Robust Dynamic Controllers for Output Regulation: Optimization-Based Synthesis and Event-Triggered Implementation

MOHAMMAD SAEED SARAFRAZ, ANTON V. PROSKURNIKOV, MOHAMMAD SALEH TAVAZOEI, AND PEYMAN MOHAJERIN ESFAHANI

Abstract. We investigate the problem of *practical robust output regulation*: keeping the state of an uncertain dynamical system uniformly bounded while the system output eventually resides within a prescribed ball centered at a desired target. We consider uncertain systems that are possibly nonlinear and the uncertainty of the linear part is modeled element-wise through a parametric family of matrix boxes. Justifying that a static proportional controller may not be able to accomplish our desired objective, we develop an optimization program characterizing the coefficients of a dynamic output control whose optimal value determines the precision of the output regulation task. We further propose a sampled-time event-triggered redesign of this controller ensuring that the output regulation is still certified and its precision is computationally available. The objective of this study is motivated by an application on behavioral control of a network of selfish agents in a non-cooperative environment.

1. Introduction

Output regulation control of uncertain systems is a fundamental problem in control systems theory, which has been linked to a wide range of recent real-world applications [29]. There are two fundamental approaches in the classical control theory to deal with systems under uncertainties: robust control, and adaptive control [47, 10]. The main advantage of the robust control perspective is often on the computational side, yielding control synthesis tools particularly suitable to deal with high-dimensional uncertainties and dynamics.

Robust control problems can be formulated in different settings where the uncertainties are posed in time or frequency domains. Regardless of the setting, the previous studies have revealed that the stability analysis and robust tracking are often provably hard and finding a viable solution is challenging. For example, it is well-known that checking the robust stability of the systems under structured frequency-domain uncertainties is NP-hard [43, 33]. Similarly, robust stability analysis of systems under parametric uncertainties in time-domain is also known to be provably hard [30, 44].

A standard control-theoretic approach toward stability builds on Lyapunov function methods. Classical examples following into this category include the H_2- and H_∞-controllers. These problems are reformulated as an optimization program in which the decision variables constitute the parameters of the Lyapunov function and controllers [5, 13]. These programs are often intractably nonconvex and one has to resort to scalable approximation techniques. Randomized algorithms [6] and robust optimization [2] prove to be powerful tools to deal with the challenges in this context. Recent progress of modern optimization techniques as well as the developments of computing technology also reinforces the applicability of these optimization-based tools in control systems problems.

Date: March 28, 2020.

The authors are with the Electrical Engineering Department, Sharif University of Technology, Iran ({Sarafraz,Tavazoei}@ee.sharif.ir), Department of Electronics and Telecommunications, Politecnico di Torino, Italy and Institute for Problems in Mechanical Engineering of the Russian Academy of Sciences, St. Petersburg, Russia (anton.p.1982@ieee.org), and the Delft Center for Systems and Control, TU Delft, The Netherlands (P.MohajerinEsfahani@tudelft.nl).

In real-world control systems, due to communication/computation limitation, continuously measuring the system output and updating the control signal may not be possible. A classical solution approach is periodic sampling, a technique building the foundation of digital control [11]. While periodic sampling turns out to be useful to deal with the communication constraints, the sampling rate required to guarantee the stability and desired performance in a practical setting can be too small. This limitation becomes even more pronounced when the system is in a steady-state, and thus input update may be unnecessary causing waste of resources [32]. This observation is the driving force of techniques known as event-triggered control. The main idea of such techniques is to introduce a triggering criteria, typically in the form of an error function along with a threshold, in order to update the control law update only when it is necessary [40, 27]. This setting may still require continuous measurements, a shortcoming that is overcome by introducing a "sampling-based event-triggered" method in which the measurements are only available at specific time instances whereas the control law is updated based on the triggering mechanism [8, 16, 12]. Event-triggering techniques also prove to be an effective tool to implement optimization-based control techniques [20].

The problem investigated in this article lies at the interface of these two topics: (i) robust control, and (ii) event-triggered mechanism. More specifically, we opt to answer this question:

Would it be possible to provide a scalable computational framework to synthesize an output controller in order to regulate an uncertain, possibly nonlinear, dynamical system that admits a sampling-based event-triggered redesign along with provably performance guarantees?

Related literature on robust control. Modeling the uncertainty in the frequency domain, either in the structured formats [9, 36] or unstructured ones [49, 21], is a prevalent approach. Robust control design techniques build on powerful results such as small-gain theorem or positive-real method can deal with such an uncertainty in the frequency domain, e.g., it allows to address norm bounded uncertainties in the dynamics. Despite the benefits of such a modeling framework in frequency domain, the critical limitation is the applicability of the results when the uncertainties are modeled in the state-space. A prominent robust control approach for systems with parametric multi-dimensional uncertainties is Lyapunov-based methods. These methods are typically reformulated into optimization programs along with conservative approximation techniques in the form of linear matrix inequalities (LMIs) [13, 17]. This synthesis perspective has recently attracted a renewed attention due to the significant developments in optimization algorithms and in particular solving LMIs [35].

Element-wise approximation of uncertainties is one of the most powerful frameworks for problems with different sources of uncertainties. Despite its natural and rich framework to model real-world applications, the resulting optimization problems are often provably NP-hard, and as such, we have to resort to tractable approximations methods for numerical purposes. For example, utilizing scaling variables to bound the effect of interval uncertainties is an interesting trick, yielding a conservative, yet computationally tractable, solution for the problem. Alternatively, one can also invoke randomized algorithms to deal with uncertain variables and robust reformulation of LMI constraints [34, 42, 24]. This approach leads to probabilistic guarantees, and when the number of randomized samples tends to infinity, one can ensure that the desired solution will be recovered. Alternatively, an effective step toward robust control builds upon techniques developed by the robust optimization community. Given the literature of robust optimization, a relatively recent approximation tool is developed by [4] which is particularly suitable to address independent sources of uncertainties. The optimization-based framework proposed in this article exploits this result in the context of output regulation.

Related literature on event-triggered control of uncertain systems. The second part of this study is concerned with event-triggered control, a powerful technique to address potential communication limitation on the measurement or actuation side. The main body of this literature with regards to uncertain systems has been developed in the context of adaptive controls [48, 45]. A difficult task in the presence of system uncertainty is to provide rigorous performance bounds. In fact, due to the lack of knowledge about the underlying system, it may be impossible to ensure asymptotic guarantees for the output regulation. A common practice in such a setting is to resort to a weaker notion known as "*practical stability*". Recently, the works [25, 26] investigate this notion in the context of uncertain systems under the assumption that a common Lyapunov function exists. Other recent related works include [41] in which the system uncertainty is described via norm-bounded uncertainties and considers continuous measurements, [46] where the practical stability is guaranteed based on a feedback domination approach for sufficiently large feedback gain and sufficiently small periodic sampling time.

Our contributions. The main objective of this article is to investigate the output regulation task toward a prescribed target when the system is potentially nonlinear and uncertain. In contrast to the existing literature mentioned above, we have a particular emphasis on the computational aspect of this design while the dynamics uncertainties are modelled element-wise and through box constraints, a feature that, to our best knowledge, has not been studied before. We also opt to design a sample-time event-triggered mechanism to implement the proposed controller. More specifically, the technical contributions of the article are summarized as follows:

- (*i*) **Dynamic structure and inherent hardness:** We justify that a static output feedback gain may not be adequate to accomplish the desired output regulation task, and that, we require to extend the search domain to a richer class of dynamic output controllers so as to locate the closed-loop equilibrium appropriately (Section 3.1 and Lemma 3.1). We further show that from a computational viewpoint the desired output regulation problem is strongly NP hard (Proposition 3.2).
- (*ii*) **Robust dynamic controller under element-wise/box uncertainties:** Exploiting the robust optimization techniques of [4], we provide a sufficient condition and then develop an optimization framework to synthesize a dynamic output controller that enjoys a provable practical stability (Theorem 3.3). As a by product, we also show that given any fixed controller, the proposed optimization program reduces to a tractable convex optimization that can be viewed as a computational certification tool for the practical stability (Corollary 3.4).
- (*iii*) **Sampled-time event-triggered mechanism:** We propose easy-to-compute sufficient conditions along with a general triggering condition under which the proposed output controller can be implemented through aperiodic measurements and event-based actuation updates (Theorem 4.2). The proposed mechanism offers explicit computable maximal inter sampling bound and regulation error bound, and subsumes the existing approaches [40, 16] as a special case (Corollary 4.5 and Remark 4.3).
- (*iv*) **Numerical algorithm:** Leveraging recent results from [23], we propose a numerical algorithm to deal with nonlinearities of the proposed optimization programs concerning the control synthesis of the output regulation task (Algorithm 2).

The robust output regulation is a rich problem framework emerging in a wide range of applications. A particular application motivating our study is behavioral control of a network of selfish consumers, aiming to optimize their own objective functions. Due to several reasons including limitations of the processors' powers, the consumers may not be able to make instantaneous optimal decisions, and it is reasonable to model their behavior through a set of gradient-flow based differential equations. The control of such a system naturally falls into the category of our problem where the uncertainty of the dynamics is indeed referred to the lack of central supplier's knowledge about consumers' individual objective functions. In this context, exchanging the information and updating the environmental parameters between the supplier and consumers are practically confined to a predefined schedule, making the study of event-triggered controllers particularly relevant.

The remainder of the article is organized as follows: The problem is formulated along with some basic assumptions in Section 2. The robust control method is developed in Section 3, and the sampled-time event-triggered mechanism is presented in Section 4. Section 5 discusses an algorithm to tackle the proposed optimization programs, and further provides several numerical examples to illustrate the theoretical results. Section 6 concludes the article.

Notation. The symbols $\mathbb{R}, \mathbb{R}_{>0}, \mathbb{R}_{\geq 0}, \mathbb{R}^n$ and $\mathbb{N}$ denote the sets of real, positive real, non-negative real, real-valued vectors and positive integer numbers, respectively. The symbol $\mathbb{R}^{m\times n}$ specifies the set of real-valued $m \times n$ matrices, and the set of $n \times n$ symmetric matrices is denoted by $\mathbb{S}^n$. Moreover, the set of $n \times n$ positive-definite (semi-definite) symmetric matrices is denoted by $\mathbb{S}^n_{\succ 0}$ ($\mathbb{S}^n_{\succeq 0}$). For two symmetric matrices

A, B, we write $A \succ B$ (respectively, $A \succeq B$) if $A - B \in \mathbb{S}^n_{\succ 0}$ (respectively, $\mathbb{S}^n_{\succeq 0}$). For a square matrix A, we denote $[A]^\dagger = A + A^\top$. The symbol $\mathfrak{Diag}\{A_1, A_2, ..., A_n\}$ denotes the block diagonal matrix with blocks $A_1, A_2, ..., A_n$. For briefness in notations, the matrix $\left[\begin{smallmatrix} A & B^\top \\ B & C \end{smallmatrix}\right]$ is shown by $\left[\begin{smallmatrix} A & * \\ B & C \end{smallmatrix}\right]$. We use $e_1 = (1, 0, \ldots, 0)^\top, \ldots, e_m = (0, 0, \ldots, 1)^\top$ to denote the standard coordinate basis of $\mathbb{R}^m$. Also, $\mathbb{1}_m \in \mathbb{R}^m$ denotes the vector whose elements are all equal to 1. The matrix I_n is denoted for the identity matrix in $\mathbb{R}^{n \times n}$ and its index will be omitted when its dimension can be computed with respect to other matrices dimension.

