A Virtual Closed Loop Method for Closed Loop Identification ⋆

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Abstract

Indirect methods for identification of linear plant models on the basis of closed-loop data are based on the use of (reconstructed) input signals that are uncorrelated with the noise. This generally requires exact (linear) controller knowledge. On the other hand, direct identification requires exact plant and noise modelling (system in the model set) in order to arrive at accurate results, although the controller can be nonlinear. In this paper, a generalized approach to closed-loop identification is presented that includes both methods as special cases and which allows novel combined methods to be generated. Besides providing robustness with respect to inexact controller knowledge, the method does not rely on linearity of the controller nor on exact noise modeling. The generalization is obtained by balancing input-noise decorrelation against noise whitening in a user-chosen flexible fashion. To this end, a user-chosen virtual controller is used to parametrize the plant model, thereby generalizing the dual-Youla method to cases where knowledge of the controller is inexact. Asymptotic bias and variance results are presented for the method. Also, the benefits of the approach are demonstrated via simulation studies.

Key words: System identification, Closed Loop Identification.

1 Introduction

Identification of dynamic systems operating in the presence of feedback has received considerable attention in the system identification literature (see e.g. [14,24,26,7,21]). In many situations, there exist strong economic and/or safety reasons for requiring process data to be collected under closed loop conditions. Even open-loop stable plants are often subject to non-stationary disturbances and long term drift that favour a closed loop experiment for data collection. Additionally, it has been found that for several model applications (as e.g. model-based control design) and experimental constraints (e.g. output power constraints), a closed-loop identification experiment is often the optimal experimental setup, see e.g. [27,19,9,18,3].

Unfortunately, handling closed-loop data in identification induces additional difficulties, see e.g. [26,7].

In the prediction error framework [21], the two principle methods for closed loop identification can be characterized as follows:

- Direct identification: Here the plant is identified directly on the basis of plant input and output data taken from within the closed-loop. The presence of feedback is ignored. Consistent plant models can be identified under the condition that the noise dynamics are modelled exactly. Exact knowledge of the controller is not necessary and the controller may be nonlinear.
- Indirect identification: Here a plant object [1] is identified (usually the complementary sensitivity) between the reference input and plant output signals. Subsequently an equivalent plant model is retrieved from the identified object. Consistent plant models can be identified under the condition that the controller is linear and exactly known. This holds for several variants of the indirect method, including the dual-Youla approach [16,22,26],

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[1] Here and in the sequel, we use the term "*plant object*" to describe a transfer function that depends on the (open-loop) plant.

the method based on tailor-made parametrization [28], and bias-elimination least-squares methods (BELS) [29,11].

A third category, called joint input-output methods (see e.g. [24]), can be considered as an indirect method in the context of the current paper.

In practice, one would ideally like to have identification tools that combine the advantageous properties of both direct and indirect methods. In particular one would like to be able to handle the following situations:

- When a (slightly) nonlinear controller is present in the loop, e.g. a linear controller that saturates regularly, or that is not exactly known;
- When the noise disturbances are non-stationary or cannot be modelled exactly by stationary white noise filtered through a linear time-invariant system.

In the above situations, both direct and indirect methods fail to provide consistent plant models and the number of alternatives is very limited. The Projection Method [8] was proposed to deal with non-linear controllers, by identifying non-causal FIR models to approximate non-linear sensitivity functions. Alternatively, Instrumental Variable methods [25,11] can handle non-linear controllers and yield consistent plant models irrespective of noise under-modeling.

In this paper we develop a novel approach that takes a more generalized perspective. Realizing that all indirect methods use controller knowledge to exactly decorrelate the identification input signal from the noise, we will focus on this decorrelation and develop a generalized approach that is robust against controller nonlinearity and inexact controller knowledge.

A central issue in our development is the choice of a virtual controller [1,12] which approximates the real (possibly nonlinear) controller. This virtual controller will be deployed only for input signal construction to be used in identification. The more accurate the virtual controller, the less noise correlation will be present in the identification input. As a result, the bias due to inexact noise model will be reduced. In this way our method generalizes both the direct and indirect method of closed-loop identification. Our approach leads to a sliding mechanism between these two extremes. The user can make an appropriate choice depending on his/her faith in either the quality of the noise model, or the available knowledge and linearity of the controller.

The current paper completes and generalizes the analysis originally presented in [4], see also [1] and [12].

The remainder of the paper is organized as follows: In section 2 we introduce closed loop identification in a general non-linear setting. In section 3 we describe the new approach for identification of closed loop systems. In section 4 we show how the virtual closed loop (VCL) method generalizes known schemes for closed loop identification. In section 5 we analyze the spectra of signals appearing in the VCL method. In section 6 we show how the choice of parameters in the virtual closed loop affects asymptotic bias in the identification of systems operating in closed loop. In section 7 we analyze the accuracy of the estimates provided by VCL. In section 8 we present a numerical example. Finally in section 9 we draw conclusions.

2 Closed-loop identification setup

We consider a data generating system $\mathcal{S}$:

$$y_t = G_o(q)u_t + v_t \qquad (1)$$
$$v_t = H_o(q)w_t$$

where q is the forward-shift operator, y_t, u_t, and w_t are the output, input and noise respectively, $G_o(q)$, and $H_o(q)$ are linear transfer functions, with H_0 stable, stably invertible and monic (i.e. $\lim_{|z|\to\infty} H_o(z) = 1$). The noise w_t is assumed to be a stochastic process with variance σ_w^2. In the case that the real system is operating with a non-linear controller, the noise sequence is assumed to be a martingale difference sequence (MDS), and for the case of linear controllers we assume that w_t is white noise. The system is assumed to operate in a stabilized closed loop (see Figure 1), with

$$u_t = C(r_t - y_t),$$

Here, r_t is an external reference signal. Throughout the paper, we will take C to be a non-linear controller . However, at times, it will be convenient to consider the linear case so that we can relate our work to earlier literature. In these cases, we will use the notation C_l to denote C. In addition, we will, at other times, wish to consider a linear controller which is “close” (in some sense) to a non-linear controller. In this case, we will use the notation C_l^a.

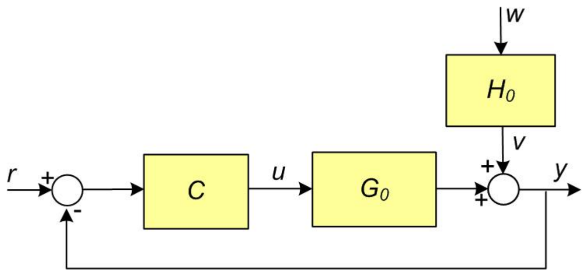

Fig. 1. Closed-loop system configuration

In order to have a well-defined closed-loop, it is further assumed that either C or G_o contains, at least, a one step time delay.
Spectral densities of signals are denoted by

$$\Phi_{uw}(\omega) = \sum_{\tau=-\infty}^{\infty} R_{uw}(\tau)e^{-j\omega\tau}$$

with $R_{uw}(\tau) := \bar{\mathrm{E}}\{u_t w_{t-\tau}\}$ and $\Phi_u = \Phi_{uu}$.

Our goal is the identification of a (consistent) plant model of G_o on the basis of closed-loop data. There are key differences between direct and indirect methods of identification. The different conditions for arriving at consistent estimates of G_o are listed in Table 1.

	direct	indirect
Exact LTI noise model	yes	no
C linear	no	yes
C exactly known	no	yes

Table 1
Required conditions for direct and indirect identification to provide consistent plant model estimates.

The clear distinction between the two situations raises the question whether one can combine the two approaches in a generalized method that is robust with respect to all three conditions; i.e. a method that is robust with respect to slight deviations from the assumptions of having exact controller knowledge, controller linearity, or exact LTI noise models. Such a method is developed in the next section.

