\,) \overline{N}, \;\; \overline{L} := \begin{pmatrix} I & 0 \\ \overline{G}_o & \overline{F}_o \end{pmatrix} \text{ and } \overline{N} := \begin{pmatrix} \overline{F}_o^* & -\overline{H}_o \\ 0 & I \end{pmatrix} \tag{53}$$

where $\overline{S} = \overline{Q}^{-1} (\overline{F}_o^{\times})^* \overline{Q}^{(-1)}$. Now we follow the proof of Theorem 3 of [Walker 1990]. First assume that there exists a similarity transformation $T \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ such that

$$\begin{pmatrix} I & 0 \\ -\gamma^{-2} M_c & I \end{pmatrix} \overline{L} = LT \;\text{ and }\; \begin{pmatrix} I & 0 \\ -\gamma^{-2} M_c^{(-1)} & I \end{pmatrix} \overline{N} = NT \tag{54}$$

Then

$$\begin{array}{l} S(\, Q \quad P\,) L = (\, Q^{(-1)} \quad P^{(-1)}\,) N \Rightarrow S(\, Q \quad P\,) LT = (\, Q^{(-1)} \quad P^{(-1)}\,) NT \Rightarrow \\ S(\, Q \quad P\,) \begin{pmatrix} I & 0 \\ -\gamma^{-2} M_c & I \end{pmatrix} \overline{L} = (\, Q^{(-1)} \quad P^{(-1)}\,) \begin{pmatrix} I & 0 \\ -\gamma^{-2} M_c^{(-1)} & I \end{pmatrix} \overline{N} \end{array}$$

Together with (53) this implies that there exists a $R \in \mathcal{D}(\mathcal{B}, \mathcal{B})$ such that

$$(\, Q \quad P\,) \begin{pmatrix} I & 0 \\ -\gamma^{-2} M_c & I \end{pmatrix} = R^{-1} (\, \overline{Q} \quad \overline{P}\,)$$

Then

$$\overline{Q}^{-1} R = (Q - \gamma^{-2} P M_c)^{-1} \text{ and } R^{-1} \overline{P} = P \Rightarrow \overline{M}_o = \overline{Q}^{-1} \overline{P} = (Q - \gamma^{-2} P M_c)^{-1} P = (I - \gamma^{-2} M_o M_c)^{-1} M_o$$

Hence, the only part that is left to prove is the existence of T. Write T as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \tag{55}$$

then it follows from (54) that

$$\begin{pmatrix} I & 0 \\ \overline{G}_o - \gamma^{-2} M_c & \overline{F}_o \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ G_o T_{11} + F_o T_{21} & G_o T_{12} + F_o T_{22} \end{pmatrix}$$

and $\begin{pmatrix} \overline{F}_o^* & -\overline{H}_o \\ -\gamma^{-2} M_c^{(-1)} \overline{F}_o^* & \gamma^{-2} M_c^{(-1)} \overline{H}_o + I \end{pmatrix} = \begin{pmatrix} F_o^* T_{11} - H_o T_{21} & F_o^* T_{12} - H_o T_{22} \\ T_{21} & T_{22} \end{pmatrix}$

Therefore, T exists and is of the form (55) with $T_{11} = I$, $T_{12} = 0$, $T_{21} = -\gamma^{-2} M_c^{(-1)} \overline{F}_o^*$, and $T_{22} = \gamma^{-2} M_c^{(-1)} \overline{H}_o + I$ if

$$\overline{F}_o = F_o(I + \gamma^{-2} M_c^{(-1)} \overline{H}_o) \tag{56}$$

$$\overline{F}_o^* = F_o^* + \gamma^{-2} H_o M_c^{(-1)} \overline{F}_o^* \tag{57}$$

$$\overline{H}_o = H_o(I + \gamma^{-2} M_c^{(-1)} \overline{H}_o) \tag{58}$$

$$\overline{G}_o = G_o - \gamma^{-2} F_o M_c^{(-1)} \overline{F}_o^* + \gamma^{-2} M_c \tag{59}$$

We can prove (56), (57), (58), and (59) by straightforward, but huge and technical calculations. We give some of the steps in these calculations.