2. Problem Statement and Motivation

2.1. **Problem description**

Consider the control system

$$\begin{cases} \dot{x}(t) = A^\star x(t) + B^\star u(t) + k^\star\big(x(t)\big) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, and $y(t) \in \mathbb{R}^{n_y}$ are the state, the control input, and the output vector, respectively. The matrices $A^\star$ and $B^\star$ represent the linear part of the state dynamics, and the function $k^\star : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ encapsulates the nonlinearity of the dynamics. Throughout this article we assume that the control system (1) admits a unique solution trajectory $x(\,\cdot\,)$. In order to synthesize an appropriate control signal, which will be formally defined later in this section, we assume that we have access to the measurement signal y. To this end, we consider the case where the matrices $A^\star, B^\star$ and the nonlinearity $k^\star : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ in the system (1) are not known exactly. Our main control objective is to stabilize (1) in the "Lagrange" sense (i.e., all solutions are bounded) and steer the output trajectory of (1) to an ε-neighborhood of a desired target $y^{\mathrm{d}} \in \mathbb{R}^{n_y}$. Formally speaking, we aim to ensure that for any initial condition $x(0) \in \mathbb{R}^{n_x}$

$$\sup_{t \geq 0} \|x(t)\| < \infty, \quad \text{and} \quad \limsup_{t \to \infty} \|y(t) - y^{\mathrm{d}}\| \leq \varepsilon\,. \tag{2}$$

The special case of $\varepsilon = 0$ is the standard output stability notion or target control, and the relaxed notion when $\varepsilon > 0$ is often referred to as "ε-practical output stability" [28, Definition 1]. Throughout the text, we impose the following standing assumptions on the control system (1) and the target value y^{d}.

Assumption 2.1. *[Uncertainty characterization] Consider control system (1) and suppose $y^{\mathrm{d}} \in \mathbb{R}^{n_y}$ is a desired output target.*

(i) (Box uncertainty:) The matrices $A^\star$ and $B^\star$ are known up to a box constraint, i.e., there are known nominal matrices A, B such that

$$|A^\star - A| \leq A_b, \quad |B^\star - B| \leq B_b, \tag{3}$$

where the inequalities are understood element-wise, and $A_b = \left[a_{b_{ij}}\right]_{ij}$, $B_b = \left[b_{b_{ij}}\right]_{ij}$ are the respective uncertainty bounds.

(ii) (Bounded nonlinearity:) The function $k^\star$ satisfies

$$\|k^\star(x_1) - k^\star(x_2)\| \leq \; k_b, \qquad \forall x_1, x_2 \in \mathbb{R}^{n_x} \tag{4}$$

where $k_b \geq 0$ is a known constant.

(iii) (Existence of an equilibrium:) There exists a pair $(x^{\mathrm{d}}, u^{\mathrm{d}}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ such that

$$y^{\mathrm{d}} = Cx^{\mathrm{d}} \quad \textit{and} \quad A^\star x^{\mathrm{d}} + k^\star(x^\star) = -B^\star u^{\mathrm{d}}\,.$$

Assumptions 2.1(i) and 2.1(ii) are concerned with the model uncertainties (linear and nonlinear) and the available prior information about the nominal values and the mismatch level. We note that Assumption 2.1(ii) (with an appropriate k_b) holds if and only if the nonlinearity of the dynamics is globally bounded. However, the "incremental" form of this boundedness condition is convenient in some situations as it may allow to reduce the constant k_b. For instance, if $k^\star(x)$ is an arbitrarily large constant, then one can still introduce

$k_b = 0$. With regards to Assumption 2.1(iii), the first algebraic condition essentially asks that the desired target y^{d} belongs to the image of the output matrix C, which is also an expected requirement. The second algebraic condition requirement in Assumption 2.1(iii) is rather unconventional and is aimed to guarantee the existence of an equilibrium in the dynamical system (1). A simple sufficient condition to ensure this is when the input matrix $B^\star$ is full rank.

As discussed above, the aim of this paper is to design a control input to steer the output of the system (1) to a desired target y^{d} as prescribed in (2). In this study, we will focus on output controllers that compute the input $u(t)$ based on the output measurements $\{y(s),\, 0 \le s \le t\}$. We start with a simpler controller that have continuous access to the output of the system; this is the focus of Section 3. We then proceed in Section 4 with a more sophisticated setting of sampled-time event-triggered emulation, or redesign of the continuous-time controller. Formal definitions of the latter setting are delegated to Section 4.

Problem 2.2. *Consider the system* (1) *under Assumption 2.1, and let* $y^{\mathrm{d}} \in \mathbb{R}^{n_y}$ *and* $\varepsilon \ge 0$ *be a desired target and regulation precision, respectively.*

(i) **Control synthesis:** *Synthesize an output control* $y_{[0,t]} \mapsto u(t)$,[1] $t \ge 0$, *in order to ensure the* ε-*practical output regulation in the sense of* (2).

(ii) **Sampled-time event-based emulation:** *Given a prescribed series of measurement sample-times, design a triggering mechanism to update the control along with a guaranteed precision of the desired output regulation* (2).

It is worth noting that the viability of the sampled-time emulation in (ii) reflects a certain robustness level of the controller in the task (i).

2.2. Motivation

In this subsection, from a practical point of view, the motivation of study on the problems discussed in the previous section is described. To this end, consider a network with N consumers such that each of the consumers in this network aims to determine its consumption value by finding the optimal value of a specific cost function with respect to other consumers' outputs and the supplier's policy. Considering the consumer i, the aforementioned objective is mathematically equivalent to find the value of $x_i^\star$ from

$$x_i^\star = \underset{x_i \in \mathbb{R}^{n_{xi}}}{\operatorname{argmin}} f_i(x_i, y_{-i}^\star, u), \tag{5}$$

where $u \in \mathbb{R}^{n_u}$ denotes the supplier's policy and $y_{-i}^\star$ is the vector of other consumers' outputs relating to their consumption. (The optimization problem (5) is similar to the problem of finding the *Nash equilibrium* for a N-player game which has been investigated in literature, see e.g. [14, 38]). Assume that the aim of the supplier is to determine the supply policy u such that the solution of the optimization problem (5) equals the desired value x_i^d. This scenario can imagine the situation of providing the active power for up and down regulation services, e.g., energy peak shaving during peak hours and load leveling at non-peak hours [7, 15]. For example, $x_i^\star$ may be practically considered as the power/water consumption of the consumer i in a power/water network system, and the supply policy u can be the pricing incentive or penalty aiming to regulate some predefined levels of the resource consumptions of the consumers. In this situation, $y_i^\star$ can be considered as the output of the consumer i, which is related to the resource consumption value of this consumer.

In practical cases, consumers may deal with limitation of processing power. In this situation, it can be assumed that the consumers cannot quickly solve the optimization problem (5), and they need to deploy a variant of gradient algorithms such as (6) to seek the optimal value of (5):

$$\dot{x}_i(t) = -\nabla_{x_i} f_i(x_i(t), y_{-i}(t), u(t)). \tag{6}$$

[1] The notation $y_{[0,t]}$ is the restriction of the function y to the set $[0,t]$, that is, $\{y(s) : s \in [0,t]\}$.

It is also important to note that the arguments akin to the dynamics (5) are also prevalent in the research works using gradient algorithms to find the Nash equilibrium in a multi-player game; see, for instance, [37] and the references therein. Consider a special case in which function $f_i(\cdot)$ has quadratic form with unknown parameters for all $1 \le i \le N$ (similar to that assumed in [39]), i.e.,

$$f_i(x_i, y_{-i}, u) = x_i^\top A_i^\star x_i + x_i^\top A_{-i}^\star y_{-i} + x_i^\top B_i^\star u + k_i^\star x_i. \tag{7}$$

In this case, (6) is simplified as

$$\dot{x}_i(t) = A_i^\star x_i(t) + A_{-i}^\star y_{-i}(t) + B_i^\star u(t) + k_i^\star. \tag{8}$$

Although $A_i^\star$, $A_{-i}^\star$, $B_i^\star$, and $k_i^\star$ are unknown for the supplier, as an assumption the supplier is aware that these unknown parameters belong to some known sets for all $1 \le i \le N$. Suppose that the notation $x(t)$ is used to represent $\left[x_1^\top(t), ..., x_N^\top(t)\right]^\top$, and $B^\star$ and $k^\star$ denote $\left[B_1^{\star\top}, ..., B_N^{\star\top}\right]^\top$ and $\left[k_1^{\star\top}, ..., k_N^{\star\top}\right]^\top$, respectively. Furthermore, assume that $A^\star$ is used instead of proper combination of matrices $A_i^\star, A_{-i}^\star$. Considering these notations, if a linear transformation between the output and the vector state of each consumer in the form $y_i(t) = C_i x_i(t)$ is considered, then the system (8) can be rewritten in the form (1). In this situation, considering bounds on $A_i^\star$, $A_{-i}^\star$, and $B_i^\star$ leads to find A_b and B_b, such that A_b and B_b satisfy (3). Moreover, (4) will be met due to constant form of k^*. Therefore, Assumption 2.1 is satisfied for the aforementioned special case.

3. Control Synthesis: a Linear Dynamical System Perspective

The focus of this section is Problem 2.2(i). We opt to design a continuous-time output controller $y_{[0,t]} \mapsto u(t)$, $t \ge 0$, with the aim to ensure the ε-practical output regulation as defined in (2). This objective will be accomplished in two steps: First, we choose the controller's structure in such a way that (2) is implied by ε-stability of some *equilibrium* of the closed-loop system. The existence of such an equilibrium seems natural, if one is interested in the ε-practical stability with arbitrarily small accuracy ε. Second, we provide a computational framework of designing the controller's parameters in a way that the aforementioned stability condition is guaranteed.

A possible control architecture, and perhaps the simplest form, is a "proportional" controller in the form of $u(t) = D_c y(t) + \eta$ where $D_c \in \mathbb{R}^{n_u \times n_y}$ and $\eta \in \mathbb{R}^{n_u}$ are the control synthesis parameters. It is, however, not difficult to see that such a control class is not rich enough to serve the purpose. More specifically, dynamics of the closed-loop systems are

$$\dot{x}(t) = (A^\star + B^\star D_c C)x + k^\star\big(x(t)\big) + D_c\eta\,.$$

Since the matrices $A^\star, B^\star$ are only partially known, the equilibrium of the latter system (if it exists) is also uncertain, and in general fails to be compatible with the requirement (2). This is particularly restrictive since we cannot opt for steering the system output to an arbitrarily close neighborhood of a desired target y^{d}. To address this issue, we enhance the design structure to a dynamic setting where we have a better control over the equilibrium of the closed-loop dynamics.

3.1. Design closed-loop equilibrium

Consider now a more general *dynamic* controller

$$\begin{cases} \dot{w}(t) = A_c w(t) + B_c y(t) + \xi \\ u(t) = C_c w(t) + D_c y(t) + \eta, \end{cases} \tag{9}$$

where matrices $A_c, C_c \in \mathbb{R}^{n_u \times n_u}$, $B_c, D_c \in \mathbb{R}^{n_u \times n_y}$ and $\xi, \eta \in \mathbb{R}^{n_u}$ are the design parameters. These additional parameters in (9) enable one to control the equilibrium of the closed-loop system (1) together with (9) despite the uncertainty in the system's parameters. Indeed, the state of the closed-loop dynamics of the system (1) together with the controller (9) is $\big(x(t), w(t)\big)$. The next lemma provides sufficient conditions under which we can ensure that the closed-loop system admits an equilibrium whose output coincides with any desired target y^{d}.

