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A virtual closed loop method for closed loop identification[☆]

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ABSTRACT

Indirect methods for the identification of linear plant models on the basis of closed loop data are based on the use of (reconstructed) input signals that are uncorrelated with the noise. This generally requires exact (linear) controller knowledge. On the other hand, direct identification requires exact plant and noise modelling (system in the model set) in order to achieve accurate results, although the controller can be non-linear. In this paper, a generalized approach to closed loop identification is presented that includes both methods as special cases and which allows novel combined methods to be generated. Besides providing robustness with respect to inexact controller knowledge, the method does not rely on linearity of the controller nor on exact noise modelling. The generalization is obtained by balancing input-noise decorrelation against noise whitening in a user-chosen flexible fashion. To this end, a user-chosen virtual controller is used to parametrize the plant model, thereby generalizing the dual-Youla method to cases where knowledge of the controller is inexact. Asymptotic bias and variance results are presented for the method. Also, the benefits of the approach are demonstrated via simulation studies.

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1. Introduction

Identification of dynamic systems operating in the presence of feedback has received considerable attention in the system identification literature (see e.g. Forssell & Ljung, 1999; Goodwin & Payne, 1977; Ljung, 1999; Söderström & Stoica, 1989; Van den Hof, 1998). In many situations, there exist strong economic and/or safety reasons for requiring process data to be collected under closed loop conditions. Even open loop stable plants are often subject to non-stationary disturbances and long term drift that favour a closed loop experiment for data collection. Additionally, it has been found that, for several model applications (as e.g. model-based control design) and experimental constraints (e.g. output power constraints), a closed loop identification experiment is often the optimal experimental setup; see e.g. Agüero and Goodwin (2007), Gevers (2005), Hjalmarsson (2005), Hjalmarsson, Gevers, and De Bruyne (1996) and Van den Hof and Schrama (1995).

Unfortunately, handling closed loop data in identification leads to additional difficulties; see e.g. Forssell and Ljung (1999) and Van den Hof (1998).

In the prediction error framework (Ljung, 1999), the two principle methods for closed loop identification can be characterized as follows.

- Direct identification: here the plant is identified directly on the basis of plant input and output data taken from within the closed loop. The presence of feedback is ignored. Consistent model estimates can be identified under the condition that the noise dynamics are modelled exactly. Exact knowledge of the controller is not necessary and the controller may be non-linear.
- Indirect identification: here a plant object[1] is identified (usually the complementary sensitivity) between the reference input and plant output signals. Subsequently, an equivalent plant model is retrieved from the identified object. Consistent plant models can be identified under the condition that the controller is linear and exactly known. This holds for several variants of the indirect method, including the Dual-Youla approach (Hansen, Franklin, & Kosut, 1989; Schrama, 1991; Van den Hof, 1998), the method based on tailor-made parametrization (Van Donkelaar & Van den Hof, 2000), and bias-elimination least-squares method (BELS) (Zheng & Feng, 1995).

A third category, called joint input–output methods (see e.g. Söderström & Stoica, 1989) can be considered as an indirect method in the context of the current paper.

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[1] Here and in the following, we use the term "*plant object*" to describe a transfer function that depends on the (open loop) plant.

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In practice, one would ideally like to have identification tools that combine the advantageous properties of both direct and indirect methods. In particular, one would like to be able to handle the following situations.

- When a (slightly) non-linear controller is present in the loop, e.g. a linear controller that saturates regularly, or that is not exactly known.
- When the noise disturbances are non-stationary or cannot be modelled exactly by stationary white noise filtered through a linear time-invariant system.

In the above situations, neither direct nor indirect methods provide consistent model estimates and the number of alternatives is very limited. The Projection Method (Forssell & Ljung, 2000) was proposed to deal with non-linear controllers, by identifying non-causal FIR models to approximate non-linear sensitivity functions. Alternatively, Instrumental Variable methods (Gilson & Van den Hof, 2005; Söderström, Stoica, & Trulsson, 1987) can handle non-linear controllers and yield consistent plant models irrespective of noise under-modelling.

In this paper we develop a novel approach that takes a more generalized perspective. Realizing that all indirect methods use controller knowledge to exactly decorrelate the identification input signal from the noise, we will focus on this decorrelation and develop a generalized approach that is robust against controller non-linearity and inexact controller knowledge.

A central issue in our development is the choice of a virtual controller (Agüero, 2005; Agüero & Goodwin, 2004; Goodwin et al., 2008) which approximates the real (possibly non-linear) controller. This virtual controller will be deployed only for input signal construction to be used in identification. The more accurate the virtual controller, the less noise correlation will be present in the identification input. As a result, the bias due to having an inexact noise model will be reduced. In this way our method generalizes both the direct and indirect method of closed loop identification. Our approach leads to a sliding mechanism between these two extremes. The user can make an appropriate choice depending on his/her faith in either the quality of the noise model, or the available knowledge and linearity of the controller.

The current paper completes and generalizes the analysis originally presented in Agüero, Goodwin, and Van den Hof (2008), see also Agüero and Goodwin (2004) and Goodwin et al. (2008).

The remainder of the paper is organized as follows: In Section 2 we introduce closed loop identification in a general non-linear setting. In Section 3 we describe the new approach for identification of closed loop systems. In Section 4 we show how the virtual closed loop (VCL) method generalizes known schemes for closed loop identification. In Section 5 we analyse the spectra of signals appearing in the VCL method. In Section 6 we show how the choice of parameters in the virtual closed loop method affects the asymptotic bias and the estimation accuracy of the identification for systems operating in closed loop. We also analyse the accuracy of the estimates provided by VCL. In Section 7 we provide general guidelines to design filters necessary to implement the identification procedure using VCL. In Section 8 we illustrate how to use VCL to identify a simple system. Finally in Section 9 we draw conclusions.