We start with (58).

$$\overline{H}_o - H_o(I + \gamma^{-2} M_c^{(-1)} \overline{H}_o) = B_1^* \tilde{H} B_1$$

where

$$\begin{aligned} \tilde{H} &= \gamma^2 U_1^{-1} - \gamma^2 U_1^{-1} D_{12}(D_{12}^* U_1^{-1} D_{12})^{-1} D_{12}^* U_1^{-1} - I + D_{12}(D_{12}^* D_{12})^{-1} D_{12}^* - B_1 M_c^{(-1)} B_1^* U_1^{-1} + \\ &\quad B_1 M_c^{(-1)} B_1^* U_1^{-1} D_{12}(D_{12}^* U_1^{-1} D_{12})^{-1} D_{12}^* U_1^{-1} + D_{12}(D_{12}^* D_{12})^{-1} D_{12}^* B_1 M_c^{(-1)} B_1^* U_1^{-1} - \\ &\quad D_{12}(D_{12}^* D_{12})^{-1} D_{12}^* B_1 M_c^{(-1)} B_1^* U_1^{-1} D_{12}(D_{12}^* U_1^{-1} D_{12})^{-1} D_{12}^* U_1^{-1} \\ &= D_{12}(D_{12}^* D_{12})^{-1} D_{12}^* (-I - B_1 M_c^{(-1)} B_1^* U_1^{-1} + \gamma^2 U_1^{-1})(D_{12}^* U_1^{-1} D_{12})^{-1} D_{12}^* U_1^{-1} = \quad \text{by (66)} \end{aligned}$$

Now we prove (57).

$$\begin{aligned} \overline{F}^* &- F^* - \gamma^{-2} H M_c^{(-1)} \overline{F}^* = \gamma^{-2} B_1^* \Big[\gamma^2 D_{12}(D_{12}^* D_{12})^{-1} D_{12}^* - \gamma^2 I + B_1 M_c^{(-1)} B_1^* - \\ &\quad D_{12}(D_{12}^* D_{12})^{-1} D_{12}^* B_1 M_c^{(-1)} B_1^*\Big] U_1^{-1} D_{12}(D_{12}^* U_1^{-1} D_{12})^{-1} D_{12}^* U_1^{-1} B_1 M_c^{(-1)} A^* \\ &\quad + \gamma^{-2} B_1^* \left[-U_1 + D_{12}(D_{12}^* D_{12})^{-1} D_{12}^* U_1\right] U_1^{-1} D_{12}(D_{12}^* U_1^{-1} D_{12})^{-1} C_2^* = 0 \quad \text{by (66)} \end{aligned}$$

Equation (56) can be easily obtained from (58) and (57).

$$\overline{F} - F(I + \gamma^{-2} M_c^{(-1)} \overline{H}_o) \quad = \quad \overline{F} - (\overline{F} - \gamma^{-2} \overline{F} M_c^{(-1)} \overline{H}_o)(I + \gamma^{-2} M_c^{(-1)} \overline{H}_o) = 0$$

Finally, we obtain (59) by using (19), (66), (68), and (74).

$$\begin{aligned} \overline{G} &- \gamma^{-2} M_c - G + \gamma^{-2} F M_c^{(-1)} \overline{F}^* \\ &= \gamma^{-2} \left[-U_3 U_2^{-1} U_3^* + (C_2 + \tilde{B}_1 U_1^{-1} D_{12})(D_{12}^* U_1^{-1} D_{12})^{-1}(C_2^* + D_{12}^* U_1^{-1} \tilde{B}_1^*) - \right. \\ &\quad A M_c^{(-1)} A^* - \tilde{B}_1 U_1^{-1} \tilde{B}_1^* - C_1 C_1^* + U_3 U_2^{-1} U_3^* + C_1 C_1^* + \\ &\quad (A - C_2(D_{12}^* D_{12})^{-1} D_{12}^* B_1) M_c^{(-1)} \left(A^* + B_1^* U_1^{-1} \tilde{B}_1 - \right. \\ &\quad \left.\left. B_1^* U_1^{-1} D_{12}(D_{12}^* U_1^{-1} D_{12})^{-1}(C_2^* + D_{12}^* U_1^{-1} \tilde{B}_1^*)\right)\right] - C_2(D_{12}^* D_{12})^{-1} C_2^* = 0 \end{aligned}$$