3 A generalized approach to closed-loop identification

3.1 Introduction

In our generalized approach we consider the setup of Figure 2, where linear, causal and stable filters $F_1 \cdots F_4$ are introduced to generate signals

$$x_t = F_1(q)u_t + F_2(q)y_t \quad (2)$$
$$z_t = F_3(q)u_t + F_4(q)y_t. \quad (3)$$

The signals x_t and z_t are bounded since they are generated by passing bounded signals through stable filters. They can be used as input and output signals in a generalized identification scheme.

The proposed identification approach amounts to identifying a plant-related object through a linear transfer function between the signals x_t and z_t, by applying a model structure:

$$\varepsilon_t(\theta) = K(q,\theta)^{-1}[z(t) - R(q,\theta)x(t)] \quad (4)$$

leading to estimates $\hat{R} = R(q,\hat{\theta}_N)$ and $\hat{K} = K(q,\hat{\theta}_N)$, and subsequently to derive an equivalent plant model $\hat{G}$ from $\hat{R}$ by applying the principle of tailor-made parametrization.

It is apparent that the identification results will depend on the choice of the filters $F_1 \cdots F_4$. For example, by appropriately choosing F_1 and F_2 one can tune the presence of noise w_t in the generalized input signal x_t, and thereby one can influence the resulting bias.

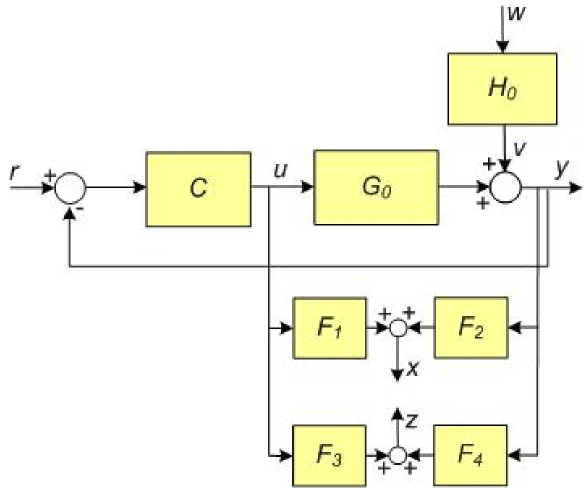

Fig. 2. Generalized scheme for closed-loop identification from input x_t to output z_t.

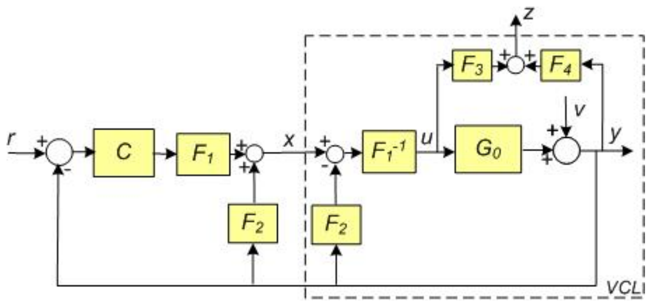

Fig. 3. Virtual Closed Loop.

3.2 The virtual closed-loop

Under, the additional condition that F_1 is stably invertible, the closed-loop diagram of Figure (2) can be redrawn as shown in Figure (3). This alternative view shows the filters F_1 and F_2 are used to construct the generalized reference input x_t via $x_t = F_1 u_t + F_2 y_t$ (where F_2/F_1 acts as a virtual controller that compoensatges for the original controller in the construction of x_t). Additionally, the resulting transfer function between x_t and z_t contains a (virtual) closed-loop plant object, where again the same controller F_2/F_1 is involved. Since the (linear) filter F_2/F_1 can be chosen freely by the user, it does not have any direct relation with the implemented controller C. Therefore the scheme is referred to by the name "Virtual Closed Loop". Note also that stability is not an issue since we already know that all signals are bounded. Moreover, in the case that the "true" controller is linear and equal to the virtual controller ($C_l = \bar{C} = F_2/F_1$) then $x_t = F_2 r_t$ is uncorrelated with the noise w_t. In the remainder of the paper, we will explore the implications of the configuration of Figure 2 in system identification.

3.3 System equations and filter conditions

In order to analyse the setup, we use equations (1), (2) and (3) to obtain the following set of equations describing the virtual closed loop system:

$$\begin{bmatrix} 1 & -F_3 & -F_4 \\ 0 & F_1 & F_2 \\ 0 & -G_o & 1 \end{bmatrix} \begin{bmatrix} z_t \\ u_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 \\ x_t \\ v_t \end{bmatrix} \tag{5}$$

Solving for z, u and y we have the following:

$$z_t = \frac{F_3 + F_4 G_o}{F_1 + F_2 G_o} x_t + \frac{F_1 F_4 - F_2 F_3}{F_1 + F_2 G_o} H_o w_t \tag{6}$$

$$u_t = \frac{1}{F_1 + F_2 G_o} x_t - \frac{F_2}{F_1 + F_2 G_o} H_o w_t \tag{7}$$

$$y_t = \frac{G_o}{F_1 + F_2 G_o} x_t + \frac{F_1}{F_1 + F_2 G_o} H_o w_t \tag{8}$$

We then write

$$z_t = R_o x_t + K_o w_t \tag{9}$$

with

$$R_o = \frac{F_3 + F_4 G_o}{F_1 + F_2 G_o}, \quad K_o = \frac{F_1 F_4 - F_2 F_3}{F_1 + F_2 G_o} H_o \tag{10}$$

In order for R_o and K_o to satisfy the usual conditions for applying prediction error identification methods the filters $F_1 \cdots F_4$ have to satisfy certain regularity conditions.

Assumption 1 *We assume that the filters F_i, $i = 1 \ldots 4$ satisfy the following:*

(1) F_1 is biproper;
(2) $F_2 G_o$ is strictly proper;
(3) G_o is stabilized by the controller F_2/F_1;
(4) $M := F_1 F_4 - F_2 F_3$ is stably invertible;
(5) $\lim_{|z| \to \infty} (F_1 F_4 - F_2 F_3) = \lim_{|z| \to \infty} (F_1 + F_2 G_o)$

$\triangledown\triangledown\triangledown$

Proposition 1 *The transfer functions R_o and K_o are causal and stable, while additionally K_o is stably invertible and monic, under the conditions in Assumption 1.*

Proof
Conditions (1) and (2) together with the causality of all filters $F_1, \cdots F_4$ guarantee that R_o and K_o are causal.
In order to show stability of R_o and K_o, we express the filters F_i and the plant G_o as a coprime polynomial factorizations: $F_i = N_i D_i^{-1}$, $i = 1, \cdots 4$, and $G_o = B_o / A_o$. Then

$$R_o = \frac{D_1 D_2}{D_3 D_4} \frac{(N_3 D_4 A + N_4 D_3 B)}{(N_1 D_2 A + N_2 D_1 B)} \tag{11}$$

Stability of R_o and K_o follows if $N_1 D_2 A + N_2 D_1 B$ has all zeros within the unit circle. This is guaranteed when G_o is stabilized by the controller $N_2 D_1 / N_1 D_2 = F_2 / F_1$, see condition (3).
Inverse stability of K_o is guaranteed by condition (4), while condition (5) guarantees K_o to be monic. $\square$

Notice that the condition to make K_o minimum phase is not necessary since, in the estimation procedure, we always use the minimum-phase spectral factor of the noise spectrum. However, for simplicity of the presentation, we assume that the filters F_i are such that K_o is minimum-phase.