2. Closed loop identification setup

We consider a data generating system $\mathcal{S}$:

$$\begin{aligned} y_t &= G_o(q)u_t + v_t \\ v_t &= H_o(q)w_t \end{aligned} \tag{1}$$

where q is the forward-shift operator, y_t, u_t, and w_t are the output, input and noise respectively, $G_o(q)$, and $H_o(q)$ are linear transfer functions, with H_o stable, stably invertible and monic

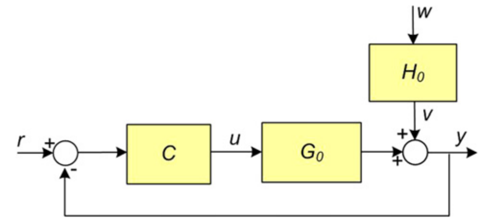

Fig. 1. Closed loop system configuration.

Table 1
Required conditions for direct and indirect identification to provide consistent plant model estimates.

	Direct	Indirect
Exact LTI noise model	Yes[a]	No
C linear	No	Yes
C exactly known	No	Yes

[a] Notice that for estimation techniques such as instrumental variables, it is possible to identify the system G_o using direct identification without modelling the noise transfer function.

(i.e. $\lim_{|z|\to\infty} H_o(z) = 1$). The noise w_t is assumed to be zero mean Gaussian white noise with variance σ_w^2. The system is assumed to operate in a stabilized closed loop (see Fig. 1). In the case that the controller is linear the input signal satisfies:

$$u_t = C(r_t - y_t), \tag{2}$$

for non-linear controller the relationship in (2) has to be understood as a non-linear dynamic map between the tracking error $r_t - y_t$ and the input signal u_t.

Here, r_t is an external reference signal. Throughout the paper, we will not restrict the controller, C, to be linear. However, at times, it will be convenient to consider the linear case so that we can relate our work to earlier literature. In these cases, we will use the notation C_l to denote C. In addition, we will, at other times, wish to consider a linear controller which is "close" (in some sense) to a non-linear controller. In this case, we will use the notation C_l^a.

In order to have a well-defined closed loop, it is further assumed that either C or G_o contains, at least, a one step time delay.

Spectral densities of signals are denoted by

$$\Phi_{uw}(\omega) = \sum_{\tau=-\infty}^{\infty} R_{uw}(\tau)\mathrm{e}^{-j\omega\tau}$$

with[2] $R_{uw}(\tau) := \bar{\mathrm{E}}\{u_t w_{t-\tau}\}$ and $\Phi_u = \Phi_{uu}$.

In order to be able to define signal spectra and cross-spectra we assume that the following assumption holds (Forssell & Ljung, 1999; Ljung, 1999):

Assumption 1. The signals w_t, r_t, u_t, y_t in the closed loop system defined by (1)–(2) are jointly quasi-stationary. □

Our goal is the identification of a (consistent) plant model for G_o on the basis of closed loop data. There are key differences between direct and indirect methods of identification. The different conditions for arriving at consistent estimates of G_o are listed in Table 1.

The clear distinction between the two situations raises the question as to whether one can combine the two approaches in a generalized method that is robust with respect to all three conditions; i.e. a method that is robust with respect to slight deviations from the assumptions of having exact controller knowledge, controller linearity, or exact LTI noise models. Such a method is developed in the next section.

[2] Here and in what follows, we use the operator $\bar{\mathrm{E}}\{(\cdot)\} = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathrm{E}\{(\cdot)\}$ where $\mathrm{E}\{(\cdot)\}$ is the expected value operator.

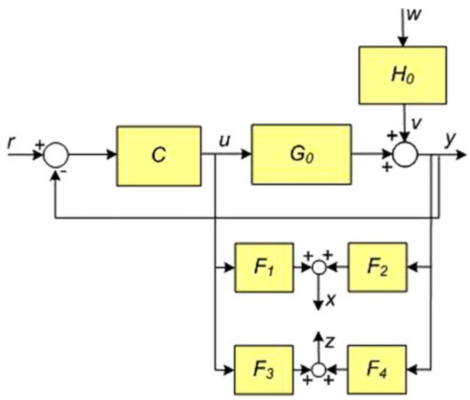

Fig. 2. Generalized scheme for closed loop identification from input x_t to output z_t.

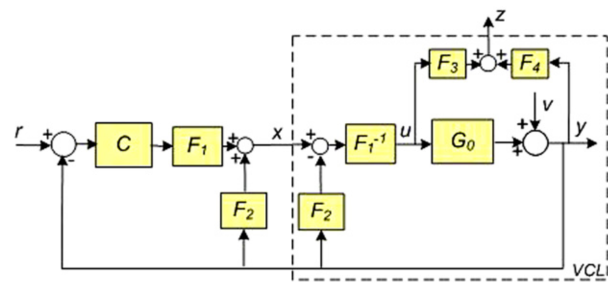

Fig. 3. Virtual closed loop.

3. A generalized approach to closed loop identification

3.1. Introduction

In our generalized approach we consider the setup of Fig. 2, where linear, causal and stable filters $F_1 \cdots F_4$ are introduced to generate signals

$$x_t = F_1(q)u_t + F_2(q)y_t \tag{3}$$

$$z_t = F_3(q)u_t + F_4(q)y_t. \tag{4}$$

The signals x_t and z_t will be used as input and output signals in a generalized identification scheme.

Notice that Assumption 1 and the relationships (3)–(4) imply that not only are all the signals in the closed loop (1)–(2) quasi-stationary but also the signals x_t and z_t.

The proposed identification approach amounts to identifying a plant-related object (parametrized by a vector θ that defines the model for the system) through a linear transfer function between the signals x_t and z_t, by applying a model structure:

$$\varepsilon_t(\theta) = K(q, \theta)^{-1}[z(t) - R(q, \theta)x(t)] \tag{5}$$

leading to estimates $\hat{R} = R(q, \hat{\theta}_N)$ and $\hat{K} = K(q, \hat{\theta}_N)$, and subsequently, to derive an equivalent plant model $\hat{G}$ from $\hat{R}$ by applying the principle of tailor-made parametrization. (More details will be provided in Section 3.4.)

It is apparent that the identification results will depend on the choice of the filters $F_1 \cdots F_4$. For example, by appropriately choosing F_1 and F_2 one can tune the presence of noise w_t in the generalized input signal x_t, and thereby one can influence the resulting bias.