□

The relation of this theorem now means that we can reformulate Theorem 23 as follows

Corollary 30. *Let T be a locally finite operator with state space realization in Eqs. (21). Let (A, B_2) be* *uniformly stabilizable, (C_2, A) uniformly detectable, $D_{21}^* D_{21} \gg 0$, $D_{12} D_{12}^* \gg 0$, and $\Gamma_c = \gamma^2 I_{\mathcal{M}_1}$ be a prescribed disturbance attenuation level with $\gamma > 0$. For this Γ_c, let M_c be a solution to the Riccati equation (19) satisfying the corresponding conditions. Let $\Gamma_o = \gamma^2 I_{\mathcal{N}_1}$ and let M_o be a solution to the Riccati equation (24) satisfying the corresponding conditions. Assume that the coupling condition $\gamma^2 I - M_c M_o \gg 0$ is fulfilled. Then there exists an controller that solves the robust output feedback problem.* ☐

8. COMPUTATIONAL CONSIDERATIONS: AN APPLICATION

8.1. Algorithmic summary

The solution provided by Theorem 23 leads to the following algorithm to solve the output feedback problem:

Given: The state space representation of the plant T, now specified on a local time scale as:

$$\begin{array}{rcccccc} x_{k+1} & = & x_k A_k & + & w_k B_{1,k} & + & u_k B_{2,k} \\ z_k & = & x_k C_{1,k} & + & & + & u_k D_{21,k} \\ y_k & = & x_k C_{2,k} & + & w_k D_{12,k} & + & u_k D_{22,k} \end{array}$$

and a prescribed disturbance attenuation level $\gamma > 0$.

Do the following:

STEP 1: Determine an initial condition for the Riccati equation (19), now denoted on a local time scale as:

$$\begin{aligned} M_{c,k} = {} & A_k M_{c,k+1} A_k^* + \left[C_{1,k} D_{E,k}^* + A_k M_{c,k+1} B_{E,k}^* \right] \\ & \left(\begin{bmatrix} \gamma I & 0 \\ 0 & 0 \end{bmatrix} - D_{E,k} D_{E,k}^* - B_{E,k} M_{c,k+1} B_{E,k}^* \right)^{-1} \left[D_{E,k} C_{1,k}^* + B_{E,k} M_{c,k+1} A_k^* + C_{1,k} C_{1,k}^* \right] \end{aligned} \tag{60}$$

with $D_{E,k} = \begin{bmatrix} 0 \\ D_{21,k} \end{bmatrix}$ and $B_{E,k} = \begin{bmatrix} B_{1,k} \\ B_{2,k} \end{bmatrix}$. Then solve this equation recursively such that,

$$U_{1,k} = \gamma I - B_{1,k} M_{c,k+1} B_{1,k}^* > 0 \quad \text{and} \quad M_{c,k} \geq 0$$

STEP 2: Calculate the system matrices of the LTV system $\overline{T}$ given in equation (33) and the Feedback gain F_k as follows:

$$\begin{array}{rclrcl} \tilde{B}_{1,k} & = & A_k M_{c,k+1} B_{1,k}^* & \tilde{B}_{3,k} & = & B_{2,k} M_{c,k+1} B_{1,k}^* \\ \tilde{B}_{2,k} & = & C_{11,k} D_{21,k}^* + A_k M_{c,k+1} B_{2,k}^* & U_{3,k} & = & \tilde{B}_{2,k} + \tilde{B}_{1,k} U_{1,k}^{-1} \tilde{B}_{3,k}^* \\ U_{2,k} & = & B_{2,k} M_{c,k+1} B_{2,k}^* + \tilde{B}_{3,k} U_{1,k}^{-1} \tilde{B}_{3,k}^* + D_{21,k} D_{21,k}^* & F_k & = & -U_{3,k} U_{2,k}^{-1} \\ \overline{A}_k & = & A_k + \tilde{B}_{1,k} U_{1,k}^{-1} B_{1,k} & \overline{B}_{1,k} & = & \sqrt{\gamma} U_{1,k}^{-1} B_{1,k} \\ \overline{B}_{2,k} & = & \tilde{B}_{3,k} U_{1,k}^{-1} B_{1,k} + B_{2,k} & \overline{C}_{1,k} & = & U_{3,k} U_{2,k}^{-\frac{1}{2}} \\ \overline{C}_{2,k} & = & C_{2,k} + \tilde{B}_{1,k} U_{1,k}^{-1} D_{12,k} & \overline{D}_{21,k} & = & U_{2,k}^{\frac{1}{2}} \\ \overline{D}_{12,k} & = & \sqrt{\gamma} U_{1,k}^{-\frac{1}{2}} D_{12,k} & \overline{D}_{22,k} & = & \tilde{B}_{3,k} U_{1,k}^{-1} D_{12,k} + D_{22,k} \end{array}$$

STEP 3: Determine an initial condition to the Riccati equation (34), again denoted on a local time scale as:

$$\begin{aligned} \overline{M}_{o,k+1} = {} & \overline{A}_k \overline{M}_{o,k} \overline{A}_k^* + \left[\overline{B}_{1,k}^* \overline{D}_{E,k} + \overline{A}_k^* \overline{M}_{o,k} \overline{C}_{E,k} \right] \\ & \left(\begin{bmatrix} \gamma I & 0 \\ 0 & 0 \end{bmatrix} - \overline{D}_{E,k}^* \overline{D}_{E,k} - \overline{C}_{E,k}^* \overline{M}_{o,k} \overline{C}_{E,k} \right)^{-1} \left[\overline{D}_{E,k}^* \overline{B}_{1,k} + \overline{C}_{E,k}^* \overline{M}_{o,k} \overline{A}_k + \overline{B}_{1,k}^* \overline{B}_{1,k} \right] \end{aligned} \tag{61}$$

with $\overline{D}_{E,k} = \begin{bmatrix} 0 & \overline{D}_{12,k} \end{bmatrix}$ and $\overline{C}_{E,k} = \begin{bmatrix} \overline{C}_{1,k} & \overline{C}_{2,k} \end{bmatrix}$. Then solve this equation recursively such that,

$$\gamma I - \overline{C}^*_{1,k}\overline{M}_{o,k}\overline{C}_{1,k} > 0 \quad \text{and} \quad \overline{M}_{o,k+1} \geq 0$$

STEP 4: Calculate the gain $\overline{L}$ of the controller $\overline{K}$:

$$L_k = -\begin{bmatrix} 0 & I \end{bmatrix} \left(\begin{bmatrix} \gamma I & 0 \\ 0 & 0 \end{bmatrix} - \overline{D}^*_{E,k}\overline{D}_{E,k} - \overline{C}^*_{E,k}\overline{M}_{o,k}\overline{C}_{E,k} \right)^{-1} \left[\overline{D}^*_{E,k}\overline{B}_{1,k} + \overline{C}^*_{E,k}\overline{M}_{o,k}\overline{A}_k + \overline{B}^*_{1,k}\overline{B}_{1,k}\right]$$

STEP 5: Calculate the state space representation of the controller $\overline{K}$ at the time instant k:

$$\begin{aligned} \hat{x}_{k+1} &= \hat{x}_k \left(\overline{A}_k + F_k\overline{B}_{2,k} - (\overline{C}_{2,k} + F_k\overline{D}_{22,k})\overline{L}_k\right) + y_k\overline{L}_k \\ u_k &= \hat{x}_k F_k \end{aligned} \tag{62}$$

Based on this algorithmic description a number of remarks hold:

Remarks 31. *1. The $\mathcal{H}_\infty$ output feedback controller in equation (62) has the same (time-varying) state dimension as that of the given plant T.*

2. Since as shown in Theorems 14 and 18, the solutions of the two Riccati equations (60) and (61), that determine the $\mathcal{H}_\infty$ controller in equation (62) are unique provided the conditions of the theorem are satisfied, the key information in the above algorithm is:

   (a) The determination of the initial conditions to both Riccati equations.