3.4 Identification setup

The virtual closed-loop system will be identified by applying a direct identification method to the virtual-closed-loop system (9). On the basis of the generalized input signal x_t, and the generalized output signal z_t, we identify a model in a Box-Jenkins type model structure, with a prediction error

$$\varepsilon_t = K(q, \eta)^{-1} [z_t - R(q, \rho) x_t] \tag{12}$$

where

$$R(q, \rho) = \frac{F_3 + F_4 G(q, \rho)}{F_1 + F_2 G(q, \rho)}. \tag{13}$$

The parametrization of $R(q, \rho)$ is a tailor-made parametrization in which the parameters of the plant model G are used to parametrize the virtual closed-loop plant $R(q, \rho)$. In the parametrization (12), the plant and noise models are parametrized independently. The parameters ρ, and η are estimated by minimizing a quadratic criterion:

$$V_N = \frac{1}{N} \sum_{t=1}^{N} \varepsilon_t^2. \tag{14}$$

Notice that once estimates $\hat{G}$, $\hat{K}$ for G_o and K_o have been obtained, one can define an estimate for H_o as follows:

$$\hat{H} := \frac{F_1 + F_2 \hat{G}}{F_1 F_4 - F_2 F_3} \hat{K} \tag{15}$$

In the subsequent analysis we will analyze the impact of the following two issues:

(1) x_t is not, in general, an exogenous signal but is potentially correlated with the noise w_t.
(2) The class of models used for $K(q,\eta)$ may not include the true noise model K_o e.g. we might decide to use a fixed noise model $K \neq K_o$.

4 Particular cases and general properties

We will first establish that the Virtual Closed Loop method generalizes known methods for closed loop identification.

- Direct identification (see e.g. [21]) is obtained by the choice $F_1 = F_4 = 1$, and $F_2 = F_3 = 0$. This results in

$$x = u,\ z = y,\ R_o = G_o,\ K_o = H_o.$$

- Traditional indirect identification ([24]) is obtained when the controller is linear and the choice [2] $F_1 = C_l^{-1}$, $F_2 = F_4 = 1$, $F_3 = 0$ is made where C_l is the (assumed known and linear) true controller. This results in

$$x = r,\ z = y,\ R_o = \frac{G_o C_l}{1 + G_o C_l},\ K_o = \frac{1}{1 + G_o C_l} H_o.$$

- In the Dual Youla method ([16,22,27]) the plant model parametrization is based on an auxiliary model G_x of G_o with rational coprime factorization N_x/D_x that is stabilized by the present (assumed known and linear) controller C_l with rational coprime factorization N_c/D_c. This method is obtained by choosing

$$F_1 = D_c/M;\ F_2 = N_c/M;\ F_3 = -N_x/M;\ F_4 = D_x/M$$

  with $M = N_c N_x + D_c D_x$.
- The "whitening procedure" (see e.g. [21]) is obtained by the choice $F_1 = F_4 = F$ and $F_2 = F_3 = 0$. In this case we have

$$x = u,\ z = y,\ R_o = G_o,\ K_0 = F H_o.$$

  Note that if $F \approx H_o^{-1}$, then we might consider using a fixed filter $K = 1$ in the estimates.

The different methods and the respective choice for the filters $F_1 \cdots F_4$ are listed in Table 2. The identification objects and input and output signals are collected in Table 3. Note that the indirect and Dual-Youla methods require the controller C to be linear.

The following observations reflect some of the main properties associated with selecting the filters $F_1 \cdots F_4$ in our method:

[2] If C_l^{-1} is non-causal and the reference signal is available, then one can directly define $x_t = r_t$.

	F_1	F_2	F_3	F_4
Direct	1	0	0	1
Indirect	C_l^{-1}	1	0	1
Dual-Youla (DY)	D_c/M	N_c/M	$-N_x/M$	D_x/M

Table 2
Particular choice for the filters $F_1, \cdots, F_4$ lead to specific closed-loop identification methods. $M = N_c N_x + D_c D_x$.

- If the model sets are flexible enough to capture the real plant and noise dynamics of R_o and K_o respectively, then all methods provide consistent estimates of G_o and H_o.
- If the model sets for R_o are chosen flexible enough to represent the real plant dynamics, (and no statement is made with respect to model sets for K_o), then the plant estimates $\hat{G}$ will contain an asymptotic bias that is determined by Φ_{xw}. This bias is zero when $\Phi_{xw} = 0$.
- The input signal x_t for identification is uncorrelated with the noise w_t, i.e. $\Phi_{xw} = 0$, if the controller is linear and the virtual controller is chosen as $\overline{C} := F_2/F_1 = C_l$.
- If the auxiliary model F_3/F_4 is stabilized by the virtual controller $\overline{C}$, then any identified model $\hat{R}$ of R_o that is stable, will through (13) correspond to an equivalent plant model $\hat{G}$ that is stabilized by the virtual controller $\overline{C}$.
- The virtual closed-loop method incorporates a generalized Dual-Youla method, where the controller that is used for the plant parametrization is not necessarily chosen equal to the present (possibly nonlinear) controller C, but is a user-chosen linear approximation thereof in the form of the virtual controller $\overline{C}$.

In [7, lemma 3], it is established that traditional indirect identification can be thought as a direct identification method where the noise model is parametrized in terms of the open loop process G and the true controller, C. The analysis presented in [7, lemma 3] assumes that the true controller is linear and exactly known. We next, generalize this result for the case of the VCL method.

Lemma 2 *VCL identification is equivalent to direct identification with the following prediction model:*

$$y_t = G(q,\rho)u_t + \bar{H}(q,\rho,\eta)\epsilon_t \tag{16}$$

where the noise model is given by:

$$\bar{H}(q,\rho,\eta) = K(q,\eta)\frac{F_1 + F_2 G(q,\rho)}{F_1 F_4 - F_2 F_3} \tag{17}$$

Proof: *In the VCL method the prediction error is given by:*

$$\epsilon_t = \frac{1}{K}\left[z_t - \frac{F_3 + F_4 G}{F_1 + F_2 G} x_t\right] \tag{18}$$

Using the input-output relationship and re-arranging terms

	input x	output z	R_o	K_o
Direct	u	y	G_o	H_o
Indirect	$C_l^{-1}u + y$	y	$\dfrac{C_l G_o}{1 + C_l G_o}$	$\dfrac{H_o}{1 + C_l G_o}$
Dual-Youla (DY)	$\dfrac{D_x^{-1}}{1 + C_l G_x}(u + Cy)$	$\dfrac{D_c^{-1}}{1 + C_l G_x}(y - G_x u)$	$\dfrac{(G_o - G_x)D_x}{D_c(1 + C_l G_o)}$	$\dfrac{D_c^{-1} H_o}{1 + C_l G_o}$
VCL	$F_1 u + F_2 y$	$F_3 u + F_4 y$	$\dfrac{F_3 + F_4 G_o}{F_1 + F_2 G_o}$	$\dfrac{F_1 F_4 - F_2 F_3}{F_1 + F_2 G_o} H_o$

Table 3
Overview of input / output signals and of objects of identification for closed-loop identification methods.

we obtain:

$$\epsilon_t = \frac{1}{K} \frac{M}{F_1 + F_2 G} [y_t - G u_t] \quad (19)$$

$$M = F_1 F_4 - F_2 F_3 \quad (20)$$

The result follows since the same prediction error is obtained from direct identification with noise model $\bar{H}(q, \rho, \eta)$. □

The previous lemma shows that VCL is equivalent to shaping the noise model for the system to be identified. This lemma also shows that most indirect identification methods can be considered as a modified version of direct identification.

Notice that the set of poles of the extended noise model $\bar{H}$ contains the poles of G. This property also appears in ARMAX and ARX models and is actually the key enabling tool that allows one to identify unstable processes G_o.

Even though, VCL can be understood as a special case of direct identification, the key point is that a systematic procedure exists to modify the effective noise model in order to reduce the bias due to under-modeling and due to signal correlation arising from the closed loop nature of the data. Note that, in the usual direct identification for Box-Jenkins (BJ) models, the noise model and the plant are independently parametrized. This means, that it is not possible, in general, to identify unstable systems using a Box-Jenkins parametrization. By way of contrast, in VCL, even though we are using BJ models, the identification procedure when viewed in the direct identification setting is not a Box-Jenkins model, but has a very particular structure.