3.2. The virtual closed loop

Under, the additional condition that F_1 is stably invertible (note that this is not required in the method), the closed loop diagram of Fig. 2 can be redrawn as shown in Fig. 3. This alternative view shows how the filters F_1 and F_2 are used to construct the generalized reference input x_t via $x_t = F_1u_t + F_2y_t$ (where F_2/F_1 acts as a virtual controller that compensates for the original controller in the construction of x_t). Additionally, the resulting transfer function between x_t and z_t contains a (virtual) closed loop plant object, where again the same controller F_2/F_1 is involved. Since the (linear) filter F_2/F_1 can be chosen freely by the user, it does not have any direct relation to the implemented controller C. Therefore the scheme is referred to by the name "Virtual Closed Loop". We also define the *virtual controller* as $\bar{C}(q) = F_2(q)/F_1(q)$. Note also that stability is not an issue since we already know that all signals are bounded (in a suitable statistical sense). Moreover, in the case that the "true" controller is linear and equal to the virtual controller ($C_l = \bar{C} = F_2/F_1$) then $x_t = F_2r_t$ is uncorrelated with the noise w_t. In the remainder of the paper, we will explore the implications of the configuration of Fig. 2 in system identification.

3.3. System equations and filter conditions

In order to analyse the setup, we use Eqs. (1), (3) and (4) to obtain the following set of equations describing the virtual closed loop system:

$$\begin{bmatrix} 1 & -F_3 & -F_4 \\ 0 & F_1 & F_2 \\ 0 & -G_o & 1 \end{bmatrix} \begin{bmatrix} z_t \\ u_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 \\ x_t \\ v_t \end{bmatrix}. \tag{6}$$

Solving for z, u and y we have the following:

$$z_t = \frac{F_3 + F_4G_o}{F_1 + F_2G_o}x_t + \frac{F_1F_4 - F_2F_3}{F_1 + F_2G_o}H_ow_t \tag{7}$$

$$u_t = \frac{1}{F_1 + F_2G_o}x_t - \frac{F_2}{F_1 + F_2G_o}H_ow_t \tag{8}$$

$$y_t = \frac{G_o}{F_1 + F_2G_o}x_t + \frac{F_1}{F_1 + F_2G_o}H_ow_t. \tag{9}$$

We then write

$$z_t = R_ox_t + K_ow_t \tag{10}$$

with

$$R_o = \frac{F_3 + F_4G_o}{F_1 + F_2G_o}, \qquad K_o = \frac{F_1F_4 - F_2F_3}{F_1 + F_2G_o}H_o. \tag{11}$$

In order for R_o and K_o to satisfy the usual conditions for applying prediction error identification methods the filters $F_1 \cdots F_4$ have to satisfy certain regularity conditions.

Assumption 2. We assume that the filters $\{F_i\}_{i=1,\ldots,4}$ satisfy the following:

(1) F_1 is biproper;
(2) F_2G_o is strictly proper;
(3) G_o is stabilized by the controller F_2/F_1;
(4) $M := F_1F_4 - F_2F_3$ is stably invertible. □

Proposition 3. *The transfer functions R_o and K_o are causal and stable under the conditions in Assumption 2.*

Proof. Conditions (1) and (2) together with the causality of all filters $F_1, \ldots, F_4$ guarantee that R_o and K_o are causal (i.e. $\lim_{z\to\infty} R_o(z)$, and $\lim_{z\to\infty} K_o(z)$ are constants).

In order to show stability of R_o and K_o, we express the filters $\{F_i\}_{i=1,\ldots,4}$ and the plant G_o as a coprime polynomial factorizations: $F_i = N_iD_i^{-1}$, $i = 1, \ldots, 4$, and $G_o = B_o/A_o$. Then

$$R_o = \frac{D_1D_2}{D_3D_4}\frac{(N_3D_4A_o + N_4D_3B_o)}{(N_1D_2A_o + N_2D_1B_o)}. \tag{12}$$

Stability of R_o and K_o follows if $N_1D_2A_o+N_2D_1B_o$ has all zeros within the unit circle (see e.g. Goodwin, Graebe, & Salgado, 2001, page 127). This is guaranteed when G_o is stabilized by the controller $N_2D_1/N_1D_2 = F_2/F_1$, see condition (3). □

Remark 4. Notice that since the noise w_t is Gaussian distributed, then without loss of generality we assume that K_o is minimum-phase. In addition, in the estimation procedure we assume that K is also minimum-phase. □

3.4. Identification setup

The virtual closed loop system will be identified by applying a direct identification method to the virtual closed loop system (10). On the basis of the generalized input signal x_t, and the generalized output signal z_t, we identify a model in a Box–Jenkins type model structure, with a prediction error

$$\varepsilon_t = K(q,\eta)^{-1}[z_t - R(q,\rho)x_t] \tag{13}$$

where

$$R(q,\rho) = \frac{F_3 + F_4G(q,\rho)}{F_1 + F_2G(q,\rho)} \tag{14}$$

represents the family of models used to estimate R_o and $K(q,\eta)$ represents the family of models used to estimate K_o.

The parametrization of $R(q,\rho)$ is a tailor-made parametrization in which the parameters of the plant model G are used to parametrize the virtual closed loop plant $R(q,\rho)$ (Forssell & Ljung, 1999; Van Donkelaar & Van den Hof, 2000). In the parametrization (13), the plant and noise models are parametrized independently. The parameters ρ, and η are estimated by minimizing a quadratic criterion:

$$V_N = \frac{1}{N}\sum_{t=1}^{N}\varepsilon_t^2. \tag{15}$$

The following condition on the filters $\{F_i\}_{i=1,\dots,4}$, the true system G_o, and the class of models G is assumed to hold:

$$\lim_{|z|\to\infty}\frac{(F_1F_4 - F_2F_3)}{(F_1 + F_2G_o)} = \lim_{|z|\to\infty}\frac{(F_1F_4 - F_2F_3)}{(F_1 + F_2G)} = 1.$$

This condition implies that K_o and K are monic.