   (b) The proper selection of γ.

   As in the time-invariant case, the (optimal) selection of γ is done iteratively. In the following subsection we discuss for a particular application how the initial conditions to equation. (60-61) can be determined.

3. The recursive Riccati equations (60-61) run opposite in time, namely equation (60) backward in time and equation (61) forward in time. Also since the system matrices in equation (61) are specified by the solution to equation (60), one has to complete the recursions with equation (60) first and completely. Therefore, when one wants to calculate the $\mathcal{H}_\infty$ controller $\overline{K}$ at a particular time instant k, a complete description of the plant for all time instances larger then k is necessary. This is a fundamental drawback when one wants to use the present solution in an adaptive controller strategy.

8.2. Robust control of a LTV system changing from one operation point to another

A widely occurring control problem in a time-varying context is the control of a plant making a transition from one operation point in its operation envelope to another. Examples are an airplane changing from cruise flight to landing configuration or when changing the throughput of a distillation column in the process industry, etc.

In this section, we highlight how to determine the initial conditions to both Riccati equations (60-61) for this particular situation.

Let the system T have the time-invariant state space representation:

$$\begin{aligned} x_{k+1} &= x_k A_{-\infty} + w_k B_{1,-\infty} + u_k B_{2,-\infty} \\ z_k &= x_k C_{1,-\infty} + \phantom{w_k B_{1,-\infty}} + u_k D_{21,-\infty} \\ y_k &= x_k C_{2,-\infty} + w_k D_{12,-\infty} + u_k D_{22,-\infty} \end{aligned} \tag{63}$$

for $k \in (-\infty, 0)$ and the time-invariant state space representation:

$$\begin{array}{rcllllll} x_{k+1} & = & x_k A_\infty & + & w_k B_{1,\infty} & + & u_k B_{2,\infty} \\ z_k & = & x_k C_{1,\infty} & + & & + & u_k D_{21,\infty} \\ y_k & = & x_k C_{2,\infty} & + & w_k D_{12,\infty} & + & u_k D_{22,\infty} \end{array} \tag{64}$$

Then we can determine an initial condition to equation (60) by using existing solutions for the discrete time-invariant $\mathcal{H}_\infty$ control problem [Stoorvogel and Weeren 1993] to determine the stabilizing solution of equation (60) for:

$$M_{c,k} = M_{c,k+1} = M_{c,\infty} \;\; A_k = A_\infty \;\; C_{1,k} = C_{1,\infty} \;\text{ etc.}$$

When γ is chosen properly, and since the solution to equation (60) is unique, it will converge to the stabilizing solution of this equation for:

$$M_{c,k} = M_{c,k+1} = M_{c,-\infty} \;\; A_k = A_{-\infty} \;\; C_{1,k} = C_{1,-\infty} \;\text{ etc.}$$

Say that convergence has occurred for $k \leq -N$, then the LTV system $\overline{T}$ is calculated as indicated in STEP 3 of the algorithm summarized in Section 7.1 for $k = -N : 1 : 0$. Again we can use existing solutions to the time-invariant case to compute the stabilizing solution to (61).

Hence, we conclude that for this particular (simple) application, the $\mathcal{H}_\infty$ output feedback controller is time-variant in the time interval $[-N, 0]$.