5 Signal spectra analysis for the VCL

5.1 Preliminary definitions and assumptions

Definition 3 *If the transfer function $X(z)$ is given by:*

$$X(z) = \ldots + x_{-1} z^1 + x_0 + x_1 z^{-1} + x_2 z^{-2} + \ldots \quad (21)$$

then the causal part is given by [23, page 197]:

$$[X(z)]_+ := x_0 + x_1 z^{-1} + x_2 z^{-2} + \ldots \quad (22)$$

and the anti-causal part by:

$$[X(z)]_- := X(z) - [X(z)]_+ = x_{-1} z^1 + x_{-2} z^2 + \ldots \quad (23)$$

▽▽▽

For the analysis presented in the sequel we use the following assumptions:

Assumption 4 *The input u_t is a $\sigma_{t-1}^{w,r}$-measurable function (i.e. depends only on the past values of w_t and r_t).* ▽▽▽

Assumption 5 *One of the following two conditions holds:*

- *w_t is a martingale difference sequence (MDS), i.e. $\mathrm{E}\left\{w_t | \sigma_{t-1}^w\right\} = 0$, with finite variance, $\mathrm{E}\left\{w_t^2 | \sigma_{t-1}^w\right\} = \sigma_w^2$.*
- *w_t is zero mean white noise with variance σ_w^2, and the input u_t is an affine function of the past values of w_t, i.e.:*

$$u_t = \sum_{k=1}^{\infty} f_k w_{t-k} + g_k \quad (24)$$

where f_k and g_k are functions of the past values of the reference signal, and do not depend on w_t or its past values. ▽▽▽

Notice that the required conditions on the noise sequence hold automatically when w_t is Gaussian distributed.

The previous assumptions are common in the analysis of stochastic linear systems [15]. In addition, it will be required that

Assumption 6 *The PEM estimates obtained by minimizing the cost function in* (12) *converge (see [20] for details).* ▽▽▽

Necessary conditions to satisfy assumption 6 for the case of non-linear controllers are (among others) that the noise sequence w_t be a MDS, with finite fourth order moment.

We further assume some extra conditions in order to have a well defined closed loop system.

Assumption 7 *One of the following two conditions holds:*

- *the true plant, G_o, and its model, G, are strictly causal,*
- *the true controller, C, and the virtual controller, $\overline{C}$ are strictly causal.* ▽▽▽

5.2 Signal spectra in the VCL framework

Spectral analysis is a common tool in system identification (see e.g. [24,21]). Here, unless otherwise stated we assume that the controller C is non-linear. Preliminary results covering the special case when C is linear are given in Appendix B.

Proposition 2 *(i) In the generalized setup the following properties of spectral densities hold:*

$$\Phi_{xw} = (F_1 + F_2 G_o)\Phi_{uw} + F_2 H_o \sigma_w^2 \tag{25}$$

$$= \frac{F_1}{\bar{S}_o}\Phi_{uw} + F_2 H_o \sigma_w^2 \tag{26}$$

$$\Phi_x = |F_1|^2 \begin{bmatrix} 1 + G_o\overline{C} & \overline{C}H_o \end{bmatrix} \Phi_{\mathcal{X}} \begin{bmatrix} 1 + G_o\overline{C} \\ \overline{C}H_o \end{bmatrix}^* \tag{27}$$

$$= |F_1|^2 \begin{bmatrix} \bar{S}_o^{-1} & \overline{C}H_o \end{bmatrix} \Phi_{\mathcal{X}} \begin{bmatrix} \bar{S}_o^{-1} \\ \overline{C}H_o \end{bmatrix}^* \tag{28}$$

$$= |F_1 + F_2 G_o|^2 \Phi_u + |F_2 H_o|^2 \sigma_w^2 + 2\mathrm{Re}\left\{(F_1 + F_2 G_o) H_o^* \Phi_{uw}\right\} \tag{29}$$

$$\Phi_z = \begin{bmatrix} F_3 + F_4 G_o & F_4 H_o \end{bmatrix} \Phi_{\mathcal{X}} \begin{bmatrix} F_3 + F_4 G_o & F_4 H_o \end{bmatrix}^* \tag{30}$$

where

$$\Phi_{\mathcal{X}} = \begin{bmatrix} \Phi_u & \Phi_{uw} \\ \Phi_{wu} & \sigma_w^2 \end{bmatrix} \tag{31}$$

and

$$\overline{C} = \frac{F_2}{F_1} \tag{32}$$

$$\bar{S}_o = \frac{1}{1 + G_o\overline{C}} \tag{33}$$

(ii) In the special case, when the controller $C = C_l$ is linear, the above expressions simplify to:

$$\Phi_{xw} = F_1(\overline{C} - C_l) H_o S_o \sigma_w^2 \tag{34}$$

$$\Phi_x = |F_1 C[1 + G_o S_o (C_l - \overline{C})]|^2 \Phi_r + |F_1 S_o (C_l - \overline{C}) H_o|^2 \sigma_w^2 \tag{35}$$

$$\Phi_z = |F_3 C_l S_o + F_4 T_o|^2 \Phi_r + |F_4 - F_3 C_l|^2 |S_o H_o|^2 \sigma_w^2 \tag{36}$$

where

$$S_o = \frac{1}{1 + G_o C_l} \tag{37}$$

$$T_o = 1 - S_o \tag{38}$$

Proof: *Immediate from the definitions of signals.* □

5.3 Cross-spectrum between x_t and w_t

A common problem in the identification of closed loop systems is that the input of the system to be identified is correlated with the noise [21]. We next analyze the impact that the choice of the different filters F_i has on the cross-spectrum Φ_{xw}.

The following result provides a condition such that the cross-spectrum Φ_{xw} is identically zero.

Lemma 8 *The cross-spectrum between x and w vanishes if*

$$F_1(1 + G_o\overline{C})\Phi_{uw} + F_2 H_o \sigma_w^2 = 0 \tag{39}$$

Moreover (39) *holds if the virtual controller is given by:*

$$\overline{C} = -\frac{\Phi_{uw}}{G_o \Phi_{uw} + H_o \sigma_w^2} \tag{40}$$

Proof: *Immediate from equations* (25) *and* (32). □

An implication of Lemma 8 is that it is possible to reduce the correlation between x_t and w_t by adjusting $\bar{C}$ irrespective of the linearity of the true controller.

We next specialize to the case when the true controller has a linear approximation which we denote C_l^a.

Corollary 9 *If the input of the real system is given by the following relationship:*

$$u_t = C_l^a(q)(r_t - y_t) + \xi_t \tag{41}$$

where $C_l^a(q)$ is a linear controller, and ξ_t captures the remaining terms due to non-linearities, then the condition (40) *can be re-written as:*

$$\overline{C} = \bar{\beta} C_l^a + (1 - \bar{\beta})(-G_o) \tag{42}$$

where

$$\bar{\beta} = \frac{1}{1 + \frac{G_o}{H_o}\frac{\Phi_{\xi w}}{\sigma_w^2}} \tag{43}$$

Proof: *Re-writing equation* (40)*, and using the relationship between input and output signals we obtain that the input signal is given by:*

$$u_t = C_l^a S_o^a r_t + S_o^a \xi_t - C_l H_o S_o^a w_t \tag{44}$$

where S_o^a is the sensitivity function calculated using C_l^a. Finally, using the expression for Φ_{xw} in Proposition 2, and re-arranging terms we obtain the result. □

From the previous corollary we see that, if the controller is slightly non-linear ($\Phi_{\xi w}$ small), then $\bar{\beta} \approx 1$, and thus one can reduce Φ_{xw} by choosing the virtual controller as a linear approximation of the true non-linear controller ($\bar{C} \approx C_l^a$).