Notice that once estimates $\hat{G}$, $\hat{K}$ for G_o and K_o have been obtained, one can define an estimate for H_o as follows:

$$\hat{H} := \frac{F_1 + F_2\hat{G}}{F_1F_4 - F_2F_3}\hat{K}. \tag{16}$$

We make the following assumption:

Assumption 5. The vector of parameters $\theta = [\rho^T\eta^T]^T$, the input (x_t), noise (w_t) and reference (r_t) satisfy regularity conditions such that the solution, $\hat{\theta}_N$ of the optimization problem in (15) converges (a.s.) to θ_*. □

Assumption 5 is necessary for asymptotic statistical analysis to hold (see Ljung, 1978; White, 1996, for details). Sufficient conditions on the true system, signals and the parametrized family of models such that Assumption 5 holds have to be obtained for every particular case. Moreover, the asymptotic statistical analysis presented in Ljung (1978) also holds when the noise w_t is a i.i.d. sequence not necessarily Gaussian distributed.

In the subsequent analysis we will analyse the impact of the following two issues:

(1) x_t is not, in general, an exogenous signal but is potentially correlated with the noise w_t.
(2) The class of models used for $K(q,\eta)$ may not include the true noise model K_o e.g. we might decide to use a fixed noise model $K \neq K_o$.

Table 2
Particular choice for the filters $F_1,\dots,F_4$ leads to specific closed loop identification methods. $M = N_cN_x + D_cD_x$.

	F_1	F_2	F_3	F_4
Direct	1	0	0	1
Indirect	C_l^{-1}	1	0	1
Dual-Youla (DY)	D_c/M	N_c/M	$-N_x/M$	D_x/M

4. Particular cases and general properties

We will first establish that the Virtual Closed Loop method generalizes known methods for closed loop identification.

- Direct identification (see e.g. Ljung, 1999) is obtained by the choice $F_1 = F_4 = 1$, and $F_2 = F_3 = 0$. This results in

  $x = u, \quad z = y, \quad R_o = G_o, \quad K_o = H_o.$
- Traditional indirect identification (Söderström & Stoica, 1989) is obtained when the controller is linear and the choice[3] $F_1 = C_l^{-1}$, $F_2 = F_4 = 1$, $F_3 = 0$ is made where C_l is the (assumed known and linear) true controller. This results in

  $$x = r, \quad z = y, \quad R_o = \frac{G_oC_l}{1+G_oC_l},$$
  $$K_o = \frac{1}{1+G_oC_l}H_o.$$

  Notice that it is necessary to incorporate an extra signal in the traditional indirect identification (the reference $x = r$). However, if the "true" controller is linear, then it is possible to reconstruct the reference from the input/output signals.
- In the Dual-Youla method (Hansen et al., 1989; Schrama, 1991; Van den Hof & Schrama, 1995) the plant model parametrization is based on an auxiliary model G_x of G_o with rational coprime factorization N_x/D_x that is stabilized by the present (assumed known and linear) controller C_l with rational coprime factorization N_c/D_c. This method is obtained by choosing

  $$F_1 = D_c/M; \quad F_2 = N_c/M; \quad F_3 = -N_x/M;$$
  $$F_4 = D_x/M$$

  with $M = N_cN_x + D_cD_x$.
- The "whitening procedure" (see e.g. Forssell, 1999; Ljung, 1999) is obtained by the choice $F_1 = F_4 = F$ and $F_2 = F_3 = 0$. In this case we have

  $x = Fu, \quad z = Fy, \quad R_o = G_o, \quad K_0 = FH_o.$

  Note that if $F \approx H_o^{-1}$, then we might consider using a fixed filter $K = 1$ in the estimates.

The different methods and the corresponding choices for the filters $F_1\cdots F_4$ are listed in Table 2. The identification objects and input and output signals are collected in Table 3. Note that the indirect and Dual-Youla methods require the controller C to be linear.

The following observations reflect some of the main properties associated with selecting the filters $F_1\cdots F_4$ in our method:

- If the model sets are flexible enough to capture the real plant and noise dynamics of R_o and K_o respectively, then all methods provide consistent estimates of G_o and H_o.
- If the model sets for R_o are chosen flexible enough to represent the real plant dynamics, (and no statement is made with respect to model sets for K_o), then the plant estimates $\hat{G}$ will contain an asymptotic bias that is determined by Φ_{xw}. This bias is zero when $\Phi_{xw} = 0$.

[3] If C_l^{-1} is non-causal and the reference signal is available, then one can directly define $x_t = r_t$.

Table 3
Overview of input/output signals and of objects of identification for closed loop identification methods.

	Input x	Output z	R_o	K_o
Direct	u	y	G_o	H_o
Indirect	$C_l^{-1}u+y$	y	$\frac{C_lG_o}{1+C_lG_o}$	$\frac{H_o}{1+C_lG_o}$
Dual-Youla (DY)	$\frac{D_x^{-1}}{1+C_lG_x}(u+Cy)$	$\frac{D_c^{-1}}{1+C_lG_x}(y-G_xu)$	$\frac{(G_o-G_x)D_x}{D_c(1+C_lG_o)}$	$\frac{D_c^{-1}H_o}{1+C_lG_o}$
VCL	F_1u+F_2y	F_3u+F_4y	$\frac{F_3+F_4G_o}{F_1+F_2G_o}$	$\frac{F_1F_4-F_2F_3}{F_1+F_2G_o}H_o$

- The input signal x_t for identification is uncorrelated with the noise w_t, i.e. $\Phi_{xw}=0$, if the controller is linear and the virtual controller is chosen as $\bar{C}:=F_2/F_1=C_l$.
- If the auxiliary model F_3/F_4 is stabilized by the virtual controller $\bar{C}$, then any identified model $\hat{R}$ of R_o that is stable will, through (14), correspond to an equivalent plant model $\hat{G}$ that is stabilized by the virtual controller $\bar{C}$.
- The virtual closed loop method incorporates a generalized Dual-Youla method, where the controller that is used for the plant parametrization is not necessarily chosen equal to the present (possibly non-linear) controller C, but is a user-chosen linear approximation thereof in the form of the virtual controller $\bar{C}$.

In Forssell and Ljung (1999, Lemma 3), it is established that traditional indirect identification can be thought as a direct identification method where the noise model is parametrized in terms of the open loop process G and the true controller, C. The analysis presented in Forssell and Ljung (1999, Lemma 3) assumes that the true controller is linear and exactly known. We next, generalize this result for the case of the VCL method.