Lemma 3.1 (Closed-loop equilibrium)**.** *Let Assumption 2.1(iii) hold and the matrix C_c have full column rank. Then, setting*

$$A_c = 0 \quad \textit{and} \quad \xi = -B_c y^{\mathrm{d}} .$$

the closed-loop system (1)*,* (9) *has an equilibrium state $(x^\star, w^\star)$ such that $Cx^\star = y^{\mathrm{d}}$.*

Proof. Assumption 2.1(iii) implies the existence of a pair (x^*, u^{d}) such that $Cx^\star = y^{\mathrm{d}}$ and $A^\star x^\star + B^\star u^{\mathrm{d}} + k^\star(x^\star) = 0$. Since the matrix C_c is also full rank, there exists $w^\star \in \mathbb{R}^{n_u}$ such that $C_c w^\star + D_c x^\star + \eta = u^{\mathrm{d}}$. Therefore, the point $(x^\star, w^\star) \in \mathbb{R}^{n_x+n_u}$ obeys the algebraic equations

$$\begin{cases} A^\star x^\star + B^\star(C_c w^\star + D_c C x^\star + \eta) + k^\star(x^\star) = 0, \\ A_c w^\star + B_c C x^\star + \xi = B_c(y^{\mathrm{d}} - Cx^\star) = 0, \end{cases} \tag{10}$$

and as such, being a point of closed-loop system's equilibrium. □

In view of Lemma 3.1, the controller's parameters B_c, D_c, η do not influence the *existence* of an equilibrium compatible with the predefined output value. Moreover, while B_c and D_c may influence the stability of the transient behavior of the closed-loop system, the vector η does not play any role on this aspect either. Therefore, for the sake of notational simplicity and with no loss of generality, we set $\eta = -D_c C x^\star$, where x^* is the equilibrium from Lemma 3.1. This slight modification only shifts the second component of the closed-loop equilibrium $w^\star$, which plays no role. Then, controller dynamics (9) under the requirements of Lemma 3.1 and the choice of η shapes into

$$\begin{cases} \dot{w}(t) = B_c\big(y(t) - y^{\mathrm{d}}\big) \\ u(t) = C_c w(t) + D_c\big(y(t) - y^{\mathrm{d}}\big) . \end{cases} \tag{11}$$

We note that the proposed dynamic controller (11) may be considered as a (multidimensional) extension of the conventional PI controller.

3.2. Closed-loop stability of transient behavior

The goal of this section is to design the controller parameters B_c, C_c, D_c such that the the equilibrium $(x^\star, w^\star)$ from Lemma 3.1 is (practically) stable. To this end, we introduce the augmented state vector of the closed-loop system

$$z(t) := \begin{bmatrix} x(t) - x^\star \\ w(t) - w^\star \end{bmatrix} . \tag{12}$$

Based on the system dynamics in (1) together with the controller (11), the dynamics of the state vector z is

$$\dot{z} = \big[\bar{A} + J^\top \Delta A J + (\bar{B} + J^\top \Delta B J) F \bar{C}\big]\, z + J^\top(k^\star(J^\top z) - k^\star(x^\star)), \tag{13}$$

where $\Delta A = A^\star - A$ and $\Delta B = B^\star - B$ represent the uncertainty in the linear part of the system dynamics, and matrices $\bar{A}, \bar{B}, \bar{C}, F, J$ are defined as

$$\bar{A} := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \bar{B} := \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \bar{C} := \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, J := \begin{bmatrix} I_{n_x} & 0_{n_x \times n_u} \end{bmatrix}, F := \begin{bmatrix} D_c & C_c \\ B_c & 0 \end{bmatrix} \tag{14}$$

We highlight that the matrix F collects all the design variables of the controller. The goal of the controller design is to provide the (practical) stability of the system (13) for all admissible uncertainties $\Delta A, \Delta B, k^\star(\,\cdot\,)$ that meet Assumption 2.1. Unfortunately, it turns out that the exact characterization of such an F is provably intractable. In fact, a simpler question of only checking the stability of the system (13) for a given F is also difficult. This is formalized in the next proposition.

Proposition 3.2 (Intractability)**.** *Consider the system* (1) *under Assumption 2.1, and let the control signal follow the dynamics* (11)*. Then, given the control parameters (matrix F in* (14)*), checking whether the output target stability* (2) *holds for some $\varepsilon \geq 0$ is strongly NP hard and equivalent to*

$$\forall|\Delta A| \leq A_b, \forall|\Delta B| \leq B_b \quad \exists P \in S_{\succ 0}^{n_x+n_u} : \left[P\left(\bar{A} + J^\top \Delta A J + (\bar{B} + J^\top \Delta B J)F\bar{C}\right)\right]^\dagger \preceq 0. \tag{15}$$

Proof. Recall that the nonlinear term in the dynamics (13) is uniformly bounded due to Assumption 2.1(ii). Therefore, thanks to the classical result of [18, Theorem 9.1], the stability of the system (13) is equivalent to the stability of the linear part described as

$$\dot{z} = \left[\bar{A} + J^\top \Delta A J + (\bar{B} + J^\top \Delta B J)F\bar{C}\right] z\,. \tag{16}$$

From the classical linear system theory, we know that the stability of (16) is equivalent to the existence of a quadratic Lyapunov function $V(z) = z^\top P z$, where the symmetric positive definite matrix P may in general depend on the uncertainty in the dynamics. This assertion can be mathematically translated to checking whether the given controller parameter F satisfies (15); note that the order of the quantifies implies that the matrix P may depend on the uncertain parameter ΔA and ΔB. The assertion (15) is indeed a special case of the problem of an interval matrix's stability [30], which is proven to be strongly NP-hard [1, Corollary 2.6]. □

A useful technique to deal with the assertion similar to (15) is to choose a so-called common Lyapunov function [31]. Namely, we aim to find a positive-definite matrix P for all possible model parameters, i.e., the assertion (15) is replaced with a more conservative requirement as follows:

$$\exists P \in S_{\succ 0}^{n_x+n_u} \quad \forall|\Delta A| \leq A_b, \forall|\Delta B| \leq B_b : \left[P\left(\bar{A} + J^\top \Delta A J + (\bar{B} + J^\top \Delta B J)F\bar{C}\right)\right]^\dagger \preceq 0. \tag{17}$$

Note that the only difference between (15) and the conservative assertion in (17) is the order of quantifiers between the Lyapunov matrix P and the linear dynamics uncertainties ΔA and ΔB. The argument (17) is a special subclass of problems known as the "matrix cube problems" [3]. While this class of problems is also provably hard [3, Proposition 4.1], the state-of-the-art in the convex optimization literature offers an attractive sufficient condition where the resulting conservatism is bounded *independently* of the size of the problem [4]. Building on these developments, we will provide an optimization framework to design the controller coefficients along with a corresponding common Lyapunov function.

Theorem 3.3 (Robust control & common Lyapunov function)**.** *Consider the system* (1)*, satisfying Assumption 2.1, and the controller* (11)*. Consider the optimization program*

$$\begin{cases} \max & \alpha\zeta^{-1} \\ \text{s.t.} & \alpha \in \mathbb{R}, \quad \zeta, \kappa_{ij}, \mu_{ik} \in \mathbb{R}_{>0}, \quad P \in \mathbb{S}_{\succ 0}^{n_x+n_u}, \quad C_c \in \mathbb{R}^{n_u \times n_u}, \quad B_c, D_c \in \mathbb{R}^{n_u \times n_y} \\ & F = \begin{bmatrix} D_c & C_c \\ B_c & 0 \end{bmatrix}, \quad M = \left[P\bar{A} + P\bar{B}F\bar{C}\right]^\dagger + \alpha I \\ & G_1 = \mathfrak{Diag}\left\{-\kappa_{ij} a_{b_{ij}}^{-2}\right\}_{i,j}, \quad G_2 = \mathfrak{Diag}\left\{-\mu_{ik} b_{b_{ik}}^{-2}\right\}_{i,k}, \quad G_3 = \mathfrak{Diag}\left\{-\mu_{ik}^{-1}\right\}_{i,k} \\ & H_1 = PJ^\top(\mathbb{1}_{n_x} \otimes I_{n_x}), \quad H_2 = \bar{C}^\top F^\top J^\top \begin{bmatrix} \mathbb{1}_{n_u} \otimes e_1 & \dots & \mathbb{1}_{n_u} \otimes e_{n_x} \end{bmatrix} \\ & \begin{bmatrix} M + \sum_{i,j} \kappa_{ij} J^\top e_j^\top e_j J & * & * & * & * \\ H_1^\top & G_1 & * & * & * \\ H_1^\top & 0 & G_2 & * & * \\ H_2^\top & 0 & 0 & G_3 & * \\ JP & 0 & 0 & 0 & -\zeta I, \end{bmatrix} \preceq 0 \end{cases} \tag{18}$$

where α_*, ζ_* and P_* denote the optimal solutions of corresponding decision variables. Then, the controller provides ε_c-practical output regulation (2) *where*

$$\varepsilon_c = k_b\|\bar{C}\|\sqrt{\frac{\lambda_{\max}(P_*)}{max\{0, \alpha_*\zeta_*^{-1}\}\lambda_{\min}(P_*)}}\,. \tag{19}$$

In particular, if $k_b = 0$ (that is, the nonlinear term vanishes to a constant) and $\alpha_* > 0$, then the closed-loop system is exponentially stable and $\lim_{t\to\infty} y(t) = y^{\mathrm{d}}$.

Proof. Consider the closed-loop system (13) and a quadratic Lyapunov function $V(z) = z^\top P z$. The derivative of V along the system trajectories of (13) can be described by

$$\frac{1}{2}\frac{d}{dt}V(z) = z^\top P\left(\bar{A} + \bar{B}F\bar{C}\right)z + z^\top P\left(J^\top \Delta A J + J^\top \Delta B J F\bar{C}\right)z + z^\top P J^\top (k^\star(J^\top z) - k^\star(x^\star)),$$

where the last term involving the nonlinear term can be estimated by invoking Young's inequality as follows:

$$2z^\top P J^\top\left(k^\star(J^\top z) - k^\star(x^\star)\right) \le \zeta^{-1} z^\top P J^\top J P z + \zeta\left\|k^\star(J^\top z) - k^\star(x^\star)\right)\|^2 \le \zeta^{-1} z^\top P J^\top J P z + \zeta k_b^2\,.$$

Notice that the parameter $\zeta \in \mathbb{R}_{>0}$ is a positive scalar, and the last line is an immediate consequence of (4) granted due to Assumption 2.1(ii). In the light of the latter estimate, one can observe that if the inequality

$$\left[P(\bar{A} + \bar{B}F\bar{C}) + P(J^\top \Delta A J + J^\top \Delta B J F\bar{C}) + \frac{\zeta^{-1}}{2} P J^\top J P\right]^\dagger \preceq -\alpha I, \tag{20}$$

holds for some $\alpha \in \mathbb{R}_{>0}$, then the dynamics of the Lyapunov function value along with system trajectories are

$$\frac{1}{2}\frac{d}{dt}V(z) \le -\alpha\|z\|^2 + \zeta k_b^2 \le \frac{-\alpha}{\lambda_{\max}(P_*)}V(z) + \zeta k_b^2.$$