6 Bias analysis for the VCL method

It is well known that the cost function in PEM is asymptotically (in the number of data points) given by [3] [21]:

$$\begin{aligned} V_N \to & \int \Phi_\epsilon && (45) \\ =& \sigma_w^2 + \int \frac{\Phi_u}{|H|^2} \left| (G_o - G) + (H_o - H)\frac{\Phi_{wu}}{\Phi_u} \right|^2 \\ &+ \int \frac{|H_o - H|^2}{|H|^2} \left(\sigma_w^2 - \frac{|\Phi_{wu}|^2}{\Phi_u} \right) && (46) \end{aligned}$$

The term $B_G = \frac{\Phi_{wu}}{\Phi_u}(H_o - H)$ is usually called the bias-pull for the estimates of G_o [7]. In order to obtain the asymptotic value for the estimate of G_o, it is typically assumed that the structure of G is sufficiently complex so that the cost function achieves its minimal value for every causal transfer function G. Then, by splitting the bias-pull in terms of its causal and anti-causal parts, we have that, for BJ models, the estimate of G_o tends to

$$\hat{G} \to G_o + \frac{H}{M_u}\left[(H_o - H)\frac{M_u}{H}\frac{\Phi_{wu}}{\Phi_u} \right]_+ \tag{47}$$

where M_u is stable, minimum-phase and is such that $\Phi_u = M_u M_u^*$. This factorization is usually called canonical factorization (see e.g. [17]).

The previous result has been shown in [7,21] for direct identification. We will see, in the sequel, that a similar result can also be obtained for the VCL.

[3] Here and in the sequel the symbol $\int$ denotes $\frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega$.

Lemma 10 *Under assumptions 4, 5, 6, and 7, the cost function given in* (14) *and* (12) *is asymptotically (in the number of data points) given by:*

$$\begin{aligned} \int \Phi_\epsilon =& \sigma_w^2 + \int \left|\frac{M}{K}\right|^2 \begin{bmatrix} X & 1 \end{bmatrix} \begin{bmatrix} \Phi_m & \Phi_{mn} \\ \Phi_{nm} & \Phi_n \end{bmatrix} \begin{bmatrix} X^* \\ 1 \end{bmatrix} && (48) \\ =& \sigma_w^2 + \int \left|\frac{M}{K}\right|^2 \Phi_m \left| X + \frac{\Phi_{nm}}{\Phi_m} \right|^2 \\ &+ \int \left|\frac{M}{K}\right|^2 \left[\Phi_n - \frac{|\Phi_{nm}|^2}{\Phi_m} \right] && (49) \end{aligned}$$

where

$$X = \frac{G_o - G}{F_1 + F_2 G} \tag{50}$$

$$m_t = u_t + \bar{C}\frac{H_o}{1 + \bar{C}G_o} w_t \tag{51}$$

$$n_t = \left[\frac{H_o}{F_1 + F_2 G_o} - \frac{K}{M} \right] w_t \tag{52}$$

Proof: *From lemma 2, we have that the prediction error in* (12) *is given by:*

$$\begin{aligned} \epsilon_t &= \bar{H}^{-1}[y_t - G u_t] && (53) \\ &= \bar{H}^{-1}[G_o u_t + H_o w_t - G u_t] && (54) \end{aligned}$$

Hence,

$$\begin{aligned} \epsilon_t &= \frac{M}{K}\left[\frac{\tilde{G}}{F_1 + F_2 G}u_t + \frac{H_o}{F_1 + F_2 G}w_t - \frac{K}{M}w_t \right] + w_t && (55) \\ &= \eta_t + w_t && (56) \end{aligned}$$

where

$$\begin{aligned} \eta_t &= \frac{K}{M}\left[X u_t + \left(\frac{H_o}{F_1 + F_2 G} - \frac{K}{M} \right) w_t \right] && (57) \\ &= \frac{K}{M}\left[X m_t + n_t \right] && (58) \end{aligned}$$

Using assumption 4 we have that η_t depends on past values of w_t. Then, using assumption 5 and lemma 17 (see Appendix), we have that η_t is uncorrelated with w_t, and

$$\int \Phi_\epsilon = \sigma_w^2 + \int \Phi_\eta \tag{59}$$

Finally, calculating the spectrum of η_t and completing squares we obtain (49). □

We then have the following result:

Theorem 11 *Using Virtual Closed Loop identification for a Box-Jenkins model (i.e. when $G(q,\rho)$ and $K(q,\eta)$ are independently parametrized), the asymptotic estimate, $\hat{G}$, of G_o satisfies:*

$$\hat{G} = G_o\bar{\lambda} - \bar{C}^{-1}(1-\bar{\lambda}) \tag{60}$$

where

$$\bar{\lambda} = \frac{1}{F_2\frac{K}{MM_m}\left[\frac{MM_m}{K}\frac{\Phi_{nm}}{\Phi_m}\right]_+ + 1} \tag{61}$$

and M_m is the canonical factor of Φ_m, i.e. $\Phi_m = M_m M_m^*$. Moreover, the bias-pull is given by:

$$B_G = (\bar{\lambda}-1)\left[G_o + \bar{C}^{-1}\right] \tag{62}$$

Proof: *Splitting the integrand in the cost function in terms of its causal and anti-causal parts we have that the part of the cost function that depends on X is given by:*

$$\sigma_w^2 + \int\left|\frac{MM_m}{K}X + \left[\frac{MM_m}{K}\frac{\Phi_{nm}}{\Phi_m}\right]_+\right|^2 \tag{63}$$

Then, we have that the causal solution of the optimization problem is given by:

$$X = \frac{K}{MM_m}\left[\frac{MM_m}{K}\frac{\Phi_{nm}}{\Phi_m}\right]_+ \tag{64}$$

Finally, using equation (50)*, and re-arranging terms we obtain the result.* □

The estimates of G_o depend on the cross-spectrum Φ_{nm}. This can be calculated as the conjugate of the following expression:

$$\Phi_{mn} = \left[\frac{K_o - K}{M}\right]^*\left[\Phi_{uw} + \left(\frac{F_2H_o}{F_1+F_2G_o}\right)\sigma_w^2\right] \tag{65}$$

$$= \left[\frac{H_o - H}{F_1+F_2G_o}\right]^*\left[\Phi_{uw} + \left(\frac{F_2H_o}{F_1+F_2G_o}\right)\sigma_w^2\right] \tag{66}$$

where H is defined as follows:

$$H := \frac{F_1+F_2G_o}{M}K \tag{67}$$

Note the similarities between the term on the right hand side of (65) and the cross-spectrum Φ_{xw} in (25).

Remark 12 *We see from Theorem 11 that, in the case of virtual closed loop identification, the estimate of G_o is biased towards the negative inverse of the virtual controller. This bias will be small provided we can make $\bar{\lambda}$ close to* 1*; i.e. make $\frac{\Phi_{nm}}{\Phi_m}$ small. It is not surprising that a sufficient condition to have $\Phi_{nm}\approx 0$ is the same we found in section 5.3 in order to reduce the correlation between x_t and w_t. Moreover, we can ensure that $\frac{\Phi_{nm}}{\Phi_m}$ is small (relative to* 1*) provided we choose $\bar{C}$ "close to" the true controller even if the latter is nonlinear and / or ill-defined.* ▽▽▽

Theorem 11 provides a basis for choosing suitable values for F_1, F_2, F_3, F_4. In particular, we see that the asymptotic bias is small under either of the following two conditions

- $H_o - H$ is small (i.e. $K_o - K$ is small),
- $\bar{C} - C_l^a$ is small

Note that this holds on a frequency by frequency basis, so it suffices for $\bar{C}$ to be near the linear approximation of true controller when $H_o - H$ is large or for $H_o - H$ to be small when $\bar{C}$ is a poor representation of the true controller.