Lemma 6. *VCL identification is equivalent to direct identification with the following prediction model:*

$$y_t=G(q,\rho)u_t+\bar{H}(q,\rho,\eta)\epsilon_t \tag{17}$$

where the noise model is given by:

$$\bar{H}(q,\rho,\eta)=K(q,\eta)\frac{F_1+F_2G(q,\rho)}{F_1F_4-F_2F_3}. \tag{18}$$

Proof. In the VCL method the prediction error is given by:

$$\epsilon_t=\frac{1}{K}\left[z_t-\frac{F_3+F_4G}{F_1+F_2G}x_t\right]. \tag{19}$$

Using the input–output relationship and re-arranging terms we obtain:

$$\epsilon_t=\frac{1}{K}\frac{M}{F_1+F_2G}[y_t-Gu_t] \tag{20}$$

$$M=F_1F_4-F_2F_3. \tag{21}$$

The result follows since the same prediction error is obtained from direct identification with noise model $\bar{H}(q,\rho,\eta)$. □

The previous lemma shows that VCL is equivalent to shaping the noise model for the system to be identified. This lemma also shows that most indirect identification methods can be considered as a modified version of direct identification.

Notice that the set of poles of the extended noise model $\bar{H}$ contains the poles of G. This property also appears in ARMAX and ARX models and is actually the key enabling tool that allows one to identify unstable processes G_o.

Even though, VCL can be understood as a special case of direct identification, the key point is that a systematic procedure exists to modify the effective noise model in order to reduce the bias due to under-modelling and due to signal correlation arising from the closed loop nature of the data. Note that, in the usual direct identification for Box–Jenkins (BJ) models, the noise model and the plant are independently parametrized. This means, that it is not possible, in general, to identify unstable systems using a Box–Jenkins parametrization. By way of contrast, in VCL, even though we are using BJ models, the identification procedure when viewed in the direct identification setting is not a Box–Jenkins model, but has a very particular structure.

5. Signal spectra analysis for the VCL

5.1. Preliminary definitions and assumptions

Definition 7 (*Söderström (2002, page 197)*)**.** If the transfer function $X(z)$ is given by:

$$X(z)=\cdots+x_{-1}z^1+x_0+x_1z^{-1}+x_2z^{-2}+\cdots \tag{22}$$

where $z\in\mathcal{A}\subset\mathbb{C}$, and $\mathcal{A}$ includes the unit circle, then the causal part is given by:

$$[X(z)]_+:=x_0+x_1z^{-1}+x_2z^{-2}+\cdots \tag{23}$$

and the anti-causal part by:

$$[X(z)]_-:=X(z)-[X(z)]_+=x_{-1}z^1+x_{-2}z^2+\cdots.\quad\Box \tag{24}$$

We assume some extra conditions in order to have a well-defined closed loop system.

Assumption 8. One of the following two conditions holds:

- the true plant, G_o, and its model, G, are strictly causal,
- the true controller, C, and the virtual controller, $\bar{C}$ are strictly causal. □

Note that Assumption 8 imposes a constraint in the class of models G and K.

5.2. Cross-spectrum between x_t and w_t

A common problem in the identification of closed loop systems is that the input to the system to be identified is correlated with the noise (Ljung, 1999). We next analyse the impact that the choice of the different filters $\{F_i\}_{i=1,\ldots,4}$ has on the cross-spectrum Φ_{xw} given by:

$$\Phi_{xw}=(F_1+F_2G_o)\Phi_{uw}+F_2H_o\sigma_w^2. \tag{25}$$

The following result provides a condition such that the cross-spectrum Φ_{xw} is identically zero.

Lemma 9. *The cross-spectrum between x and w vanishes if*

$$F_1(1+G_o\bar{C})\Phi_{uw}+F_2H_o\sigma_w^2=0. \tag{26}$$

Moreover (26) holds if the virtual controller is given by:

$$\bar{C}=-\frac{\Phi_{uw}}{G_o\Phi_{uw}+H_o\sigma_w^2}. \tag{27}$$

Proof. Immediate from Eqs. (25) and $\bar{C}=\frac{F_2}{F_1}$, $\bar{S}_o=\frac{1}{1+G_o\bar{C}}$. □

An implication of Lemma 9 is that it is possible to reduce the correlation between x_t and w_t by adjusting $\bar{C}$ irrespective of the linearity of the true controller.

We next specialize to the case when the true controller has a linear approximation which we denote C_l^a.

Corollary 10. *If the input of the real system is given by the following relationship:*

$$u_t = C_l^a(q)(r_t - y_t) + \xi_t \tag{28}$$

where $C_l^a(q)$ is a linear controller, and ξ_t is a quasi-stationary signal that might depend on r_t, y_t and their past values, then condition (27) can be re-written as:

$$\bar{C} = \bar{\beta} C_l^a + (1 - \bar{\beta})(-G_o^{-1}) \tag{29}$$

where

$$\bar{\beta} = \frac{1}{1 + \frac{G_o}{H_o}\frac{\Phi_{\xi w}}{\sigma_w^2}}. \tag{30}$$

Proof. Re-writing Eq. (27), and using the relationship between input and output signals we observe that the input signal is given by:

$$u_t = C_l^a S_o^a r_t + S_o^a \xi_t - C_l H_o S_o^a w_t \tag{31}$$

where S_o^a is the sensitivity function calculated using C_l^a. Finally, considering that $\Phi_{xw} = \frac{F_1}{\bar{S}_o}\Phi_{uw} + F_2 H_o \sigma_w^2$, and re-arranging terms we obtain the result. $\square$

Notice that, if the true controller is non-linear, then $C_l^a(q)$ in (28) represents a linear approximation of the true controller, and ξ_t captures the remaining terms due to non-linearities.

From the previous corollary we see that, if the controller is slightly non-linear ($\Phi_{\xi w}$ small), then $\bar{\beta} \approx 1$, and thus one can reduce Φ_{xw} by choosing the virtual controller as a linear approximation of the true non-linear controller ($\bar{C} \approx C_l^a$).

6. Bias and accuracy analysis for the VCL method

Bias and variance are fundamental concepts to assess the performance of estimators (Stuart, Ord, & Arnold, 1999). We next, analyse the bias and variance of the estimators in the VCL setup.