9. CONCLUDING REMARKS

The ∞-horizon $\mathcal{H}_\infty$ output feedback control problem for LTV systems under standard assumptions has been addressed in the present paper. The strategy of the solution follows that outlined in [Doyle *et. al.* 1989]. However, contrary to [Doyle *et. al.* 1989], which derives a solution for the continuous time-invariant counterpart based on operator theoretic results of mixed Hankel-Toeplitz operators, the bounded real lemma in the proper time-varying context plays the key role in solving the $\mathcal{H}_\infty$ output feedback problem.

Taking into account that the latter lemma plays a fundamental role in the solution of a large number of engineering problems, such as demonstrated e.g. in [Anderson and Vongpanitlerd 1973] for the time-invariant case and later on in [van der Veen 1993b] for the time-varying case, it might be expected that the solution devised in this way becomes more easily accessible to the practitioner engineer interested in the theoretical background.

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APPENDIX

The proof of Theorem 14:

Proof (The strategy of the first part of the proof follows that given in Theorem 3.1. of [De Souza and Xie 1992].) It follows from Theorem 11 that the operator F solves the $\mathcal{H}_\infty$ static state feedback control problem if and only if there exists a unique operator $M \in \mathcal{D}(\mathcal{B},\mathcal{B})$ which is a solution of,

$$\begin{aligned} M \;\; = \;\; & (A+FB_2)M^{(-1)}(A+FB_2)^* + \left[(C_1+FD_{21})D_{11}^* + (A+FB_2)M^{(-1)}B_1^*\right] \\ & (\Gamma_c I - D_{11}D_{11}^* - B_1 M^{(-1)} B_1^*)^{-1} \left[D_{11}(C_1^* + D_{21}^* F^*) + B_1 M^{(-1)}(A^* + B_2^* F^*)\right] + \\ & (C_1+FD_{21})(C_1^* + D_{21}^* F^*) \end{aligned} \tag{65}$$

such that

$$U_{1M} = \Gamma_c I - D_{11}D_{11}^* - B_1 M^{(-1)} B_1^* \gg 0, \quad M \geq 0 \tag{66}$$

and the operator $A_c^{\times}{}'$ given as:

$$A_c^{\times}{}' = A + FB_2 + \left[(C_1+FD_{21})D_{11}^* + (A+FB_2)M^{(-1)}B_1^*\right] U_{1M}^{-1} B_1 \tag{67}$$

satisfies $\ell_{A_c^{\times}{}'} < 1$.

Let us define the following quantities:

$$\tilde{B}_{3M} = D_{21}D_{11}^* + B_2 M^{(-1)} B_1^* \tag{68}$$

$$\tilde{B}_{1M} = C_1 D_{11}^* + A M^{(-1)} B_1^* \tag{69}$$

$$\tilde{B}_{2M} = C_1 D_{21}^* + A M^{(-1)} B_2^* \tag{70}$$

$$U_{3M} = \tilde{B}_{2M} + \tilde{B}_{1M} U_{1M}^{-1} \tilde{B}_{3M}^* \tag{71}$$

$$U_{2M} = B_{2M} M^{(-1)} B_{2M}^* + \tilde{B}_{3M} U_{1M}^{-1} \tilde{B}_{3M}^* + D_{21} D_{21}^* \gg 0 \tag{72}$$

Then we can rewrite equation (65) as,

$$M = AM^{(-1)}A^* + \tilde{B}_{1M} U_{1M}^{-1} \tilde{B}_{1M}^* + C_1 C_1^* - U_{3M} U_{2M}^{-1} U_{3M}^* + (F + U_{3M} U_{2M}^{-1}) U_{2M} (F^* + U_{2M}^{-1} U_{3M}^*) \tag{73}$$

where F solves the $\mathcal{H}_\infty$ static state feedback problem. For the if direction of this proof we continue in the following, and for the only if direction we refer to [Halanay and Ionescu 1994]. In order to proof the if direction, define

$$F := -U_{3M} U_{2M}^{-1} \tag{74}$$

and the above Riccati equation (73) further simplifies to,

$$M = AM^{(-1)}A^* + \tilde{B}_{1M} U_{1M}^{-1} \tilde{B}_{1M}^* + C_1 C_1^* - U_{3M} U_{2M}^{-1} U_{3M}^* \tag{75}$$

