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Markov Parameter Estimation using Least Squares Lattice Recursions

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The work in this note is based on the results given in [1].

1 The VARX predictor

Consider that the dynamics of the system to be modelled can be written in the following minimal state-space model in the innovation form:

$$\mathcal{S}\begin{cases} x_{k+1} &= Ax_k + Bu_k + Ke_k, \\ y_k &= Cx_k + e_k, \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^r$, $y_k \in \mathbb{R}^\ell$, are the state, input and output vectors, and $e_k \in \mathbb{R}^\ell$ denotes the zero-mean white innovation process noise over a time $k = \{0, \ldots, N-1\}$. The state-space matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{\ell\times n}$, $D \in \mathbb{R}^{\ell\times r}$, and $K \in \mathbb{R}^{n\times\ell}$ are also called the system, input, output, direct feedthrough, and Kalman gain matrix, respectively.

Let the state-space model in (1) be rewritten in the Kalman predictor form as:

$$\begin{cases} x_{k+1} &= \tilde{A}x_k + \tilde{B}z_k, \\ y_k &= Cx_k + e_k, \end{cases} \tag{2}$$

with $\tilde{A} = A - KC$, $\tilde{B}_k = \begin{bmatrix} B & K \end{bmatrix}^T$, and $z_k = \begin{bmatrix} u_k^T & y_k^T \end{bmatrix}^T$. Considering a finite representation up to a past window p, then the power series description of the forward VARX model becomes:

$$y_k = \sum_{j=1}^{p} \Xi_{p,j} z_{k-j} + e_k, \quad \Xi_j = C\tilde{A}^j\tilde{B}, \tag{3}$$

where $\Xi_p = \begin{bmatrix} \Xi_{1,p} & \cdots & \Xi_{p,p} \end{bmatrix}$ is the set with the forward Markov parameters. We also consider the power series description of the backwards VARX model:

$$y_k = \sum_{j=-1}^{-p} \Theta_{p,-j} z_{k-j} + r_k, \tag{4}$$

where $\Theta_p = \begin{bmatrix} \Theta_{1,p} & \cdots & \Theta_{p,p} \end{bmatrix}$ is the set with the backward Markov parameters.

2 The Yule-Walker equation

Rewriting (3) and (4) gives:

$$\mathcal{I}z_k - \sum_{j=1}^{p} \Xi_{p,j} z_{k-j} = e_k, \tag{5}$$

$$\mathcal{I}z_k - \sum_{j=-1}^{-p} \Theta_{p,-j} z_{k-j} = r_k. \tag{6}$$

where $\mathcal{I} = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Multiply both sides with z_{k-t}^T and $t \in \mathbb{N}$ gives:

$$\mathcal{I}z_k z_{k-t}^T - \sum_{j=1}^{p} \Xi_{p,j} z_k z_{k-t+j}^T = e_k z_{k-t}^T, \tag{7}$$

$$\mathcal{I}z_k z_{k-t}^T - \sum_{j=-1}^{-p} \Theta_{p,-j} z_k z_{k-t+j}^T = r_k z_{k-t}^T. \tag{8}$$

By taking the expectance gives:

$$\begin{cases} \mathcal{I}\phi_t - \sum\limits_{j=1}^{p} \Xi_{p,j}\phi_{t-j} = E_p^f, & t = 0 \\ \mathcal{I}\phi_t - \sum\limits_{j=1}^{p} \Xi_{p,j}\phi_{t-j} = 0, & t \neq 0 \end{cases} \tag{9}$$

$$\begin{cases} \mathcal{I}\phi_t - \sum\limits_{j=-1}^{-p} \Theta_{p,-j}\phi_{t-j} = E_p^b, & t = 0 \\ \mathcal{I}\phi_t - \sum\limits_{j=-1}^{-p} \Theta_{p,-j}\phi_{t-j} = 0, & t \neq 0 \end{cases} \tag{10}$$

where $\phi_t = \mathrm{E}\left[z_k z_{k-t}^T\right]$, $E_p^f = \mathrm{E}\left[e_k e_k^T\right]$ and $E_p^b = \mathrm{E}\left[r_k r_k^T\right]$. These equations can be rearranged in the lifted block-Toeplitz matrix equations, the so-called Yule-Walker equations:

$$\begin{bmatrix} \mathcal{I} & -\Xi_{p,1} & \cdots & -\Xi_{p,p} \end{bmatrix} \underbrace{\begin{bmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-p} \\ \phi_1 & \phi_0 & \ddots & \phi_{-p+1} \\ \vdots & \ddots & \ddots & \vdots \\ \phi_p & \phi_{p-1} & \cdots & \phi_0 \end{bmatrix}}_{\Phi_p} = \begin{bmatrix} E_p^f & 0 & \cdots & 0 \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} -\Theta_{p,p} & \cdots & -\Theta_{p,1} & \mathcal{I} \end{bmatrix} \underbrace{\begin{bmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-p} \\ \phi_1 & \phi_0 & \ddots & \phi_{-p+1} \\ \vdots & \ddots & \ddots & \vdots \\ \phi_p & \phi_{p-1} & \cdots & \phi_0 \end{bmatrix}}_{\Phi_p} = \begin{bmatrix} 0 & \cdots & 0 & E_p^b \end{bmatrix} \tag{12}$$