The above observation implies that $\limsup_{t\to\infty} V\big(z(t)\big) \le \lambda_{\max}(P_*)\zeta k_b^2/\alpha$, which together with the simple bound $\lambda_{\min}(P_*)\|z\|^2 \le V(z)$, leads to

$$\limsup_{t\to\infty}\|y(t) - y^{\mathrm{d}}\| \le \limsup_{t\to\infty}\|\bar{C}\|\|z(t)\| \le \limsup_{t\to\infty}\|\bar{C}\|\sqrt{\frac{V\big(z(t)\big)}{\lambda_{\min}(P_*)}} \le \varepsilon_{\mathrm{c}}\,.$$

where ε_{c} is defined as in (19). Hence, the above observation indicates that under the requirement (20) for some $\alpha > 0$, the desired assertion holds. Next, we aim to replace the robust inequality (20) by a more conservative criterion, which in turn can be verified efficiently. This procedure consists of several steps. Introducing the variable $M := \left[P\bar{A} + P\bar{B}F\bar{C}\right]^\dagger + \alpha I$, the inequality (20) is rewritten as

$$-M - \zeta^{-1} P J^\top J P + \left[P J^\top \sum_{i=1}^{n_x}\Big(\sum_{j=1}^{n_x}(\delta a_{ij}) e_i^\top e_j\Big) J + P J^\top \sum_{i=1}^{n_x}\Big(\sum_{k=1}^{n_u}(\delta b_{ik}) e_i^\top e_k\Big) J F\bar{C}\right]^\dagger \succeq 0, \tag{21}$$

where the uncertainty parameters are described element wise as $\Delta A = [\delta a_{ij}]$ and $\Delta B = [\delta b_{ij}]$. Recall that the condition (21) has to hold for all uncertain parameters, i.e., it is a robust constraint. By virtue of [4, Theorem 3.1], the constraint (21) then holds if there exist such parameters D_{ij}, E_{ik}, λ_{ij}, γ_{ik}, where $i, j \in \{1, \dots, n_x\}$ and $k \in \{1, \dots, n_u\}$ that

$$\begin{gathered}\begin{bmatrix} D_{ij} - \lambda_{ij} a_{b_{ij}}^2 z^\top P J^\top e_i^\top e_i J P z & * \\ e_j J z & \lambda_{ij} I \end{bmatrix} \succeq 0,\\ \begin{bmatrix} E_{ik} - \gamma_{ik} b_{b_{ik}}^2 z^\top P J^\top e_i^\top e_i J P z & * \\ e_k J F\bar{C} z & \gamma_{ik} I \end{bmatrix} \succeq 0,\\ -z^\top\left(M + \zeta^{-1} P J^\top J P\right)z \ge \textstyle\sum_{i,j} D_{ij} + \sum_{i,k} E_{ik}.\end{gathered} \tag{22}$$

By deploying the standard Schur complement in the first two inequalities of (22), we arrive at

$$\begin{aligned}&\lambda_{ij}, \gamma_{ik} > 0,\\ &D_{ij} - \lambda_{ij} a_{b_{ij}}^2 z^\top P J^\top e_i^\top e_i J P z - \lambda_{ij}^{-1} z^\top J^\top e_j^\top e_j J z \ge 0,\\ &E_{ik} - \gamma_{ik} b_{b_{ij}}^2 z^\top P J^\top e_i^\top e_i J P z - \gamma_{ik}^{-1} z^\top \bar{C}^\top F^\top J^\top e_k^\top e_k J F\bar{C} z \ge 0,\\ &-z^\top\left(M + \zeta^{-1} P J^\top J P\right)z \ge \textstyle\sum_{i,j} D_{ij} + \sum_{i,k} E_{ik}.\end{aligned} \tag{23}$$

Eliminating $\{D_{ij}\}_{i,j}, \{E_{ik}\}_{i,k}$ and using straightforward computation, the above inequalities reduces to

$$\begin{aligned}&\lambda_{ij}, \gamma_{ik} > 0,\\ &M + \zeta^{-1} P J^\top J P + \textstyle\sum_{i,j} \kappa_{ij} J^\top e_j^\top e_j J - H_1 G_1^{-1} H_1^\top - H_1 G_2^{-1} H_1^\top - H_2 G_3^{-1} H_2^\top \preceq 0,\end{aligned} \tag{24}$$

where the matrices G_1, G_2, G_3, H_1, H_2 are defined as in (18). The proof is then concluded by applying yet again the Schur complement to the inequality (24) and replace the variables $\kappa_{ij} = \lambda_{ij}^{-1}$ and $\mu_{ik} = \gamma_{ik}^{-1}$. We note that since $\zeta > 0$, then $\alpha \geq 0$ if and only the objective function $\alpha\zeta^{-1} \geq 0$. Therefore, the explicit positivity constraint over the variable α can be discarded without any impact on the assertion of the theorem. In fact, the elimination of this constraint allows the program (18) being always feasible. Finally, we also note that the second part of the assertion is a straightforward consequence of the bound (19) and the fact that asymptotic stability and exponential stability in linear system coincide. □

The optimization program (18) in Theorem 3.3 is, in general, non-convex. We however highlight two important features of this program: (i) It is a tool enabling *co-design* of a controller and a Lyapunov function for the closed-loop system, and (ii) when the control parameters are fixed, the resulting program reduces to a linear matrix inequality (LMI), which is amenable to the off-the-shelves convex optimization solvers. The latter argument is formalized in the following corollary.

Corollary 3.4 (Controller certification via convex optimization)**.** *Consider the system* (1) *where Assumption 2.1 holds and the control is set to* (11) *with the given parameters* (14)*. Consider the program*

$$\begin{cases} \max & \alpha\zeta^{-1} \\ \text{s.t.} & \alpha \in \mathbb{R}, \quad \zeta, \kappa_{ij}, \mu_{ik} \in \mathbb{R}_{>0}, \quad P \in \mathbb{S}_{\succ 0}^{n_x+n_u} \\ & M' = M + \sum_{i,j} \kappa_{ij} J^\top e_j^\top e_j J - H_2 G_3^{-1} H_2^\top \\ & \begin{bmatrix} M' & * & * & * \\ H_1^\top & G_1 & * & * \\ H_1^\top & 0 & G_2 & * \\ JP & 0 & 0 & -\zeta I \end{bmatrix} \preceq 0 \end{cases} \tag{25}$$

where the matrices $C, F, G_1, G_2, G_3, H_1, H_2$ are defined on the basis of the system and control parameters[2]*. Let α_*, ζ_*, and P_* denote an optimizer of the program* (25)*. Then, if $\alpha_* > 0$, then the output target control* (2) *is fulfilled for all $\varepsilon \geq \varepsilon_c$ as defined in* (25)*. Moreover, if $\alpha_* \leq 0$, then there exists dynamics matrices $A^\star$ and $B^\star$ such that*

$$|A^\star - A| \leq \frac{\pi}{2} A_b, \quad |B^\star - B| \leq \frac{\pi}{2} B_b,$$

and the closed-loop system is unstable.

Proof. Starting from the optimization program (18) in Theorem 3.3 and given controller parameters F, the matrix H_2 is also constant, and as such, an application of the standard Schur complement in the step (24) arrives at the desired assertion. Concerning the instability claim, we emphasize that thanks to [4, Theorem 3.1], the convex characterization of (17) (i.e., the step from (21) to (22)) is indeed tight up to the constant multiplier $\pi/2$. □

We close this section by a remark on the different sources of conservatism concerning the solution approach proposed in this section. It is needless to say that any numerical progress at the frontier of each of these sources will lead to an improvement of the solution method in this article.

Remark 3.5 (Conservatism of the proposed approach)**.** *The path from the output target control* (2) *to the numerical solution of the optimization program* (18) *constitutes three steps that are only sufficient conditions*

[2]Formally speaking, the objective function in (25) is not convex. However, since the only source of nonconvexity is the scalar variable ζ, a straightforward approach is to adjust this variable through a grid-search or bisection.

and may contribute to the level of conservatism: (i) to restrict to a common Lyapunov function, i.e., the transition from (15) *to* (17)*, (ii) to apply the state-of-the-art matrix cube problem from* (21) *to* (22)*, and (iii) to numerically solve the finite, but possibly nonconvex, optimization program* (18)*. As detailed in Corollary 3.4, the conservatism introduced by step (ii) is actually tight up to a constant independently of the dimension of the problem. With regards to the nonconvexity issue raised in step (iii), we will examine a recent approximation technique proposed by* [23] *that is particularly tailored to deal with bilinearity of a similar kind in Theorem 3.3; this will be reported in Section 5.*

4. Aperiodic event-triggered robust control

In this section, we address Problem 2.2(ii). Namely, instead of the continuous-time dynamic controller examined in the previous section, we consider a sampled-time control algorithm, arising as the *emulation* of the continuous-time feedback. Unlike its continuous-time counterpart, the controller has access to the system output $y(\cdot)$ only at *sampled* instants $\{t_s\}_{s\in\mathbb{N}}$, where the sequence t_s are supposed to be monotonically increasing and $t_s \to \infty$ as $s \to \infty$. Notice that the inter-sampling intervals $t_{s+1} - t_s$ need not be constant, i.e., we allow an arbitrary *aperiodic* sampling. Recall that the controller's state $w(t)$ evolves according the the dynamics (11). In the sampled times setting described above, the input signal to (11) (i.e., the sampled of the system output y) is frozen during the inter-sampling, leading to the controller's state evolution

$$w(t) = w(t_s) + (t - t_s)B_c(y(t_s) - y^{\mathrm{d}}), t \in [t_s, t_{s+1}). \tag{26a}$$

On the actuation side, the simplest scenario is to compute the new control input upon receiving measurement $y(t_s)$, which remains constant till the next measurement $y(t_{s+1})$ arrives:

$$u(t) = C_c w(t_s) + D_c\big(y(t_s) - y^{\mathrm{d}}\big), \quad t \in [t_s, t_{s+1}). \tag{26b}$$

Note that $u(t)$ takes a constant value within the time interval $t \in [t_s, t_{s+1})$. In a more generalized version, we however allow for an *event-triggered* strategy: Upon arrival of the new measurement $y(t_s)$, the control input is updated only if a certain criteria (i.e., triggering condition) is fulfilled. This criteria may reflect how far the plant's output or the controller's state have visibly changed since the last time that the control signal was updated. This mechanism can be formally described as follows. At the time instant t_s, let us denote the last time that the control input has been updated by $t_j < t_s$, and the vector of available information from the system and controller as

$$v(t_j, t_s) \coloneqq \big[w(t_j)^\top, y(t_j)^\top, w(t_s)^\top, y(t_s)^\top\big]^\top.$$

Inspired by [16], we consider an extended quadratic triggering condition in the form of

$$\begin{bmatrix} v(t_j, t_s) \\ 1 \end{bmatrix}^\top \mathcal{Q} \begin{bmatrix} v(t_j, t_s) \\ 1 \end{bmatrix} \geq 0. \tag{27}$$

The condition (27) is slightly more generalized than the one proposed in [16] in a way that it also supports constant thresholds; note that the information vector $v(t_j, t_s)$ is augmented by a constant 1. In view of (27), at time t_s if the condition holds, then the input value is updated and the corresponding index is updated (i.e., $j = s$). In case (27) does not hold, the control inputs remains unchanged till the next measurement time instant t_{s+1}. This procedure is summarized in Algorithm 1.

Algorithm 1 Aperiodic Event-Triggered Control (AETC)

Initialization: sample instants $\{t_s\}_{s\in\mathbb{N}}$, initial measurement y_0, initial control state $w_0 = 0$. Set $j = 0$, compute u_0 from (26b), and send it to the system (1).

Upon receiving y_s at time t_s, compute $w(t_s)$ in (26a) and verify (27).

- If (27) holds, then set $j \leftarrow s$, compute $u(t_s)$ from (26b) and send it to the system (1);
- otherwise, keep $u(t_s) = u(t_j)$ for $t \in [t_s, t_{s+1})$, i.e., nothing is required to be communicated with (1).

Set $s \leftarrow s + 1$ and go to step 2.

Remark 4.1 (Special triggering mechanisms). *If in (27) $\mathcal{Q} = 0$, the control strategy reduces to the classical sampled-time (or digital) control. Besides, as pointed out in [16], the quadratic form (27) subsumes the relative event-triggered mechanism [40]. The extension of $v(t_j, t_s)$ by a constant 1 also allows us to capture the absolute event-triggered mechanism [50] and mixed event-triggered mechanism [8]. More specifically, when $\mathcal{Q} = \widetilde{\mathcal{Q}}(q_0, q_1)$ where*

$$\widetilde{\mathcal{Q}}(q_0, q_1) := \begin{bmatrix} I & * & * & * & * \\ 0 & I & * & * & * \\ -I & 0 & I - q_1 I & * & * \\ 0 & -I & 0 & I - q_1 I & * \\ 0 & 0 & 0 & 0 & -q_0 \end{bmatrix}, \tag{28}$$

the triggering mechanism (27) is translated into the condition

$$\left\| \begin{bmatrix} w(t_s) - w(t_j) \\ y(t_s) - y(t_j) \end{bmatrix} \right\|^2 \geq q_0 + q_1 \left\| \begin{bmatrix} w(t_s) \\ y(t_s) \end{bmatrix} \right\|^2 .$$

In summary, the aperiodic event-triggered control (AETC) mechanism introduced above entails two key components: the time instants $\{t_s\}_{s\in\mathbb{N}}$, and the triggering mechanism (27) characterized by the matrix $\mathcal{Q}$. In the rest of this section, we aim to implement the dynamic controller designed in Section 3 through an AETC. To that end, we further provide sufficient conditions on these two key components so as to still ensure the desired practical regulation (2) with a prespecified precision level.