It is well known that the cost function in direct identification using PEM is asymptotically (in the number of data points) given by[4] Ljung (1999):

$$\begin{aligned} V_N &\to \frac{1}{2\pi}\int_{\pi}^{\pi} \Phi_\epsilon \\ &= \sigma_w^2 + \frac{1}{2\pi}\int_{\pi}^{\pi} \frac{\Phi_u}{|H|^2}\left|(G_o - G) + (H_o - H)\frac{\Phi_{wu}}{\Phi_u}\right|^2 \\ &\quad + \frac{1}{2\pi}\int_{\pi}^{\pi} \frac{|H_o - H|^2}{|H|^2}\left(\sigma_w^2 - \frac{|\Phi_{wu}|^2}{\Phi_u}\right). \end{aligned} \tag{32}$$

The expression in (32) is valid for different system parametrizations and holds irrespective of the conditions under which the real system is operating (i.e. open or closed loop). If the system is operating in open loop and one uses a Box–Jenkins models structure (i.e. $G(q,\theta) = G(q,\rho)$ and $H(q,\theta) = H(q,\eta)$, and $\theta = [\rho^T \eta^T]^T$), then we have that the corresponding optimization problem to determine ρ is (asymptotically in the number of data points) given by:

$$\hat{\rho} = \arg\min_{\rho} \frac{1}{2\pi}\int_{\pi}^{\pi} \frac{\Phi_u}{|H_*|^2}|G_o - G|^2 \tag{33}$$

where $H_*(q) = H(q,\eta_*)$ and η_* is the value for η that optimizes the asymptotic cost function (32). Thus, in this particular case, we have that, if the model class for G contains G_o, then one obtains a consistent estimate for G_o irrespective of possible under-modelling in the noise transfer function H_o (i.e. $H_*(q) \neq H_o(q)$). This analysis illustrates the benefits of using Box–Jenkins models for systems operating in open loop.

For systems operating in closed loop, the term $B_G = \frac{\Phi_{wu}}{\Phi_u}(H_o - H_*)$ is usually called the bias-pull for the estimates of G_o (Forssell & Ljung, 1999). We next present a definition of this term that will be useful for the analysis in the following.

Definition 11. Consider a given true system $G_o(q)$, and a given cost function $V_N(\theta)$ that converges to $V_\infty(\theta)$ as $N \to \infty$. The bias-pull is the bias that would occur in an estimated G-model if it were parametrized non-parametrically (i.e. allowing unlimited orders and non-causal dynamics) and independent of the noise model. $\square$

The importance of the bias-pull concept is that, for BJ models, it defines the limit point where an estimate of $G_o(q)$ obtained by solving an optimization problem converges, but limited to the freedom available in the parametric model.

In order to obtain the asymptotic value for the estimate of G_o, it is typically assumed that the structure of G is sufficiently complex so that the cost function achieves its minimal value for every causal transfer function G. Then, by splitting the bias-pull into its causal and anti-causal parts, we have that, for BJ models, the estimate of G_o tends to

$$\hat{G} \to G_o + \frac{H_*}{M_u}\left[(H_o - H_*)\frac{M_u}{H}\frac{\Phi_{wu}}{\Phi_u}\right]_+ \tag{34}$$

where M_u is stable, minimum-phase and is such that $\Phi_u = M_u M_u^*$. This factorization is usually called a canonical factorization (see e.g. Hassibi, Sayed, & Kailath, 1999).

The previous result has been shown in Forssell and Ljung (1999) and Ljung (1999) for direct identification. We will see, in what follows, that a similar result can also be obtained for the VCL method.

Lemma 12. *Under Assumptions 1, 5 and 8, the cost function given in (15) and (13) is asymptotically (in the number of data points) given by:*

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_\epsilon &= \sigma_w^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{M}{K}\right|^2 \Phi_m \left|X + \frac{\Phi_{nm}}{\Phi_m}\right|^2 \\ &\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{M}{K}\right|^2\left[\Phi_n - \frac{|\Phi_{nm}|^2}{\Phi_m}\right] \end{aligned} \tag{35}$$

where

$$X = \frac{G_o - G}{F_1 + F_2 G} \tag{36}$$

$$m_t = u_t + \bar{C}\frac{H_o}{1 + \bar{C}G_o} w_t \tag{37}$$

$$n_t = \left[\frac{H_o}{F_1 + F_2 G_o} - \frac{K}{M}\right] w_t. \tag{38}$$

Proof. From Lemma 6, we have that the prediction error in (13) is given by:

$$\begin{aligned} \epsilon_t &= \bar{H}^{-1}[y_t - G u_t] \qquad (39) \\ &= \bar{H}^{-1}[G_o u_t + H_o w_t - G u_t]. \qquad (40) \end{aligned}$$

Hence,

$$\begin{aligned} \epsilon_t &= \frac{M}{K}\left[\frac{G_o - G}{F_1 + F_2 G}u_t + \frac{H_o}{F_1 + F_2 G}w_t - \frac{K}{M}w_t\right] + w_t \qquad (41) \\ &= \eta_t + w_t \qquad (42) \end{aligned}$$

where

[4] Here and in what follows all integrals are with respect to the variable ω.

$$\eta_t = \frac{M}{K}\left[Xu_t + \left(\frac{H_o}{F_1+F_2G} - \frac{K}{M}\right)w_t\right] \tag{43}$$

$$= \frac{M}{K}[Xm_t + n_t]. \tag{44}$$

Using Assumption 8 we have that η_t depends on past values of w_t. Then, considering that w_t and η_t are independent, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_\epsilon = \sigma_w^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_\eta. \tag{45}$$

Finally, calculating the spectrum of η_t and completing squares we obtain (35). □

We then have the following result:

Theorem 13. *Using Virtual Closed Loop identification for a Box–Jenkins model (i.e. when $G(q,\rho)$ and $K(q,\eta)$ are independently parametrized), the bias-pull for the estimates of $G_o(q)$ is given by:*

$$B_G := \begin{cases}(\lambda-1)[G_o+\bar{C}^{-1}] & \text{if } \bar{C}\neq 0\\ F_1\dfrac{\Phi_{nm}}{\Phi_m} & \text{if } \bar{C}=0\end{cases} \tag{46}$$

where

$$\lambda = \frac{1}{1-F_2\frac{\Phi_{nm}}{\Phi_m}}. \tag{47}$$

Moreover, for a general family of models for[5] *G in the class of causal Box–Jenkins models, the asymptotic estimate, $\hat{G}$, of G_o tends to:*

$$\hat{G}\to\begin{cases}G_o\bar{\lambda}-\bar{C}^{-1}(1-\bar{\lambda}) & \text{if } \bar{C}\neq 0\\ G_o+F_1\dfrac{K_*}{MM_m}\left[\dfrac{MM_m}{K_*}\dfrac{\Phi_{nm}}{\Phi_m}\right]_+ & \text{if } \bar{C}=0\end{cases} \tag{48}$$

where

$$\bar{\lambda} = \frac{1}{1-F_2\frac{K_*}{MM_m}\left[\frac{MM_m}{K_*}\frac{\Phi_{nm}}{\Phi_m}\right]_+}. \tag{49}$$

$K_* = K(q,\eta_*)$, and M_m is the canonical factor of Φ_m, i.e. $\Phi_m = M_mM_m^*$.

Proof. Splitting the integrand in the cost function (35) in terms of its causal and anti-causal parts we have that the part of the cost function that depends on X is given by:

$$\sigma_w^2 + \frac{1}{2\pi}\int_{\pi}^{\pi}\left|\frac{MM_m}{K_*}X + \left[\frac{MM_m}{K_*}\frac{\Phi_{nm}}{\Phi_m}\right]_+\right|^2. \tag{50}$$

Then, we have that the causal solution of the optimization problem is given by:

$$X_* = -\frac{K_*}{MM_m}\left[\frac{MM_m}{K_*}\frac{\Phi_{nm}}{\Phi_m}\right]_+. \tag{51}$$

Finally, using Eq. (36), and re-arranging terms we obtain the result. □

Remark 14. Notice that the expression for the estimate $\hat{G}$ in Eq. (48) is similar to the one obtained in non-parametric identification (see e.g. Heath, 2001; Welsh & Goodwin, 2002). This is mainly due to the fact that, in non-parametric identification, the estimate of the transfer function has as many degrees of freedom as the one used in the definition of bias-pull in Definition 11. □

We next analyse the impact of the choice of the filters F_i ($i = 1\cdots 4$) on the variance of the estimates for G_o. We assume that there is no-under-modelling, i.e. there exist $\theta = \theta_o = \begin{bmatrix}\rho_o^T & \eta_o^T\end{bmatrix}^T$ such that $R(\rho_o) = R_o$ and $K(\eta_o) = K_o$. This is a standard assumption to develop the accuracy analysis for the estimates.

Remark 15. The inverse of the covariance matrix of the vector of parameters $\hat{\theta}$ is given by:

$$P_\theta^{-1} = \begin{bmatrix}A & B\\ B^T & D\end{bmatrix} \tag{52}$$

where[6]

$$A = \frac{N}{2\pi\sigma_w^2}\int_{-\pi}^{\pi}\frac{1}{|K_o|^2}\frac{\partial R}{\partial\rho}\frac{\partial R}{\partial\rho}^H\Phi_x \tag{53}$$

$$B = \frac{N}{2\pi\sigma_w^2}\int_{-\pi}^{\pi}\frac{1}{|K_o|^2}\frac{\partial R}{\partial\rho}\frac{\partial K}{\partial\eta}^H\Phi_{xw} \tag{54}$$

$$D = \frac{N}{2\pi\sigma_w^2}\int_{-\pi}^{\pi}\frac{1}{|K_o|^2}\frac{\partial K}{\partial\eta}\frac{\partial K}{\partial\eta}^H\sigma_w^2. \tag{55}$$

The inverse of the covariance of $\hat{\rho}$ is given by:

$$P_\rho^{-1} = A - BD^{-1}B^T. \tag{56}$$

In addition, P_ρ^{-1} is bounded as follows Agüero and Goodwin (2007):

- $P_\rho^{-1} \leq \frac{N}{2\pi\sigma_w^2}\int_{-\pi}^{\pi}\frac{1}{|K_o|^2}\frac{\partial R}{\partial\rho}\frac{\partial R}{\partial\rho}^H\Phi_x$. Moreover, equality holds if and only if $B=0$.
- $P_\rho^{-1} \geq \frac{N}{2\pi\sigma_w^2}\int_{-\pi}^{\pi}\frac{1}{|K_o|^2}\frac{\partial R}{\partial\rho}\frac{\partial R}{\partial\rho}^H\left[\Phi_x - \frac{|\Phi_{xw}|^2}{\sigma_w^2}\right]$. Moreover, equality holds if and only if there exists a non-frequency dependent matrix Γ such that $\Gamma\frac{\partial K}{\partial\eta} = \frac{\partial R}{\partial\rho}\Phi_{xw}$ (almost everywhere in ω), where the derivatives are evaluated at θ_o. □

The previous remark is valid for linear and non-linear controllers, and also valid for finite number of parameters (see Agüero & Goodwin, 2007, for details).

Claim 16. *In the case that the true controller, C_l is linear and equal to the virtual controller, $\bar{C}$, then the covariance of the parameters of $\hat{G}$, obtained using PEM in the VCL framework, is given by:*

$$P_\rho^{-1}\{\text{VCL}\} = \frac{N}{2\pi\sigma_w^2}\int_{-\pi}^{\pi}\frac{1}{|H_o|^2}\frac{\mathrm{d}G}{\mathrm{d}\rho}\frac{\mathrm{d}G}{\mathrm{d}\rho}^H\Phi_u^r \tag{57}$$

with

$$\Phi_u^r = |C_lS_o|^2\Phi_r. \tag{58}$$

Moreover, the covariance of the parameters of G, also satisfy the following inequality:

$$P_\rho^{-1}\{\text{Direct}\} \geq P_\rho^{-1}\{\text{VCL}\} \tag{59}$$

where $P_\rho\{\text{Direct}\}$ is the covariance obtained when using direct identification. Moreover, equality holds in (59) if and only if there exists a non-frequency dependent matrix Γ such that $\Gamma\frac{\partial H}{\partial\eta} = \frac{\partial G}{\partial\rho}\Phi_{uw}$ (almost everywhere in ω), where the derivatives are evaluated at θ_o.