With equations (70-71), this equation can be written as:

$$M = AM^{(-1)}A^* + \begin{bmatrix} \tilde{B}_{1M} & \tilde{B}_{2M} \end{bmatrix} \begin{bmatrix} U_{1M}^{-1} - U_{1M}^{-1} \tilde{B}_{3M}^* U_{2M}^{-1} \tilde{B}_{3M} U_{1M}^{-1} & -U_{1M}^{-1} \tilde{B}_{3M}^* U_{2M}^{-1} \\ -U_{2M}^{-1} \tilde{B}_{3M} U_{1M}^{-1} & -U_{2M}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{B}_{1M}^* \\ \tilde{B}_{2M}^* \end{bmatrix} + C_1 C_1^*$$

Using standard operator calculus, the inverse of the 2×2 block operator equals:

$$\begin{bmatrix} U_{1M} & -\tilde{B}_{3M}^* \\ -\tilde{B}_{3M} & \tilde{B}_{3M} U_{1M}^{-1} \tilde{B}_{3M}^* - U_{2M} \end{bmatrix}$$

and hence with the definition of U_{1M} and the quantities in equation. (68-72), the above Riccati equation becomes equal to that given in equation (19).

To conclude the proof we verify the equivalence of the operators $A_c^{\times}$, F and the operators $A_c^{\times}{}'$ and $-U_{3M}U_{2M}^{-1}$

respectively. We start with the verification of $A_c^\times$.

$$\begin{aligned}
A_c^\times &= A + \begin{bmatrix} \tilde{B}_{1M} & \tilde{B}_{2M} \end{bmatrix} \begin{bmatrix} U_{1M}^{-1} - U_{1M}^{-1}\tilde{B}_{3M}^* U_{2M}^{-1}\tilde{B}_{3M}U_{1M}^{-1} & -U_{1M}^{-1}\tilde{B}_{3M}^* U_{2M}^{-1} \\ -U_{2M}^{-1}\tilde{B}_{3M}U_{1M}^{-1} & -U_{2M}^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\
&= A - U_{3M}U_{2M}^{-1}B_2 + \tilde{B}_{1M}U_{1M}^{-1}B_1 - U_{3M}U_{2M}^{-1}\tilde{B}_{3M}U_{1M}^{-1}B_1 \\
&= (A + FB_2) + \left[(C_1 + FD_{21})D_{11}^* + (A + FB_2)M_c^{(-1)}B_1^*\right] U_{1M}^{-1}B_1 \\
&= A_c^{\times\prime}
\end{aligned}$$

Second we verify that of F given in equation (20) and $-U_{3M}U_{2M}^{-1}$.

$$\begin{aligned}
F &= \begin{bmatrix} \tilde{B}_{1M} & \tilde{B}_{2M} \end{bmatrix} \begin{bmatrix} U_{1M}^{-1} - U_{1M}^{-1}\tilde{B}_{3M}^* U_{2M}^{-1}\tilde{B}_{3M}U_{1M}^{-1} & -U_{1M}^{-1}\tilde{B}_{3M}^* U_{2M}^{-1} \\ -U_{2M}^{-1}\tilde{B}_{3M}U_{1M}^{-1} & -U_{2M}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} \\
&= -\tilde{B}_{1M}U_{1M}^{-1}\tilde{B}_{3M}^* U_{2M}^{-1} - \tilde{B}_{2M}U_{2M}^{-1} = -U_{3M}U_{2M}^{-1}
\end{aligned}$$

□

The proof of Lemma 22.