Let us fix the controller parameters to a feasible solution (B_{c*}, C_{c*}, D_{c*}) of the optimization program (18) along with the Lyapunov matrix P_*. For the brevity of the exposition, we also introduce the following notation:

$$\begin{aligned} \hat{F}_* &:= \begin{bmatrix} D_{c*} & C_{c*} \\ 0 & 0 \end{bmatrix}, \quad \beta := \|P_*\| \|B_{c*}\bar{C}\|, \quad \varrho_B := \left(\|\bar{B}\| + \|B_b\|\right)^2 \|\hat{F}_*\|^2, \\ \varrho_{AB} &:= \varrho_B \|\bar{C}\|^2 + \left(\|\bar{A}\| + \|A_b\|\right)^2, \quad \vartheta_B := \max_{|\Delta B| \leq B_b} \|P_*(\bar{B} + J^\top \Delta B J)\hat{F}_*\|, \\ \vartheta_{AB} &:= \max_{|\Delta A| \leq A_b,\, |\Delta B| \leq B_b} \|\bar{A} + J^\top \Delta A J + (\bar{B} + J^\top \Delta B J - I)(F_* - \hat{F}_*)\|, \quad \mathfrak{e}(h) := \vartheta_{AB}^{-1}(e^{\vartheta_{AB} h} - 1) \end{aligned} \tag{29}$$

Now we are ready to proceed with the main result of this section.

Theorem 4.2 (Certified robust regulation under AETC). *Consider the system (1) where Assumption 2.1 holds and the control input follows the dynamics (11). Let the matrices $(B_{c*}, C_{c*}, D_{c*}, P_*, \alpha_*, \zeta_*)$ be a feasible solution to the optimization problem (18) where $\alpha_* > 0$. Consider the AETC in Algorithm 1 with the sampled instants $\{t_s\}_{s\in\mathbb{N}}$ and the triggering mechanism (27) with the matrix $\mathcal{Q}$. Suppose*

$$\bar{h} := \sup_{s\in\mathbb{N}} (t_{s+1} - t_s) \leq h_{\max} \quad \textit{and} \quad \mathcal{Q} \preceq \widetilde{\mathcal{Q}}(q_0, q_1),$$

where $\widetilde{\mathcal{Q}}(q_0, q_1)$ is defined in (28) for some constants $q_0, q_1 \geq 0$, and $h_{\max}$ is defined in

$$h_{\max} := \vartheta_{AB}^{-1} \ln \left(1 + \vartheta_{AB} \sqrt{\frac{\alpha_*^2 \sqrt{q_1} \lambda_{\min}(P_*)[(1 + 2\sqrt{q_1})^2 \lambda_{\max}(P_*)]^{-1} - 2\vartheta_B^2 q_1 \|\bar{C}\|^2}{6\vartheta_B^2 (q_1 \varrho_B \|\bar{C}\|^4 + 6\varrho_{AB}\|\bar{C}\|^2) + 3\beta^2(\varrho_B q_1 \|\bar{C}\|^2 + \varrho_{AB})^2}} \right). \tag{30}$$

Then, the closed-loop system under AETC is ε_{d}-practical output stable in the sense of (2) where

$$\varepsilon_{\mathrm{d}}^2 = \mathfrak{f}_1(\bar{h}, q_1) q_0 + \mathfrak{f}_2(\bar{h}, q_1) k_b^2, \tag{31}$$

in which the constants $\mathfrak{f}_1, \mathfrak{f}_2$ are defined in (32a)-(32b).

$$\mathfrak{f}_1(\bar{h}, q_1) := \frac{\vartheta_B^2 \left(2 + 6\varrho_B \|\bar{C}\|^2 \mathfrak{e}^2(\bar{h})\right) \|\bar{C}\|^4 + 3\beta^2 \varrho_B \|\bar{C}\|^4 \mathfrak{e}^2(\bar{h})}{-\vartheta_B^2 \left(2q_1 \|\bar{C}\|^2 + 6q_1 \varrho_B \|\bar{C}\|^4 \mathfrak{e}^2(\bar{h}) + 6\varrho_{AB}\|\bar{C}\|^2 \mathfrak{e}^2(\bar{h})\right) - 3\beta^2 (\varrho_B q_1 \|\bar{C}\|^2 + \varrho_{AB})^2 \mathfrak{e}^2(\bar{h}) + \alpha_*^2 \frac{\sqrt{q_1}\lambda_{\min}(P_*)}{(1 + 2\sqrt{q_1})^2 \lambda_{\max}(P_*)}}, \tag{32a}$$

$$\mathfrak{f}_2(\bar{h}, q_1) := \frac{6\vartheta_B^2\|\bar{C}\|^6\mathfrak{c}^2(\bar{h}) + 3\beta^2\|\bar{C}\|^4\mathfrak{c}^2(\bar{h}) + \alpha_*\zeta_*\|\bar{C}\|^2\sqrt{q_1}(1+2\sqrt{q_1})^{-1}}{-\vartheta_B^2\left(2q_1\|\bar{C}\|^2 + 6q_1\varrho_B\|\bar{C}\|^4\mathfrak{c}^2(\bar{h}) + 6\varrho_{AB}\|\bar{C}\|^2\mathfrak{c}^2(\bar{h})\right) - 3\beta^2(\varrho_B q_1\|\bar{C}\|^2 + \varrho_{AB})^2\mathfrak{c}^2(\bar{h}) + \alpha_*^2\dfrac{\sqrt{q_1}\lambda_{\min}(P_*)}{(1+2\sqrt{q_1})^2\lambda_{\max}(P_*)}}. \tag{32b}$$

Proof. Suppose $t \in [t_s, t_{s+1})$ and let $t_j \le t_s$ be the last time instant when the control input was computed. Let $z(t)$ be the state of the closed system defined in (12), and define $e(t)$ as

$$e(t) := \begin{bmatrix} y(t_j) - y(t) \\ w(t_j) - w(t) \end{bmatrix} = \bar{C}(z(t_j) - z(t)), \tag{33}$$

where the matrix $\bar{C}$ is defined in (14). Similarly, also define $\bar{z}(t) := z(t) - z(t_s)$. Since (26a) holds and $u(t) \equiv u(t_j)$, the closed-loop dynamics evolves according to

$$\begin{aligned} \dot{z}(t) = &\left[\bar{A} + J^\top \Delta A J + (\bar{B} + J^\top \Delta B J)F_*\bar{C}\right] z(t) \\ &+ J^\top\left(k^\star\left(J^\top z(t)\right) - k^\star(x^\star)\right) + (\hat{F}_* - F_*)\bar{C}\bar{z}(t) + (\bar{B} + J^\top\Delta BJ)\hat{F}_* e(t), \qquad t \in [t_s, t_{s+1}), \end{aligned} \tag{34}$$

where the matrices $\bar{A}$, $\bar{B}$, J are defined in (14). Consider the same Lyapunov function as in the continuous-time case

$$V(z) = z^\top P_* z, \qquad P_* = P_*^\top \succ 0.$$

The time derivative of this Lyapunov function with respect to (34) can be computed by

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}V(z) = &z^\top(t)P_*\Big((\bar{B} + J^\top\Delta BJ)\hat{F}_* e(t) \\ &+ \left(\bar{A} + J^\top\Delta AJ + (\bar{B} + J^\top\Delta BJ)F_*\bar{C}\right)z(t) + (\hat{F}_* - F_*)\bar{C}\bar{z}(t) + J^\top\left(k^\star(J^\top z) - k^\star(x^\star)\right)\Big). \end{aligned} \tag{35}$$

By assumption, we know that the objective function of the program (18) is positive, i.e., $\alpha_*\zeta_*^{-1} > 0$. Invoking Young's inequality, we have

$$2z^\top(t)P_*\left(\bar{B} + J^\top\Delta BJ\right)\hat{F}_* e(t) \le \psi_1\vartheta_B^2\|z(t)\|^2 + \psi_1^{-1}\|e(t)\|^2,$$
$$2z^\top(t)P_*\left(\hat{F}_* - F_*\right)\bar{C}\bar{z}(t) \le \psi_2\beta^2\|z(t)\|^2 + \psi_2^{-1}\|\bar{z}(t)\|^2,$$

where ψ_1, ψ_2 are two positive scalars to be specified later. Thus, the derivative $\dot{V}$ from (35) can be estimated by

$$\frac{d}{dt}V(z(t)) \le -(\alpha_* - \psi_1\vartheta_B^2 - \psi_2\beta^2)\|z(t)\|^2 + \zeta_* k_b^2 + {\psi_1}^{-1}\|e(t)\|^2 + {\psi_2}^{-1}\|\bar{z}(t)\|^2. \tag{36}$$

One may also notice that since $\dot{\bar{z}}(t) = \dot{z}(t)$ and $e(t) = \bar{C}(z(t_j) - z(t_s)) - \bar{C}\bar{z}(t)$, the equation (34) is rewritten as

$$\begin{aligned} \dot{\bar{z}}(t) = &\left[\bar{A} + J^\top\Delta AJ + (\bar{B} + J^\top\Delta BJ)F_*\bar{C}\right] z(t_s) + J^\top(k^\star(J^\top z) - k^\star(x^\star)) \\ &+ (\bar{B} + J^\top\Delta BJ)\hat{F}_*\bar{C}(z(t_j) - z(t_s)) + \left[\bar{A} + J^\top\Delta AJ + (\bar{B} + J^\top\Delta BJ - I)(F_* - \hat{F}_*)\right]\bar{C}\bar{z}(t). \end{aligned} \tag{37}$$

Recall that we have assume $\bar{h} \le h_{\max}$. Leveraging similar techniques as in [19, Lemma 3], the estimate of the ODE solution of (37) yields

$$\|\bar{z}(t)\| \le \Big[\left(\|\bar{B}\| + \|B_b\|\right)\|\hat{F}_*\|\,\|e(t_s)\| + k_b + \left(\|\bar{A}\| + \|A_b\| + (\|\bar{B}\| + \|B_b\|)\|F_*\bar{C}\|\right)\|z(t_s)\|\Big]\mathfrak{c}(\bar{h}) \tag{38}$$

where the constant $\mathfrak{c}(h)$ is defined in (29). Notice now that if $\mathcal{Q} \preceq \widetilde{\mathcal{Q}}(q_0, q_1)$, we can conclude that $\|e(t_s)\|^2 \le q_0 + q_1\|\bar{C}\|^2\|z(t_s)\|^2$. This inequality automatically holds if $t_s = t_j$ since $e(t_j) = 0$. Otherwise, at time $t_s > t_j$ the triggering condition (27) cannot hold. Therefore,

$$\|e(t)\|^2 \le \left(\|e(t_s)\| + \|e(t) - e(t_s)\|\right)^2 \le 2q_0 + 2q_1\|\bar{C}\|^2\|z(t_s)\|^2 + 2\|\bar{C}\|^2\|\bar{z}(t)\|^2. \tag{39}$$