3 The Levinson-Durbin algortihm

Assuming we know the jth order predictor, the following relation exists for the $(j+1)$th order solution:

$$\begin{bmatrix} \mathcal{I} & -\Xi_{j,1} & \cdots & -\Xi_{j,j} & 0 \\ 0 & -\Theta_{j,j} & \cdots & -\Theta_{j,1} & \mathcal{I} \end{bmatrix} \Phi_{j+1} = \begin{bmatrix} E_j^f & 0 & \cdots & 0 & \Delta_{j+1}^f \\ \Delta_{j+1}^b & 0 & \cdots & 0 & E_j^b \end{bmatrix} \tag{13}$$

where

$$\Delta_{j+1}^f = \phi_{j+1} - \sum_{i=1}^{j} \Xi_{j,i}\phi_{j-i+1} \tag{14}$$

$$\Delta_{j+1}^b = \phi_{-(j+1)} - \sum_{i=-1}^{-j} \Theta_{j,-i}\phi_{j-i-1} \tag{15}$$

By introducing $R_{j+1}^f = (E_j^f)^{-1}\Delta_{j+1}^b$ and $R_{j+1}^b = \Delta_{j+1}^f(E_j^b)^{-1}$, which are the forward and backward reflection coefficients, we can introduce zeros into the positions of Δ_{j+1}^b and Δ_{j+1}^f.

$$\begin{aligned} &\begin{bmatrix} I & -R_{j+1}^b \\ -(R_{j+1}^f)^T & I \end{bmatrix} \begin{bmatrix} \mathcal{I} & -\Xi_{j,1} & \cdots & -\Xi_{j,j} & 0 \\ 0 & -\Theta_{j,j} & \cdots & -\Theta_{j,1} & \mathcal{I} \end{bmatrix} \Phi_{j+1} \\ &= \begin{bmatrix} \mathcal{I} & -\Xi_{j+1,1} & \cdots & -\Xi_{j+1,j} & -\Xi_{j+1,j+1} \\ -\Theta_{j+1,j+1} & -\Theta_{j+1,j} & \cdots & -\Theta_{j+1,1} & \mathcal{I} \end{bmatrix} \Phi_{j+1} \\ &= \begin{bmatrix} E_{j+1}^f & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & E_{j+1}^b \end{bmatrix} \end{aligned} \tag{16}$$

where

$$E_{j+1}^f = E_j^f - \Delta_{j+1}^f(E_j^b)^{-1}\left(\Delta_{j+1}^b\right)^T \tag{17}$$

$$E_{j+1}^b = E_j^b - \left(\Delta_{j+1}^b\right)^T (E_j^f)^{-1}\Delta_{j+1}^f \tag{18}$$

Note, that it can be shown that:

$$\Delta_{j+1} = \Delta_{j+1}^f = \Delta_{j+1}^b = \mathrm{E}\left[e_k r_k^T\right]. \tag{19}$$

Eliminating the covariance matrix Φ_{p+1}, leads to the following Levinson-Durbin (Lattice) recursions as:

$$\begin{aligned} &\begin{bmatrix} I & -R_{j+1}^b \\ -(R_{j+1}^f)^T & I \end{bmatrix} \begin{bmatrix} \mathcal{I} & -\Xi_{j,1} & \cdots & -\Xi_{j,j} & 0 \\ 0 & -\Theta_{j,j} & \cdots & -\Theta_{j,1} & \mathcal{I} \end{bmatrix} = \\ &\begin{bmatrix} \mathcal{I} & -\Xi_{j+1,1} & \cdots & -\Xi_{j+1,j} & -\Xi_{j+1,j+1} \\ -\Theta_{j+1,j+1} & -\Theta_{j+1,j} & \cdots & -\Theta_{j+1,1} & \mathcal{I} \end{bmatrix} \end{aligned} \tag{20}$$

There are two ways to calculate the reflection coefficients. The first method, called "sample covariance method", uses the equations (14–15) and (17–18) to calculate the reflection coefficients, where ϕ_t is replaced by its estimate:

$$\widehat{\phi}_t = \frac{1}{N-t}\sum_{k=t}^{N-1} z_k z_{k-t}^T \tag{21}$$

The second method, called "prediction error method", provides a practical way of the coefficients by replacing the expected values by time-averages:

$$\begin{aligned} R^f_{j+1} &= \left[\frac{1}{N}\sum_{k=0}^{N-1} e_{k,j}e^T_{k,j}\right]^{-1}\left[\frac{1}{N-1}\sum_{k=1}^{N-1} e_{k,j}r^T_{k-1,j}\right] \\ R^b_{j+1} &= \left[\frac{1}{N-1}\sum_{k=1}^{N-1} e_{k,j}r^T_{k-1,j}\right]\left[\frac{1}{N}\sum_{k=0}^{N-1} r_{k,j}r^T_{k,j}\right]^{-1} \end{aligned} \tag{22}$$

The first reflection coefficients can be computed directly from the data as:

$$\begin{aligned} R^f_1 &= \left[\frac{1}{N}\sum_{k=0}^{N-1} z_k z^T_k\right]^{-1}\left[\frac{1}{N-1}\sum_{k=1}^{N-1} z_k z^T_{k-1}\right] \\ R^b_1 &= \left[\frac{1}{N-1}\sum_{k=1}^{N-1} z_k z^T_{k-1}\right]\left[\frac{1}{N}\sum_{k=0}^{N-1} z_k z^T_k\right]^{-1} \end{aligned} \tag{23}$$

References

[1] Friedlander, B., "Lattice Filters for Adaptive Processing", In Proceedings of the IEEE, Volume 70, Number 8, August 1982.