Hence, it makes sense to choose F_1, F_2 such that $\bar{C} = F_2/F_1$ is close to the true controller. For example, if the true controller is a linear controller incorporating anti-windup protection for input saturation, then $\bar{C}$ could be chosen as the linear controller without anti-windup. Also note that the expressions for the bias-pull for the case when the "true" controller is linear can be easily obtained by using the results presented in Section 5.

From (6), it may be tempting to think that a good choice for F_3, F_4 would be such that $F_1F_4 = F_2F_3$ since this removes all noise from (6). However, in this case, $R_o = F_4F_3^{-1}$ i.e. we learn nothing about G_o. Thus, it is necessary to design the filters F_i such that M is different from zero in the frequency range of interest.

An alternative choice for F_3, F_4 would be to use a-priori estimates G_x, H_x for G_o, H_o to render $K_o \approx 1$. In this case, we might try using a fixed value for K (namely 1) in (12). The virtual closed loop scheme then reduces to an output error method linking the measured variable z_t to the model output $\hat{z}_t$. Of course, based on Theorem 11, bias may result if $\frac{F_1+F_2G_o}{F_1F_4-F_2F_3}$ is significantly different form H_o in frequency ranges where $\bar{C}$ is a poor approximation to the true controller.

7 Accuracy analysis for the VCL method

In this section, we analyze the impact of the choice of the filters F_i $(i = 1\cdots 4)$ on the variance of the estimates for G_o. We assume that there is no under-modelling, i.e. there exist $\theta = \theta_o = \left[\rho_o^T\ \eta_o^T\right]^T$ such that $R(\rho_o) = R_o$ and $K(\eta_o) = K_o$. This is a standard assumption to develop the accuracy analysis for the estimates.

Theorem 13 *The inverse of the covariance matrix of the vector of parameters $\hat{\theta}$ is given by:*

$$P_\theta^{-1} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \tag{68}$$

where [4]

$$A = \frac{N}{2\pi\sigma_w^2}\int_{-\pi}^{\pi}\frac{1}{|K_o|^2}\frac{\partial R}{\partial\rho}\frac{\partial R}{\partial\rho}^H\Phi_x \tag{69}$$

$$B = \frac{N}{2\pi\sigma_w^2}\int_{-\pi}^{\pi}\frac{1}{|K_o|^2}\frac{\partial R}{\partial\rho}\frac{\partial K}{\partial\eta}^H\Phi_{xw} \tag{70}$$

$$D = \frac{N}{2\pi\sigma_w^2}\int_{-\pi}^{\pi}\frac{1}{|K_o|^2}\frac{\partial K}{\partial\eta}\frac{\partial K}{\partial\eta}^H\sigma_w^2 \tag{71}$$

The inverse of the covariance of $\hat{\rho}$ is given by:

$$P_\rho^{-1} = A - BD^{-1}B^T \tag{72}$$

In addition, P_ρ^{-1} is bounded as follows:

- $P_\rho^{-1} \leq \frac{N}{2\pi\sigma_w^2}\int_{-\pi}^{\pi}\frac{1}{|K_o|^2}\frac{\partial R}{\partial\rho}\frac{\partial R}{\partial\rho}^H\Phi_x$. *Moreover, equality holds if and only if $B = 0$.*
- $P_\rho^{-1} \geq \frac{N}{2\pi\sigma_w^2}\int_{-\pi}^{\pi}\frac{1}{|K_o|^2}\frac{\partial R}{\partial\rho}\frac{\partial R}{\partial\rho}^H\left[\Phi_x - \frac{|\Phi_{xw}|^2}{\sigma_w^2}\right]$. *Moreover, equality holds if and only if there exists a non-frequency dependent matrix Γ such that $\Gamma\frac{\partial K}{\partial\eta} = \frac{\partial R}{\partial\rho}\Phi_{xw}$ (almost everywhere in ω), where the derivatives are evaluated at θ_o.*

Proof: *Essentially as in [3, lemma 1].* □

The previous theorem is general in the sense that it is valid for linear and non-linear controllers, and also valid for finite number of parameters (see [3] for details).

Lemma 14 *In the case that the true controller, C_l is linear and equal to the virtual controller, $\overline{C}$, then the covariance of the parameters of $\hat{G}$, obtained using PEM in the VCL framework, is given by:*

$$P_\rho^{-1}\{VCL\} = \frac{N}{2\pi\sigma_w^2}\int_{-\pi}^{\pi}\frac{1}{|H_o|^2}\frac{dG}{d\rho}\frac{dG}{d\rho}^H\Phi_u^r \tag{73}$$

with

$$\Phi_u^r = |C_l S_o|^2\Phi_r \tag{74}$$

Moreover, the covariance of the parameters of G, also satisfy the following inequality:

$$P_\rho^{-1}\{Direct\} \geq P_\rho^{-1}\{VCL\} \tag{75}$$

where $P_\rho^{-1}\{Direct\}$ is the covariance obtained when using direct identification. Moreover, equality holds in (75) if and only if there exists a non-frequency dependent matrix Γ such that $\Gamma\frac{\partial H}{\partial\eta} = \frac{\partial G}{\partial\rho}\Phi_{uw}$ (almost everywhere in ω), where the derivatives are evaluated at θ_o.

Proof: *The first part follows from theorem 13, the relationship of the signals in the virtual controller, and considering that, in the case that $\overline{C} = C_l$, the signal x_t and w_t are not correlated. The second part is obtained by using Theorem 13 for the case of direct identification ($F_1 = F_4 = 1$, and $F_3 = F_2 = 0$) and equation (73).* □

The previous lemma shows that most indirect identification methods provide estimates with the same covariance (provided that the true controller is linear). This lemma generalizes the results presented in [10] obtained by using the asymptotic in the number of parameters ($n \to \infty$) covariance formula (see [21]). In addition, it is shown that whenever there is no-under-modeling (i.e. no bias) then the best choice for the filters are, in general, given by direct identification (i.e. $F_1 = F_4 = 1$, and $F_2 = F_3 = 0$). Note that there exists a class of systems where direct identification and the VCL (with $\bar{C} = C$) provide estimates with the same accuracy (see [2] for an example).

Of course, in the case of under-modeling, it seems natural to choose a different value for the filters in order to deal with the usual bias-variance trade-off.

8 A numerical example

Consider a system described by:

$$y_t = a_1^o y_{t-1} + b_1^o u_{t-1} + \frac{1}{D_o(q^{-1})}w_t \tag{76}$$

where $a_1^o = 2$, $b_1^o = 1$, and $D_o(q^{-1})$ is a polynomial in the backward shift-operator, q^{-1}, given by

$$D_o(q^{-1}) = 1 + d_1^o q^{-1} + d_2^o q^{-2} \tag{77}$$

where $d_1^o = 1.956$, $d_2^o = 0.996$ and w_t is zero mean Gaussian white noise.

Since the system is unstable, we perform the identification in closed loop using the following control law:

$$u_t = \text{sat}\{r_t - 2y_t, \Delta\} \tag{78}$$

where the reference signal is zero mean Gaussian white noise with variance $\sigma_r^2 = 23$ and the function sat is given by:

$$\text{sat}\{x,\Delta\} = \begin{cases} -\Delta & x \leq -\Delta \\ x & -\Delta < x \leq \Delta \\ \Delta & x > \Delta \end{cases} \tag{79}$$

[4] Here and in the sequel x^H denotes the conjugate transpose of x.