Set the values of ψ_1 and ψ_2 as

$$\psi_1 = \sigma_1\vartheta_B^{-2}\alpha_*, \qquad \psi_2 = \sigma_2\beta^{-2}\alpha_*, \tag{40}$$

for arbitrary values of σ_1 and σ_2 with condition $0 < \sigma_1, \sigma_2$ and also, $\sigma_1 + \sigma_2 < 1$. Equations (36) together with (38)-(40) leads to

$$\dot{V}(z(t)) \leq -\alpha_*(1-\sigma_1-\sigma_2)\|z(t)\|^2 + \mathfrak{g}_1\|z(t_s)\|^2 + \mathfrak{g}_2, \tag{41}$$

where the constants $\mathfrak{g}_1, \mathfrak{g}_2$ are defined as

$$\mathfrak{g}_1 = \sigma_1^{-1}\vartheta_B^2\alpha_*^{-1}\Big(2q_1\|\bar{C}\|^2 + 6q_1\varrho_B\|\bar{C}\|^4\mathfrak{e}^2(\bar{h}) + 6\varrho_{AB}\|\bar{C}\|^2\mathfrak{e}^2(\bar{h})\Big) + 3\sigma_2^{-1}\beta^2\alpha_*^{-1}(\varrho_B q_1\|\bar{C}\|^2 + \varrho_{AB})^2\mathfrak{e}^2(\bar{h}), \tag{42a}$$

$$\mathfrak{g}_2 = \sigma_1^{-1}\vartheta_B^2\alpha_*^{-1}\Big(2q_0 + 6q_0\varrho_B\|\bar{C}\|^2\mathfrak{e}^2(\bar{h}) + 6\|\bar{C}\|^2\mathfrak{e}^2(\bar{h})k_b^2\Big) + 3\sigma_2^{-1}\beta^2\alpha_*^{-1}\left(\varrho_B q_0 + k_b^2\right)\mathfrak{e}^2(\bar{h}) + \zeta_* k_b^2. \tag{42b}$$

By substituting $t = t_{s+1}$ in (41) and defining $h_s := t_{s+1} - t_s$, it is concluded that

$$V(t_{s+1}) \leq \left(e^{\mathfrak{g}_3\lambda_{\max}^{-1}(P_*)h_s} - 1\right)\mathfrak{g}_3^{-1}\mathfrak{g}_2 + \left[e^{\mathfrak{g}_3\lambda_{\max}^{-1}(P_*)h_s} + \left(e^{\mathfrak{g}_3\lambda_{\max}^{-1}(P_*)h_s} - 1\right)\mathfrak{g}_3^{-1}\mathfrak{g}_1\frac{\lambda_{\max}(P_*)}{\lambda_{\min}(P_*)}\right]V(t_s).$$

where the new constant is defined as $\mathfrak{g}_3 = -\alpha_*(1-\sigma_1-\sigma_2)$. It can be shown that coefficient of $V(z(t_s))$ in the right hand side of last inequality is less than 1 if $h_s \leq \bar{h} < h_{\max}$. We select $\sigma_1 = \sigma_2 = \sqrt{q_1}(1+2\sqrt{q_1})^{-1}$. If $\bar{h} < h_{\max}$, then

$$\varlimsup_{t\to\infty}\|y(t)\|^2 \leq \|\bar{C}\|^2\varlimsup_{t\to\infty}\|z(t)\|^2 \leq \|\bar{C}\|^2\lambda_{\min}^{-1}(P_*)\varlimsup_{t\to\infty}V(t) \leq \|\bar{C}\|^2\frac{\mathfrak{g}_2\lambda_{\max}(P_*)}{-\mathfrak{g}_1\lambda_{\max}(P_*) - \mathfrak{g}_3\lambda_{\min}(P_*)} = \varepsilon_{\mathrm{d}}^2.$$

This implies that the system (1) is ε_{d}-practical stable and also $y(t)$ converges to a ball with center y^{d} and radius ε_{d}. □

Remark 4.3 (Explicit inter sampling bound)**.** *Theorem 4.2 offers an AETC with a more general framework including absolute and relative thresholds whose maximal inter sampling time $h_{\max}$ enjoys explicit easy-to-compute formula in* (30) *(cf.,* [16, Assumption III.1]*).*

The setting in Theorem 4.2 is clearly more stringent than the continuous measurements and actuation framework in Theorem 3.3. Therefore, it is no longer surprising that the corresponding practical stability levels in (19) and (31) satisfy $\varepsilon_{\mathrm{c}} \leq \varepsilon_{\mathrm{d}}$. The latter is essentially quantified based on three parameters: maximum inter sampling bound $h_{\max}$, and the absolute and relative triggering thresholds q_0, q_1 (cf. Remark 4.1). When $h_{\max}$ tends to 0, our setting effectively moves from the aperiodic sampled measurement framework to the continuous domain, and when the thresholds q_0 and q_1 tend to 0, the event-triggered control mechanism transfers to the continuous-time implementation. A natural question with regards to Theorems 3.3 and 4.2 is how the gap between the thresholds $\varepsilon_{\mathrm{c}}, \varepsilon_{\mathrm{d}}$ behaves as these critical parameters tend to zero.

Remark 4.4 (From discrete to continuous implementation)**.** *Let ε_{c} be defined as in* (19) *and $\varepsilon_{\mathrm{d}}(\bar{h}, q_0, q_1)$ in* (31) *as a function of the relevant parameters $\bar{h}, q_0, q_1$. With a straightforward computation, one can inspect that*

$$\lim_{q_0,q_1\to 0}\lim_{\bar{h}\to 0}\varepsilon_{\mathrm{d}}(\bar{h}, q_0, q_1) = \varepsilon_{\mathrm{c}}.$$

We note that the practical stability certificate ε_{d} of the proposed AETC in (31) may take 0 values when $k_b = q_0 = 0$. This implies that even if the system is uncertain and we have an AETC in place, we may still be able to steer the output of the system to the desired target y^{d}. This interesting outcome, however, comes at the price of a bound on the absolute threshold q_1. We close this section with the following result in this regard.

Corollary 4.5 (Relative AETC threshold for perfect tracking)**.** *Suppose the system* (1) *is linear (i.e., $k_b = 0$ in Assumption 2.1(ii)), the program* (18) *is feasible with $\alpha_* > 0$, and the absolute threshold in Theorem 4.2 is $q_0 = 0$. Then, if*

$$\sqrt{q_1}(2\sqrt{q_1}+1)^2 < \frac{\alpha_*^2\lambda_{\min}(P_*)}{2\|\bar{L}\|^2\vartheta_B^2\lambda_{\max}(P_*)},$$

then the regulation performance in (31) *is* $\varepsilon_{\mathrm{d}} = 0$, *i.e., the controller* (11) *implemented via the AETC scheme in Algorithm 1 steers the output of the system to the desired target* y^{d}.

Proof. The proof is an immediate consequence of Theorem 4.2. It only suffices to check for which values of q_1 the maximal inter sampling $h_{\max}$ in (30) is still well-defined. □

5. Numerical Method and Examples

Due to the existence of nonlinearities in the optimization problem (18), some tricks are required for solving this problem by using regular LMI solvers. On the basis of such tricks, firstly a numerical technique is proposed in this section for solving the optimization problem (18). Then, this technique is applied in numerical examples illustrating the main results of the paper.

5.1. Numerical Method

There are two types of nonlinearities in the optimization problem (18). The first type of these nonlinearities comes from cross products of decision variables and the second one comes from the appearance of inverse of some of decision variables. Since no general-purpose scheme is available to deal with bilinear matrix inequalities, one needs to resort to approximation approaches. To this end, we propose a numerical algorithm that builds on a powerful technique called "*sequential parametric convex approximation*" from [23]. We first provide two preparatory lemmas.

Lemma 5.1. *Let* $\mathcal{Y}$ *and* $\mathcal{Z}$ *be two matrices with appropriate dimensions. The inequality* $\left[\mathcal{Y}^\top \mathcal{Z}\right]^\dagger \preceq 0$ *holds if*

$$\begin{bmatrix} \left[(\mathcal{Y}-\mathcal{Y}_k)^\top \mathcal{Z}_k + \mathcal{Y}_k^\top(\mathcal{Z}-\mathcal{Z}_k) + \mathcal{Y}_k^\top \mathcal{Z}_k\right]^\dagger & * & * \\ (\mathcal{Y}-\mathcal{Y}_k)^\top & -\mathcal{U} & * \\ (\mathcal{Z}-\mathcal{Z}_k)^\top & 0 & -\mathcal{U}^{-1} \end{bmatrix} \preceq 0 \tag{43}$$

where $\mathcal{Y}_k$ *and* $\mathcal{Z}_k$ *are given matrices with the same size as* $\mathcal{Y}$ *and* $\mathcal{Z}$, *respectively, and* $\mathcal{U} \in \mathbb{S}_{\succ 0}$ *is an arbitrary matrix.*

Lemma 5.1 is essentially a combination of standard Young's inequality and Schur complement. It is worth noting that applying Young's inequality to the term $\left[\mathcal{Y}^\top \mathcal{Z}\right]^\dagger \preceq 0$ yields an alternative approximation in the form of $\mathcal{Z}^\top \mathcal{U}^{-1} \mathcal{Z} + \mathcal{Y}^\top \mathcal{U} \mathcal{Y} \preceq 0$. However, if the constant matrices $\mathcal{Y}_k$ and $\mathcal{Z}_k$ are close estimates of the variables $\mathcal{Y}$ and $\mathcal{Z}$, respectively, then the proposed approximation in (43) is more efficient. We also note that in a context of optimization problem, the matrix $\mathcal{U}$ is a degree of freedom, and that can be viewed as an additional decision variable.

The next lemma suggests an idea to deal with the inverse of a decision variable in an optimization problem by introducing a linear over-approximation for the inverse of a matrix.

Lemma 5.2. [22, Lemma 2] *If* $\mathcal{U}, \mathcal{U}_k \in \mathbb{S}^n_{\succ 0}$, *then*

$$-\mathcal{U}^{-1} \preceq -2\mathcal{U}_k + \mathcal{U}_k^{-1} \mathcal{U} \mathcal{U}_k^{-1}.$$

By some straightforward computations and using the results of Lemmas 5.1 and 5.2, one can observe that

$$\begin{bmatrix} M_k + \sum_{i,j} \kappa_{ij} J^\top e_j^\top e_j J & * & * & * \\ H_1^\top & & & \\ H_1^\top & & G_k & \\ X_k^\top & & & \end{bmatrix} \preceq 0 \Rightarrow \begin{bmatrix} M + \sum_{i,j} \kappa_{ij} J^\top e_j^\top e_j J & * & * & * & * \\ H_1^\top & G_1 & * & * & * \\ H_1^\top & 0 & G_2 & * & * \\ H_2^\top & 0 & 0 & G_3 & * \\ JP & 0 & 0 & 0 & -\zeta I \end{bmatrix} \preceq 0$$

where,

$$M_k := [P\bar{A} + P_k\bar{B}(F - F_k)\bar{C} + P\bar{B}F_k\bar{C}]^{\dagger} + \alpha I, \quad G_{3_k} := \mathfrak{Diag}\left\{(-2\mu_{ij_k} + \mu_{ij})\right\}_{i,j},$$
$$G_k := \mathfrak{Diag}\left\{G_1, G_2, G_{3_k}, -2U_k + U, -U, -\zeta I\right\}, \quad H_{2_k} := \mathfrak{Diag}\left\{\mu_{ij_k}\right\} H_2,$$
$$X_k := \begin{bmatrix} H_{2_k} & (P - P_k)U_k^{\top} & \bar{B}(F - F_k)\bar{C} & PJ^{\top} \end{bmatrix}.$$

Building on the above definitions, Algorithm 2, as a sequential approximate algorithm, can be proposed to find a stationary point for the optimization problem (18) (From [23, Proposition 3], it can be proved that Algorithm 2 converges to a stationary point of (18)).

Algorithm 2 Sequential Parametric Convex Approximation

Set $k = 0, F_k = 0, \mu_{ij_k} = 1$.

