Student Project
Optimal Control Approach to Parameter Inference for Hidden Markov Models

I. PARAMETER INFERENCE FOR HIDDEN MARKOV MODELS

Many dynamical systems that appear in several disciplines varying from mathematical finance to control systems are modelled by stochastic differential equations (SDEs) of the form

$$\mathrm{d}X_t^\kappa = f(X_t^\kappa, \kappa)\,\mathrm{d}t + \sigma(X_t^\kappa, \kappa)\,\mathrm{d}W_t, \quad X_0^\kappa = x_0, \quad 0 \le t \le T, \tag{1}$$

that describe the state evolution and are parametrized by κ representing an unknown parameter. Oftentimes, the state of the system is not directly observed, and inference of the state trajectories and parameter of the system has to be achieved based on noisy partial observations

$$Y_t^\kappa = \int_0^t h(X_s^\kappa)\,\mathrm{d}s + B_t, \tag{2}$$

where W and B are independent Brownian motions. Given a sample path $\{y_t : 0 \le t \le T\}$ of the observation process $Y_{[0,T]}^\kappa$, the objective is to select an optimal $\kappa^\star$ such that the observation process $(Y_t^{\kappa^\star})_{t\in[0,T]}$ in (2) has a high probability of reproducing the given data y. This problem can be answered by classical maximum likelihood estimation. The major difficulty is that, maximizing (even evaluating) the likelihood function is computationally difficult in the given setting. There are several approximation methods known to circumvent this computational difficulty, one quite recent method invokes optimal control [1]. Roughly speaking the method requires to solve a particular optimal control problem (OCP) in each step of an iterative method that is called the expectation maximization algorithm. As such, one is interested in efficiently solving the mentioned OCP, which basically is an open problem.

II. GOALS OF THE PROJECT

The aim of this project is to investigate the class of optimal control problems that arise from the parameter inference task sketched above, see [1] for details. One promising approach would be to use the so-called shooting method, see [2], to solve the necessary optimality conditions for the OCP provided by the Pontryagin's maximum principle.
Another more modern approach to characterize the solutions of the OCP is to use its so-called weak formulation which consists of an infinite-dimensional linear program, see [3, Chapter 10] for details. Therefore, numerical approximation schemes to such infinite-dimensional linear programs, that have been studied in the literature, can be employed to solve the mentioned OCP. This approach seems particularly promising when the data of the OCP (dynamics and costs) are described by polynomials, as then the seminal Lasserre hierarchy based on solving a sequence of semidefinite programs, is applicable [3, 4].
Of course other methods and directions, depending on the student's interest, could be analyzed and investigated. The mentioned goals here should be understood as some input thoughts and not as striclty recommended tasks.

III. REQUIREMENTS

The project is well suited for a student that enjoys mathematics. A solid background in analysis and convex optimization is required. Any knowledge in optimal control is useful, however not required.

IV. SUPERVISORS

- Tobias Sutter, Automatic Control Laboratory, ETHZ
- Peyman Mohajerin Esfahani, Automatic Control Laboratory, ETHZ
- Prof. John Lygeros, Automatic Control Laboratory, ETHZ

Interested students are highly motivated to contact any of the supervisors listed above to discuss further details about the mentioned project.

[1] T. Sutter, A. Ganguly, and H. Koeppl, "A variational approach to path estimation and parameter inference of hidden diffusion processes," ArXiv e-prints (2015), arXiv:1508.00506 [math.OC].

[2] J. Stoer, R. Bulirsch, R. Bartels, W. Gautschi, and C. Witzgall, *Introduction to Numerical Analysis*, Texts in Applied Mathematics (Springer, 2002).

[3] J. B. Lasserre, *Moments, Positive Polynomials and Their Applications*, Imperial College Press Optimization Series, Vol. 1 (Imperial College Press, London, 2010) pp. xxii+361.

[4] Jean B. Lasserre, "Global optimization with polynomials and the problem of moments," SIAM Journal on Optimization **11**, 796–817 (2001).