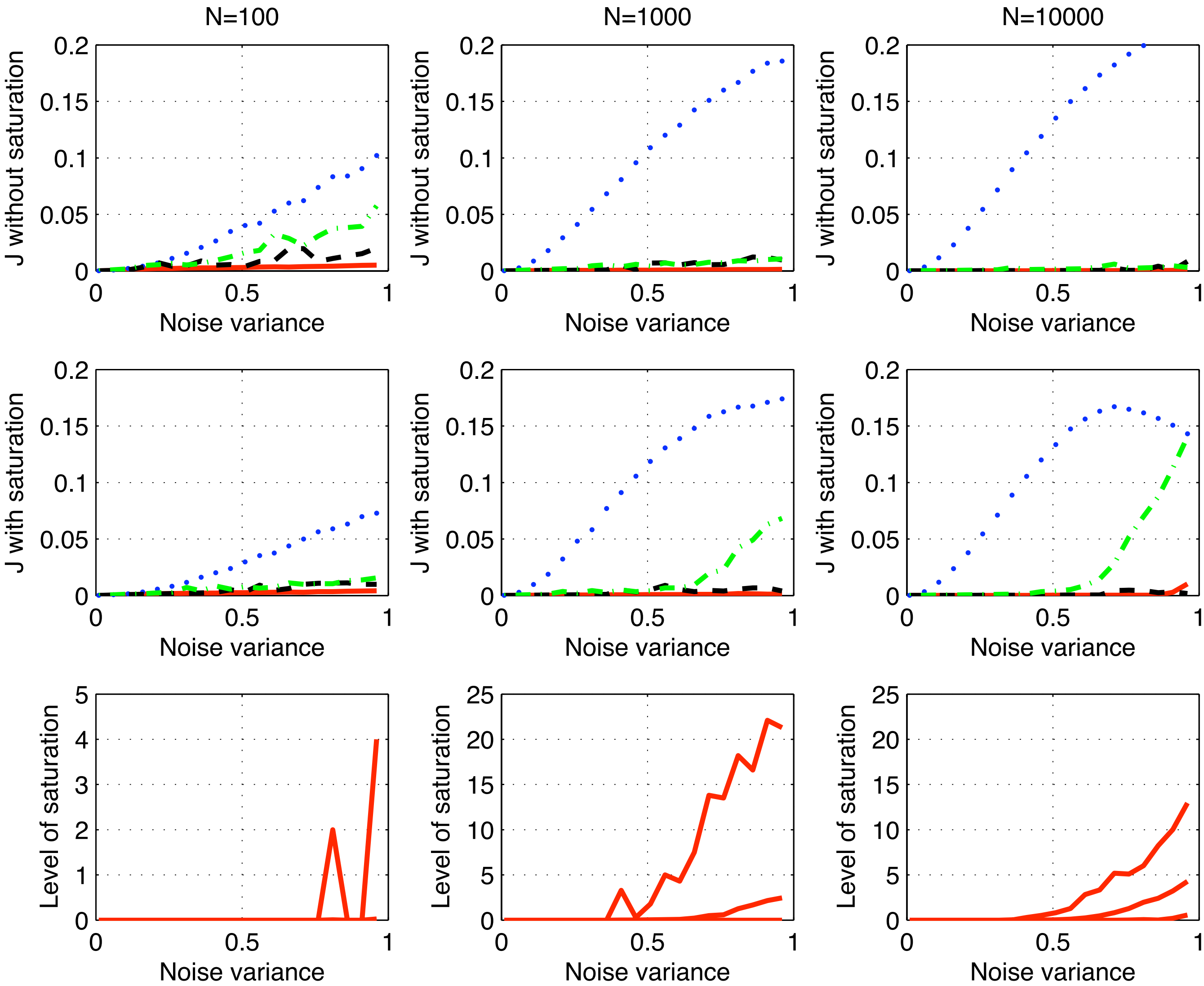

Fig. 4. First row: Plot of J with respect to noise variance σ_w^2 for 300 Monte-Carlo experiments that do no consider saturation of the controller. Second row: Plot of J with respect to noise variance σ_w^2 for 300 Monte-Carlo experiments considering a saturation level of $\Delta = 200$. Third row: Level of saturation (S, S_{max} and S_{min}). VCL-1 (dashed-green line) ,VCL-2 (dashed-black line), Direct-1 (dotted-blue line), Direct-2 (solid-red line).

Notice that this controller is the minimum variance controller for the case that there is no saturation, $D_o(q^{-1}) = 1$ and the reference is constant.

In section 7, we showed that, provided there is no undermodeling, direct identification provides the "best" estimates for the system parameters. Here, we compare the performance of direct identification with the VCL method using the following filters $F_1 = 1$, $F_2 = 2$, $F_3 = 0$, $F_4 = 1$.

For the case of direct identification we use the following model:

$$y_t = ay_{t-1} + bu_{t-1} + \frac{1}{D(q^{-1})}\varepsilon_t \tag{80}$$

where the polynomial $D(q^{-1})$ is given by

$$D(q^{-1}) = 1 + d_1q^{-1} + d_2q^{-2} + \ldots + d_{n_d}q^{-n_d} \tag{81}$$

For the case of the VCL method, we have the following model:

$$z_t = \frac{bq^{-1}}{1 + (2b - a)q^{-1}}x_t + K(q^{-1})\varepsilon_t \tag{82}$$

where b and a are the estimates of b_1^o and a_1^o respectively, and the transfer function $K(q^{-1})$ is given by

$$K(q^{-1}) = \frac{1}{1 + p_1 q^{-1} + p_2 q^{-2} + \ldots + p_{n_p} q^{-n_p}} \quad (83)$$

Notice that the true virtual closed loop transfer functions R_o and K_o are actually given by:

$$R_o(q^{-1}) = b_1^o q^{-1} \quad (84)$$

$$K_o(q^{-1}) = \frac{1}{D_o(q^{-1})} \quad (85)$$

We study the performance of the algorithms with and without noise under-modeling. For the VCL method the two cases correspond to the choice $n_p = 1$ and $n_p = 2$ respectively. We also denote the results **VCL-1** and **VCL-2** respectively. For the case of direct identification, the two cases correspond to the choice $n_d = 1$ and $n_d = 2$ respectively. We also denote the results **Direct-1** and **Direct-2** respectively.

We run 300 Monte-Carlo simulations for three different data length, $N = 100$, $N = 1000$ and $N = 10000$ and different values for the noise variance σ_w^2.

In order to compare the performance of the 4 strategies (VCL-1, VCL-2, Direct-1 and Direct-2), we calculate the average 2−norm of the models obtained, i.e.:

$$J = \frac{1}{Ns} \sum_{k=1}^{Ns} \int |\hat{G}_k - G_o|^2 \quad (86)$$

where N_s is the number of Monte-Carlo simulations, and $\hat{G}_k$ is the corresponding model obtained from the data in each experiment. We run simulations that consider saturation with $\Delta = 200$, and without saturation. For the case that the controller saturates, we calculate average, maximum and minimum percentage of saturation for each value of the noise variance:

$$S = 100 \frac{1}{Ns} \sum_{k=1}^{Ns} \frac{T_k}{N} \quad (87)$$

$$S_{max} = 100 \max_{k=1..Ns} \frac{T_k}{N} \quad (88)$$

$$S_{min} = 100 \min_{k=1..Ns} \frac{T_k}{N} \quad (89)$$

where T_k is the number of samples that the controller saturates for the experiment k and N is the data length.

We present the simulation results in figure 4.

When there is no under-modeling, we see that VCL performs similarly to **Direct-2** irrespective of saturation. However, when there is noise under-modeling, then **Direct-1** performs poorly wheter or not there is saturation. On the other hand, **VCL-1** performs well even in the presence of saturation although the performance degrades for large noise levels when saturation occurs more frequently.

9 Conclusions

In this paper we have presented a general method (the virtual closed loop method) to perform identification of systems operating in closed loop. The method is sufficiently general to be applied to different types of systems. We have presented a correlation analysis for the signals of interest, and also analyzed the asymptotic bias due to feedback and noise model mismatching and also the impact on the variance of $\hat{G}$ at different frequencies. We have shown that the new parametrization generalizes known methods for closed loop identification and also offers additional flexibility. A numerical example has confirmed the claimed merits of the approach.