Solve $$\begin{cases} (\alpha_k, \zeta_k) = \text{argmax } (\ln\alpha - \ln\zeta) \\ \text{Subject to} \\ \quad \lambda_{ij} \in \mathbb{R}_{>0}, W_{ij} \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_u)} \\ \quad \begin{bmatrix} W_{ij} - \lambda_{ij} a_{b_{ij}}^2 J^{\top} e_i^{\top} e_i J & * \\ e_j J & \lambda_{ij} \end{bmatrix} \succeq 0 \\ \quad 0 \succeq \alpha I + \bar{A} + \bar{A}^{\top} + \sum_{i,j} W_{ij} + \zeta J^{\top} J \end{cases}.$$

Solve $$\begin{cases} P_k \in \mathbb{S}_{\succ 0}^{n_x+n_u} \\ P_k\bar{A} + \bar{A}^{\top}P_k + \zeta_k^{-1} P J^{\top} J P \preceq -\alpha_k I \end{cases}.$$

Set $k = 1$.

while $\left|\alpha_k\zeta_k^{-1} - \alpha_{k-1}\zeta_{k-1}^{-1}\right| > \varepsilon$ **do**

Solve $$\begin{cases} (P_k, F_k, \mu_{ij_k}, \alpha_k, \zeta_k) = \text{argmax } (\ln\alpha - \ln\zeta) \\ \text{s.t.} \quad \kappa_{ij}, \mu_{ij} \in \mathbb{R}_{>0}, \;\; P, U \in \mathbb{S}_{\succ 0}^{n_x+n_u}, \\ \qquad C_c \in \mathbb{R}^{n_u\times n_u}, \;\; B_c, D_c \in \mathbb{R}^{n_u\times n_y} \\ \qquad \begin{bmatrix} C_k + \sum_{i,j} \kappa_{ij} J^{\top} e_j^{\top} e_j J & * & * & * \\ H_1^{\top} & & & \\ H_1^{\top} & & G_k & \\ X_k^{\top} & & & \end{bmatrix} \preceq 0 \end{cases}$$

Set $k + 1 \leftarrow k$.

5.2. Examples

This subsection is devoted to presenting two examples to confirm the applicability of the obtained results. The first example numerically evaluates the performance of the controller introduced in Theorem 3.3. Also, the second example reveals the applicability of Theorem 4.2 in solving a bidding problem in electricity markets, which is related to the motivation explained in Section 2.2.

Example 1. Consider the uncertain system (1) (at first, without the nonlinear term $k^{\star}(x(t))$), where the nominal matrices A and B are as follows (These nominal matrices are chosen from "REA2" example in *Complib* library of MATLAB).

$$A = \begin{bmatrix} 1.50 & -0.11 & 6.82 & -5.58 \\ -0.48 & -4.19 & 0.10 & 0.78 \\ 1.17 & 4.37 & -6.55 & 5.99 \\ 0.15 & 4.37 & 1.44 & -2.00 \end{bmatrix}, \quad B = \begin{bmatrix} -0.100 & -0.100 & 0.405 & -0.019 \\ 5.579 & -0.100 & 0.661 & 0.677 \\ 1.036 & -3.246 & 0.531 & 0.805 \\ 1.036 & -0.100 & -0.010 & 0.434 \end{bmatrix}.$$

Also, assume that $C = I_4$. Furthermore, consider the uncertainty bounds $A_b = B_b = \mathbb{1}_4^{\top} \otimes \mathbb{1}_4$. In this case, by solving the optimization problem (18) via Algorithm 2, the matrices B_c, C_c, and D_c are found. Applying controller (9) with the obtained matrices, the system output will asymptotically converge to desired target y^{d}. As a sample, the simulation results for the case $y^{\mathrm{d}} = \begin{bmatrix} 9 & 10 & 11 & 12 \end{bmatrix}$ are presented in Figure 1.

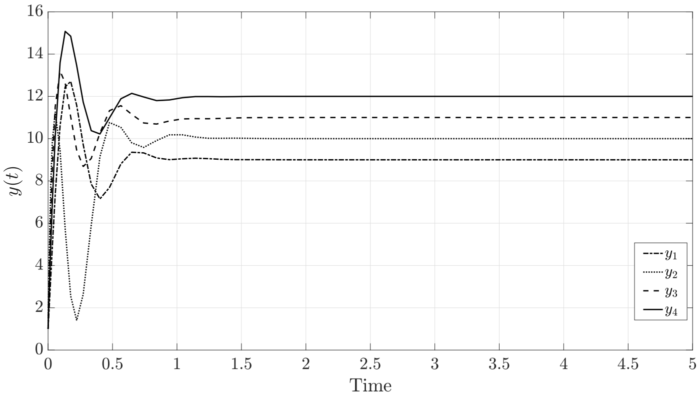

FIGURE 1. Simulation results of Example 1 ($y^{\mathrm{d}} = [9\ 10\ 11\ 12]$).

By considering the uncertainties in the form $A_b^{\upsilon} = B_b^{\upsilon} = \upsilon * \left(\mathbb{1}_4^{\top} \otimes \mathbb{1}_4\right)$ for $0 \leq \upsilon \leq 3$, the performance of the proposed robust controller is compared with a proportional one in Figure 2. As it can be seen from Figure 2, the robust controller guarantees the closed-loop stability for $0 \leq \upsilon \leq 3$, while the proportional controller may lead to instability for $1.2 \leq \upsilon$.

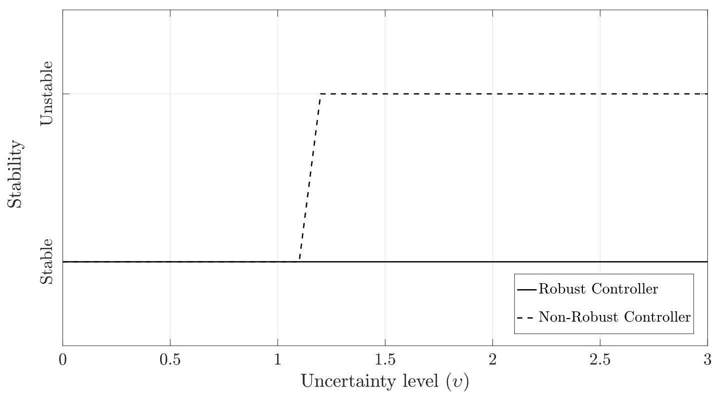

FIGURE 2. Comparing the performance of the robust controller with a proportional one in Example 1.

Moreover, Figure 3 shows the simulation results on using the proposed robust controller, where the nonlinear term $k^{\star}(x(t))$ is considered as $20\left[sin(x_1(t)) \quad \ldots \quad sin(x_4(t))\right]$ in system (1). These results confirm that the system output reaches to a ball around the desired value, and the practical converges is achieved.

Example 2. Consider an electricity market consisting of a system manager and a group of strategic generators. The manager seeks to solve the power dispatch problem and regulate the frequency of the network. On the other hand, each power generator faces with a private cost function specifying the cost of its power generation, and aims to maximize its income, which is the difference between the bid fee suggesting by the manager and its cost for power generation. It is assumed that the network manager sets the power purchase price. By this assumption, it is possible for the system manager to find the desired working point for each generator according to demand on the network and the objective of frequency regulation. Now, the problem for the system manager is to set the purchase price and send it to generators such that each generator convinces to work at a considered set point, which is desirable in the viewpoint of the system manager.

Similar to [39, Section VI], consider a 7-node power network such that each node in this network is a generator and the power generation for generator i ($i = 1, 2, ..., 7$) has a cost in the form $C_i(\mathcal{P}_i) = \frac{1}{2} q_i \mathcal{P}_i^2 + c_i \mathcal{P}_i$,

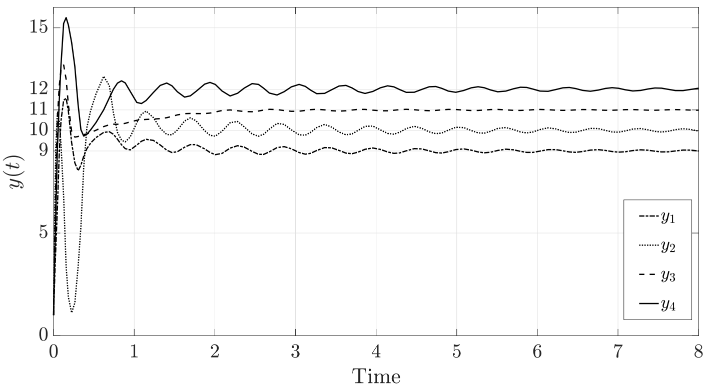

FIGURE 3. Simulation results of Example 1 in the case of the existence of the nonlinear term $k^{\star}(x(t)) = 20\,[\sin(x_1(t)) \;\; \ldots \;\; \sin(x_4(t))]$ in system (1).

where $\mathcal{P}_i$ is the power generated by ith generator. Assume that the manager sets the purchase price by tuning the parameter b_i for each generator. In this situation, the income for generator i will be equal to $b_i\mathcal{P}_i - C_i(\mathcal{P}_i)$. In this problem, generator i $(i = 1, 2, ..., 7)$ seeks its best power generation level by maximizing $b_i\mathcal{P}_i - C_i(\mathcal{P}_i)$. It is assumed that generators cannot optimize $b_i\mathcal{P}_i - C_i(\mathcal{P}_i)$ instantly, and they will use a gradient method to find the optimal value of this optimization problem. According to the point that the exact values of q_i and c_i are not known for the system manager, he will try to set parameters b_i such that the optimal value of $b_i\mathcal{P}_i - C_i(\mathcal{P}_i)$ is equal to desired values selected by him for each i $(1 \le i \le 7)$. Assume that the system manager is aware about the uncertainty ranges $1 \le c_i \le 3$ and $0.5 \le q_i \le 1.5$. Considering this assumption, he can use a controller in the form (9) with parameters obtained from optimization problem (18) to find $b_i(t)$ for $i = 1, 2, ..., 7$. Furthermore, it is assumed that the manager forces the generators to broadcast their power generation information at some specific times $(\{t_k\}_{k\in\mathbb{N}})$. To find the time in which the values of b_i should be updated, the manager uses the triggering condition in the form (27) by using matrix $\mathcal{Q}$ similar to $\widetilde{\mathcal{Q}}(q_0, q_1)$ which was introduced by (28). This condition can be simply represented as follows:

$$\sum_{i=1}^{7} (\mathcal{P}_i(t) - \mathcal{P}_i(t_j))^2 + (w_i(t) - w_i(t_j))^2 \ge q_0 + q_1 \sum_{i=1}^{7} [\mathcal{P}_i^2(t_j) + w_i^2(t_j)], \tag{44}$$

where $t_j \in \{t_k\}_{k\in\mathbb{N}} < t$ is the last broadcast time of b_is, and w is the state vector of the dynamic controller. Assuming $\sup_{k\in\mathbb{N}}\{t_{k+1} - t_k\} \le h_{\max}$, numerical simulation to verify the results of Theorem 4.2 has been done by change the threshold level in the inequality (44) in the form $q_0 = q_1 = \Xi$. Figure 4 shows how changing the threshold level (q_0, q_1) affects the distance between the steady state values of the outputs and the desired targets (defined by $\lim_{t\to\infty} \sqrt{\sum \|\mathcal{P}_i(t) - \mathcal{P}_i^*\|^2}$). The obtained results are compatible with those expected from Theorem 4.2 (Equation (31)).

Furthermore, Figure 5 shows the relation between the percentage of times in which the manager sent data (in proportion to the size of time series) and the distance between the final values and the desired ones. These simulation results reveal that by increasing the number of control signal broadcasting, the outputs will be more close to the desired targets.

6. Conclusion

In this paper, we introduced an optimization-based framework to synthesize robust dynamic controllers in order to ensure the output regularization task for systems with uncertain and potentially nonlinear dynamics. To numerically solve such an optimization problem, a sequential parametric convex approximate algorithm was proposed. We further introduced a general sampling-based event-triggered technique that paves the way to implement the proposed controller in case of sampled measurements and discontinuous actuation updates.