PROOF (⇐) From the relationship $\begin{bmatrix} z & r \end{bmatrix} = \begin{bmatrix} w & v \end{bmatrix} P$ and $v = rQ$, we derive,

$$\begin{aligned}
r &= wP_{12}(I - QP_{22})^{-1} &&= wT_{wr} \\
v &= wP_{12}(I - QP_{22})^{-1}Q &&= wT_{wr}Q \\
z &= w\left[P_{11} + P_{12}(I - QP_{22})^{-1}QP_{21}\right]
\end{aligned} \tag{76}$$

Since $P \begin{bmatrix} I & \\ & \Gamma_c \end{bmatrix} P^* = \begin{bmatrix} \Gamma_c & \\ & I \end{bmatrix}$, it follows that $P_{22}\Gamma_c P_{22}^* = I - P_{21}P_{21}^* \le I$. Hence, $I \ge P_{22}\Gamma_c P_{22}^* \gg P_{22}QQ^*P_{22}^*$ and $(I - QP_{22})^{-1} \in \mathcal{U}$. Therefore, the closed-loop system is well posed. Since $P \in \mathcal{U}^2$ and $Q \in \mathcal{U}$, internal stability follows.

Again, from the property $P \begin{bmatrix} I & \\ & \Gamma_c \end{bmatrix} P^* = \begin{bmatrix} \Gamma_c & \\ & I \end{bmatrix}$, we derive that,

$$zz^* + r\Gamma_c r^* = w\Gamma_c w^* + vv^*$$

Using the definition of r and v in terms of w as given in equation (76), this is equivalent to,

$$\begin{aligned}
zz^* &= w\left(\Gamma_c I + T_{wr}QQ^*T_{wr}^* - T_{wr}\Gamma_c T_{wr}^*\right) w^* \\
&= w\left(\Gamma_c I - T_{wr}(\Gamma_c I - QQ^*)T_{wr}^*\right) w^*
\end{aligned}$$

Since $\Gamma_c I - QQ^* \gg 0$ and T_{wr}, T_{wr}^{-1} are both in $\mathcal{U}$, it follows that,

$$zz^* < w\Gamma_c w^*$$

for $\forall w \neq 0$. This is equivalent to,

$$\Gamma_c I - T_{wz}T_{wz}^* \gg 0$$

(⇒) Assume that $\Gamma_c I - T_{wz}T_{wz}^* \gg 0$. The relation between r and w is given by $r = wP_{12}(I - QP_{22})^{-1}$. Since $P_{12}^{-1} \in \mathcal{U}$, and $(I - QP_{22}) \in \mathcal{X}$ it follows that

$$(I - QP_{22})P_{12}^{-1}\left(\Gamma_c I - T_{wz}T_{wz}^*\right)P_{12}^{-*}(I - QP_{22})^* \gg 0. \tag{77}$$

Since P fulfills condition 2 of Lemma 20, we know that

$$\begin{aligned}
&P_{12}^{-1}P_{11}P_{11}^{*}P_{12}^{-*} = P_{12}^{-1}\Gamma_c P_{12}^{-*} - \Gamma_c I\\
&P_{21}P_{21}^{*} = I - P_{22}\Gamma_c P_{22}^{*}\\
&P_{21}P_{11}^{*}P_{12}^{-*} = -P_{22}\Gamma_c
\end{aligned}$$

Furthermore, $T_{wz} = P_{11} + P_{12}(I - QP_{22})^{-1}QP_{21}$. Plugging this into equation (77), then we obtain by straightforward calculations that

$$\Gamma_c I - QQ^{*} \gg 0.$$

Now we know (as above) from condition 2 on P that $I - P_{22}QQ^{*}P_{22}^{*} \gg 0$. Furthermore, $(I - QP_{22})^{-1} \in \mathcal{U}$, and $(I - QP_{22})^{-1}Q \in \mathcal{U}$ by the internal stability of the system in Figure 6. Then Proposition 6.11 of [van der Veen 1993b] states that the projections on $\mathcal{L}_2 Z^{-1}$ of $(I - QP_{22})^{-1}$ and $(I - QP_{22})$ are isomorphisms on $\mathcal{L}_2 Z^{-1}$. Therefore, it follows that $(I - QP_{22}) \in \mathcal{U}$. This implies that $Q \in \mathcal{U}$. Also the system Q is stabilizable and detectable, since the closed loop system is internally stable. This implies that Q is internally stable. $\square$

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