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A Appendix

Lemma 15 *For any integer $\tau \in \mathbb{Z}$ we have that*

$$I = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jw\tau} d\omega = \begin{cases} 1 & if \tau = 0 \\ 0 & if \tau \neq 0 \end{cases} \quad \text{(A.1)}$$

Proof: *For $\tau = 0$ we have that:*

$$I = \frac{1}{2\pi}[\pi + \pi] = 1 \quad \text{(A.2)}$$

For $\tau \neq 0$, $\tau \in \mathbb{Z}$, we have that:

$$I = \frac{1}{j2\pi\tau} e^{j\omega\tau}|_{-\pi}^{\pi} = \frac{1}{j2\pi\tau}[cos(\tau\pi) - cos(\tau\pi)] = 0 \quad \text{(A.3)}$$

□

Lemma 16 *Let $\varepsilon_t = \eta_t + w_t$, then we have that:*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_\varepsilon = \sigma_w^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_\eta \quad \text{(A.4)}$$

$$\Leftrightarrow R_{\eta w}(0) = \bar{\mathrm{E}}\{\eta_t w_t\} = 0 \quad \text{(A.5)}$$

Proof: *The spectrum of η_t is given by:*

$$\Phi_\varepsilon = \Phi_\eta + \sigma_w^2 + \Phi_{\eta w} + \Phi_{w\eta} \quad \text{(A.6)}$$

considering that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{\eta w} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{\tau=-\infty}^{\infty} R_{\eta w}(\tau) e^{j\omega\tau} d\omega \quad \text{(A.7)}$$

$$= \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty} R_{\eta w}(\tau)\int_{-\pi}^{\pi} e^{j\omega\tau} d\omega \quad \text{(A.8)}$$

$$= R_{\eta w}(0) \quad \text{(A.9)}$$

The first equality follows by using the definition of the cross spectrum $\Phi_{\eta w}$. The second equality follows by interchanging the integral with the sum operators. The third equality follows by using lemma 15. Finally, we obtain the result by considering that

$$\Phi_{w\eta} = \Phi_{\eta w}^* \quad \text{(A.10)}$$

where $*$ denotes complex conjugate. □

Lemma 17 *If any of the following conditions holds:*

- *η_t is $\sigma_{t-1}^{w,r}$-measurable function and w_t is a martingale difference sequence (MDS),*
- *η_t is an affine function the past values of w_t, i.e.:*

$$\eta_t = \sum_{k=1}^{\infty} f_k w_{t-k} + g_k \quad \text{(A.11)}$$

where w_t is white noise and f_k and g_k are independent of w_t and its past vales.

then we have that

$$R_{\eta w}(0) = \bar{\mathrm{E}}\{\eta_t w_t\} = 0 \quad \text{(A.12)}$$

Moreover, if

$$\varepsilon_t = \eta_t + w_t \quad \text{(A.13)}$$

then

$$\int \Phi_\varepsilon = \sigma_w^2 + \int \Phi_\eta \quad \text{(A.14)}$$

Proof:

- *w_t is a martingale difference sequence i.e.*

$$\mathrm{E}\left\{w_t|\sigma^w_{t-1}\right\} = 0 \tag{A.15}$$

where σ^w_{t-1} is the sigma-algebra generated by the past values of w_t (i.e. w_{t-1}, w_{t-2}, …) and η_t is measurable random variable with respect to $\sigma^{w,r}_{t-1}$. In this case the integral of the cross spectrum between η_t and w_t is given by:

$$\begin{aligned}
\int \Phi_{\eta w} &= R_{\eta w}(0) && \text{(A.16)}\\
&= \bar{\mathrm{E}}\left\{\eta_t w_t\right\} && \text{(A.17)}\\
&= \lim_{N\to\infty}\sum_{t=1}^{N}\mathrm{E}\left\{\eta_t w_t\right\} && \text{(A.18)}\\
&= \bar{\mathrm{E}}\left\{\mathrm{E}\left\{\eta_t w_t|\sigma^{w,r}_{t-1}\right\}\right\} && \text{(A.19)}\\
&= \bar{\mathrm{E}}\left\{\eta_t \mathrm{E}\left\{w_t|\sigma^{w,r}_{t-1}\right\}\right\} && \text{(A.20)}\\
&= 0 && \text{(A.21)}
\end{aligned}$$

The first equality follows by using equations (A.7) *to* (A.9). *The second equality follows by using the definition of the cross correlation $R_{\eta w}$. The third equality follows by using the definition of the operator $\bar{\mathrm{E}}\{\}$. The fourth equality follows by using lemma 19. The fifth equality follows by considering that η_t is a $\sigma^{w,r}_{t-1}$-measurable function and using lemma 18. The sixth equality follows by considering that w_t is a martingale difference sequence.*

In addition, the integral of the spectrum of ε_t is given by

$$\int \Phi_\varepsilon = \sigma_w^2 + \int \Phi_\eta \tag{A.22}$$

- *Considering that the cross-correlation between η_t and w_t is given by:*

$$\begin{aligned}
\bar{\mathrm{E}}\left\{\eta_t w_t\right\} &= \sum_{k=1}^{\infty}\bar{\mathrm{E}}\left\{f_k w_{t-k} w_t + g_k w_t\right\} && \text{(A.23)}\\
&= \lim_{N\to\infty}\sum_{k=1}^{\infty}\sum_{t=1}^{N}\mathrm{E}\left\{f_k w_{t-k} w_t + g_k w_t\right\} && \text{(A.24)}\\
&= \lim_{N\to\infty}\sum_{k=1}^{\infty}\sum_{t=1}^{N}\mathrm{E}\left\{f_k\right\}\mathrm{E}\left\{w_{t-k} w_t\right\}\\
&\quad + \mathrm{E}\left\{g_k\right\}\mathrm{E}\left\{w_t\right\} && \text{(A.25)}
\end{aligned}$$

where we have used the definition of $\bar{\mathrm{E}}\{\}$, and that f_k and g_k are independent of w_t and its past values. Finally we obtain the result by considering that w_t is zero mean white noise.

□

Lemma 18 *If $\mathcal{W}_1$ is a sub-sigma algebra of $\mathcal{W}_2$ then,*

$$\mathrm{E}\left\{f|\mathcal{W}_1\right\} = \mathrm{E}\left\{\mathrm{E}\left\{f|\mathcal{W}_2\right\}|\mathcal{W}_1\right\} \tag{A.26}$$

Proof: *See [5, chapter 10] and [6, page 226].*

Lemma 19 *If η_t is $\sigma^{w,r}_{t-1}$-measurable and bounded, then*

$$\bar{\mathrm{E}}\left\{\eta w|\sigma^{w,r}_{t-1}\right\} = \bar{\mathrm{E}}\left\{\eta \mathrm{E}\left\{w|\sigma^{w,r}_{t-1}\right\}\right\} \tag{A.27}$$

Proof: *See [6].*

B Signals relationship in the linear case

In the case that the true controller is linear, the input and output signals are given by (see e.g. [13]):

$$y_t = T_o r_t + S_o H_o w_t \tag{B.1}$$
$$u_t = C S_o r_t - C S_o H_o w_t \tag{B.2}$$

Thus, the signal spectra are given by:

$$\begin{aligned}
\Phi_y &= |T_o|^2\Phi_r + |S_o H_o|^2\sigma_w^2 && \text{(B.3)}\\
\Phi_u &= |C S_o|^2\Phi_r + |C S_o H_o|^2\sigma_w^2 && \text{(B.4)}\\
\Phi_{uw} &= -C S_o H_o \sigma_w^2 && \text{(B.5)}\\
\Phi_{yw} &= S_o H_o \sigma_w^2 && \text{(B.6)}\\
\Phi_{yu} &= T_o (S_o C)^* \Phi_r - C^* |S_o H_o|^2 \sigma_w^2 && \text{(B.7)}
\end{aligned}$$