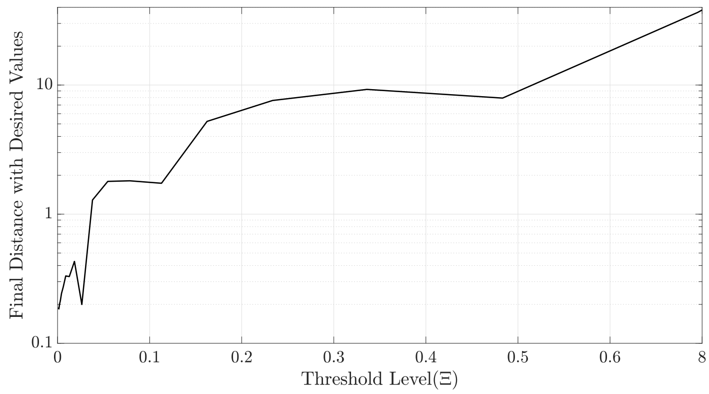

FIGURE 4. Relation between the final distance with the desired targets and the threshold level.

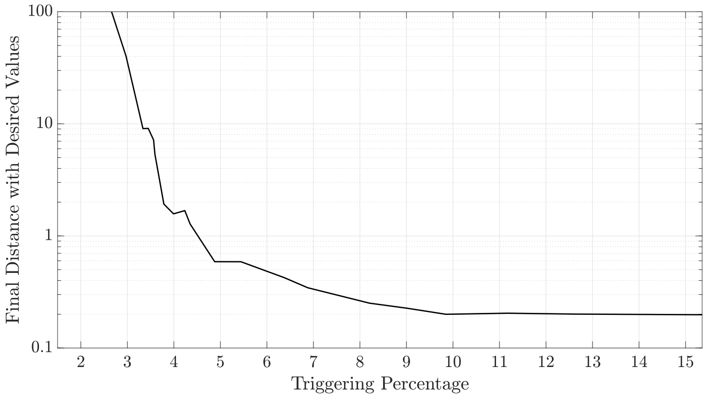

FIGURE 5. Relation between the final distance with the desired targets and the percentage of times in which data is sent by the manager.

It is remarkable that the procedure of the triggering law is decoupled from control synthesis and the key parameters such as maximal inter sampling time is explicitly computationally available.

REFERENCES

[1] Amir Ali Ahmadi and Georgina Hall. On the complexity of detecting convexity over a box. *Mathematical Programming*, pages 1–15, 2019.

[2] Aharon Ben-Tal and Arkadi Nemirovski. Robust convex optimization. *Mathematics of operations research*, 23(4):769–805, 1998.

[3] Aharon Ben-Tal and Arkadi Nemirovski. On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty. *SIAM Journal on Optimization*, 12(3):811–833, 2002.

[4] Aharon Ben-Tal, Arkadi Nemirovski, and Cornelis Roos. Extended matrix cube theorems with applications to μ-theory in control. *Mathematics of Operations Research*, 28(3):497–523, 2003.

[5] Stephen Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15. Siam, 1994.

[6] Giuseppe C Calafiore and Marco C Campi. The scenario approach to robust control design. *IEEE Transactions on Automatic Control*, 51(5):742–753, 2006.

[7] Duncan S Callaway and Ian A Hiskens. Achieving controllability of electric loads. *Proceedings of the IEEE*, 99(1):184–199, 2010.

[8] MCF Donkers and WPMH Heemels. Output-based event-triggered control with guaranteed $\mathcal{L}_\infty$-gain and improved and decentralized event-triggering. *IEEE Transactions on Automatic Control*, 57(6):1362–1376, 2012.

[9] John Doyle. Analysis of feedback systems with structured uncertainties. In *IEE Proceedings D-Control Theory and Applications*, volume 129, pages 242–250. IET, 1982.

[10] John Doyle and Gunter Stein. Multivariable feedback design: Concepts for a classical/modern synthesis. *IEEE Transactions on Automatic Control*, 26(1):4–16, 1981.

[11] Gene F Franklin, J David Powell, Michael L Workman, et al. *Digital control of dynamic systems*, volume 3. Addison-wesley Menlo Park, CA, 1998.

[12] Anqi Fu and Manuel Mazo. Traffic models of periodic event-triggered control systems. *IEEE Transactions on Automatic Control*, 64(8):3453–3460, 2018.

[13] Pascal Gahinet, Pierre Apkarian, and Mahmoud Chilali. Affine parameter-dependent lyapunov functions and real parametric uncertainty. *IEEE Transactions on Automatic control*, 41(3):436–442, 1996.

[14] Bahman Gharesifard, Tamer Başar, and Alejandro D Domínguez-García. Price-based coordinated aggregation of networked distributed energy resources. *IEEE Transactions on Automatic Control*, 61(10):2936–2946, 2015.

[15] Christophe Guille and George Gross. A conceptual framework for the vehicle-to-grid (v2g) implementation. *Energy policy*, 37(11):4379–4390, 2009.

[16] WPM Heemels Heemels, MCF Donkers, and Andrew R Teel. Periodic event-triggered control for linear systems. *IEEE Transactions on Automatic Control*, 58(4):847–861, 2013.

[17] Tetsuya Iwasaki and Shinji Hara. Generalized KYP lemma: Unified frequency domain inequalities with design applications. *IEEE Transactions on Automatic Control*, 50(1):41–59, 2005.

[18] Hassan K Khalil and Jessy W Grizzle. *Nonlinear systems*, volume 3. Prentice hall Upper Saddle River, NJ, 2002.

[19] Masako Kishida, Markus Kögel, and Rolf Findeisen. Combined event-and self-triggered control approach with guaranteed finite-gain $\mathcal{L}_2$ stability for uncertain linear systems. *IET Control Theory & Applications*, 11(11):1674–1683, 2017.

[20] Arman Sharifi Kolarijani, Sander C Bregman, Peyman Mohajerin Esfahani, and Tamás Keviczky. A decentralized event-based approach for robust model predictive control. *IEEE Transactions on Automatic Control*, 2019.

[21] Alexander Lanzon and Ian R Petersen. Stability robustness of a feedback interconnection of systems with negative imaginary frequency response. *IEEE Transactions on Automatic Control*, 53(4):1042–1046, 2008.

[22] Donghwan Lee and Jianghai Hu. A sequential parametric convex approximation method for solving bilinear matrix inequalities. In *IEEE Conference on Decision and Control (CDC)*, pages 1965–1970, 2016.

[23] Donghwan Lee and Jianghai Hu. Sequential parametric convex approximation algorithm for bilinear matrix inequality problem. *Optimization Letters*, pages 1–19, 2018.

[24] Daniel Liberzon and Roberto Tempo. Common lyapunov functions and gradient algorithms. *IEEE Transactions on Automatic Control*, 49(6):990–994, 2004.

[25] Wei Liu and Jie Huang. Event-triggered cooperative robust practical output regulation for a class of linear multi-agent systems. *Automatica*, 85:158–164, 2017.

[26] Wei Liu and Jie Huang. Event-triggered global robust output regulation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 62(11):5923–5930, 2017.

[27] Manuel Mazo and Paulo Tabuada. On event-triggered and self-triggered control over sensor/actuator networks. In *IEEE Conference on Decision and Control*, pages 435–440, 2008.

[28] Luc Moreau and Dirk Aeyels. Practical stability and stabilization. *IEEE Transactions on Automatic Control*, 45(8):1554–1558, 2000.

[29] Roberto Naldi, Michele Furci, Ricardo G Sanfelice, and Lorenzo Marconi. Robust global trajectory tracking for underactuated vtol aerial vehicles using inner-outer loop control paradigms. *IEEE Transactions on Automatic Control*, 62(1):97–112, 2016.

[30] Arkadii Nemirovskii. Several NP-hard problems arising in robust stability analysis. *Mathematics of Control, Signals and Systems*, 6(2):99–105, 1993.

[31] Tatsushi Ooba and Yasuyuki Funahashi. On a common quadratic lyapunov function for widely distant systems. *IEEE Transactions on Automatic Control*, 42(12):1697–1699, 1997.

[32] Chen Peng and Fuqiang Li. A survey on recent advances in event-triggered communication and control. *Information Sciences*, 457:113–125, 2018.

[33] Svatopluk Poljak and Jiří Rohn. Checking robust nonsingularity is NP-hard. *Mathematics of Control, Signals and Systems*, 6(1):1–9, 1993.

[34] Boris T Polyak and Roberto Tempo. Probabilistic robust design with linear quadratic regulators. *Systems & Control Letters*, 43(5):343–353, 2001.

[35] Domingos CW Ramos and Pedro LD Peres. An LMI condition for the robust stability of uncertain continuous-time linear systems. *IEEE Transactions on Automatic Control*, 47(4):675–678, 2002.

[36] Michael G Safonov. Stability margins of diagonally perturbed multivariable feedback systems. In *IEE Proceedings D-Control Theory and Applications*, volume 129, pages 251–256. IET, 1982.

[37] Jeff S Shamma and Gürdal Arslan. Dynamic fictitious play, dynamic gradient play, and distributed convergence to nash equilibria. *IEEE Transactions on Automatic Control*, 50(3):312–327, 2005.

[38] Rajani Singh and Agnieszka Wiszniewska-Matyszkiel. Discontinuous nash equilibria in a two-stage linear-quadratic dynamic game with linear constraints. *IEEE Transactions on Automatic Control*, 64(7):3074–3079, 2018.

[39] Tjerk Stcgink, Ashish Cherukuri, Claudio De Persis, Arjan Van Der Schaft, and Jorge Cortés. Integrating iterative bidding in electricity markets and frequency regulation. In *American Control Conference*, pages 6182–6187, 2018.

[40] Paulo Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9):1680–1685, 2007.

[41] Sophie Tarbouriech, Alexandre Seuret, Christophe Prieur, and Luca Zaccarian. Insights on event-triggered control for linear systems subject to norm-bounded uncertainty. In *Control Subject to Computational and Communication Constraints*, pages 181–196. Springer, 2018.

[42] Roberto Tempo, Giuseppe Calafiore, and Fabrizio Dabbene. *Randomized algorithms for analysis and control of uncertain systems: with applications*. Springer Science & Business Media, 2012.

[43] O Toker and H Ozbay. On the np-hardness of the purely complex μ computation, analysis/synthesis, and some related problems in multidimensional systems. In *American Control Conference*, volume 1, pages 447–451, 1995.

[44] Onur Toker. On the complexity of the robust stability problem for linear parameter varying systems. *Automatica*, 33(11):2015–2017, 1997.

[45] Lantao Xing, Changyun Wen, Zhitao Liu, Hongye Su, and Jianping Cai. Event-triggered output feedback control for a class of uncertain nonlinear systems. *IEEE Transactions on Automatic Control*, 64(1):290–297, 2019.

[46] Jun Yang, Jiankun Sun, Wei Xing Zheng, and Shihua Li. Periodic event-triggered robust output feedback control for nonlinear uncertain systems with time-varying disturbance. *Automatica*, 94:324–333, 2018.

[47] George Zames. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Transactions on Automatic Control*, 26(2):301–320, 1981.

[48] Cui-Hua Zhang and Guang-Hong Yang. Event-triggered adaptive output feedback control for a class of uncertain nonlinear systems with actuator failures. *IEEE Transactions on Cybernetics*, 50(1):201–210, 2018.

[49] Kemin Zhou, John Comstock Doyle, Keith Glover, et al. *Robust and optimal control*, volume 40. Prentice hall New Jersey, 1996.

[50] Yuanqiang Zhou, Dewei Li, Yugeng Xi, and Zhongxue Gan. Periodic event-triggered control for distributed networked multiagents with asynchronous communication: A predictive control approach. *International Journal of Robust and Nonlinear Control*, 29(1):43–66, 2019.