Proof. The first part follows from Remark 15, the relationship of the signals in the virtual controller, and considering that, in the case that $\bar{C} = C_l$, the signal x_t and w_t are not correlated. The second part is obtained by using Remark 15 for the case of direct identification ($F_1 = F_4 = 1$, and $F_3 = F_2 = 0$) and Eq. (57). □

Remark 17. The previous lemma shows that most indirect identification methods provide estimates with the same covariance

[5] For "general family of models", we mean that there are no system order constraints.

[6] Here and in what follows x^H denotes the conjugate transpose of x.

(provided that the true controller is linear). This lemma generalizes the results presented in Gevers, Ljung, and Van den Hof (2001) obtained by using the asymptotic in the number of parameters ($n \to \infty$) covariance formula (see Ljung, 1999). □

7. Design of the filters

The analysis in the previous section provides expressions for bias and variance for estimates obtained by VCL for systems operating in closed loop using high order model.

Theorem 13 shows a frequency by frequency expression for the bias-pull. It also shows that the bias-pull can be reduced (or eliminated) if the cross-spectrum Φ_{nm} is reduced (or equal to zero). This cross-spectrum can be calculated as the conjugate of the following expression:

$$\Phi_{mn} = \left[\frac{K_o - K_*}{M}\right]^* \left[\Phi_{uw} + \left(\frac{F_2 H_o}{F_1 + F_2 G_o}\right)\sigma_w^2\right] \tag{60}$$

$$= \left[\frac{H_o - H_*}{F_1 + F_2 G_o}\right]^* \left[\Phi_{uw} + \left(\frac{F_2 H_o}{F_1 + F_2 G_o}\right)\sigma_w^2\right] \tag{61}$$

where H_* is defined as follows:

$$H_* := \frac{F_1 + F_2 G_o}{M} K_*. \tag{62}$$

Remark 18. Note the similarities between the term on the right-hand side of (60) and the cross-spectrum Φ_{xw}. (See (25).) □

The bias-pull can be shaped by reducing any of the two terms above in a particular frequency range of interest. If $\bar{C} = 0$, then the second term in (60) is equal to Φ_{uw} (which does not depend on the filters $\{F_i\}_{i=1,\dots,4}$). Thus, if one chooses $\bar{C} = 0$ then the only way to reduce the bias in the estimates of G_o is by choosing the filters such that $\frac{1}{F_1F_4}[H_oF_4 - K_*] \approx 0$ in the frequency range of interest. This condition is similar to the "whitening" procedure. It requires knowledge of the noise model H_o. In the particular case of direct identification ($F_1 = F_4 = 1$ and $F_2 = F_3 = \bar{C} = 0$) we have that the only way to have a small bias-pull is by having a good model for H_o.

Theorem 13 provides a basis for choosing suitable values for F_1, F_2, F_3, F_4 in order to reduce the bias due to under-modelling in the noise transfer function H_0 and to the presence of a non-linear controller. In particular, we see that the asymptotic bias is small under *either* of the following two conditions

- $H_o - H$ is small (i.e. $K_o - K$ is small),
- $\bar{C} - C_l^a$ is small.

Note that this holds on a frequency by frequency basis, so it suffices for $\bar{C}$ to be near the linear approximation of the true controller when $H_o - H$ is large or for $H_o - H$ to be small when $\bar{C}$ is a poor representation of the true controller.

On the other hand, the flexibility provided by the use of all the filters $\{F_i\}_{i=1,\dots,4}$ allows us to reduce the bias-pull by minimizing both terms in (60) in the frequency range of interest. We see from Theorem 13 that, for the VCL method, the estimate of G_o is biased towards the negative inverse of the virtual controller. This bias will be small provided we can make $\bar{\lambda}$ close to 1; i.e. make $\frac{\Phi_{nm}}{\Phi_m}$ small. It is not surprising that a sufficient condition to have $\Phi_{nm} \approx 0$ is the same condition that we found in Section 5.2 in order to reduce the correlation between x_t and w_t. Moreover, we can ensure that $\frac{\Phi_{nm}}{\Phi_m}$ is small (relative to 1) provided we choose $\bar{C}$ "close to" the true controller even if the latter is non-linear and / or ill defined. In addition, the first term in (60) can also be minimized provided a model for G_o and H_o is available. Hence, it makes sense to choose F_1, F_2 such that $\bar{C} = F_2/F_1$ is close to the true controller. For example, if the true controller is a linear controller incorporating anti-windup protection for input saturation, then $\bar{C}$ could be chosen as the linear controller without anti-windup. Also note that the expressions for the bias-pull for the case when the "true" controller is linear can be easily obtained by using the results presented in Section 5.

From (7), it may be tempting to think that a good choice for F_3, F_4 would be such that $F_1F_4 = F_2F_3$ since this removes all noise from (7). However, in this case, $R_o = F_4F_3^{-1}$ i.e. we learn nothing about G_o. Thus, it is necessary to design the filters $\{F_i\}_{i=1,\dots,4}$ such that M is different from zero in the frequency range of interest.

An alternative choice for F_3, F_4 would be to use a priori estimates G_x, H_x for G_o, H_o to render $K_o \approx 1$. In this case, we might try using a fixed value for K (namely 1) in (13). In this case, the virtual closed loop scheme reduces to an output error method linking the measured variable z_t to the model output $\hat{z}_t$. Of course, based on Theorem 13, bias may result if $\frac{F_1+F_2G_o}{F_1F_4-F_2F_3}$ is significantly different from H_o in frequency ranges where $\bar{C}$ is a poor approximation to the true controller.

On the other hand, it is well known that using filtered data improves the quality of estimates (Ljung, 1999, Chapter 7). In fact, in Ljung (2003) it is shown that for the identification of a particular continuous-time system it is necessary to filter the input–output signals in order to obtain "good" estimates. In addition, filtering the data is also used in "advanced" instrumental variables techniques such as refined instrumental variables (RIV) (Gilson, Garnier, Young, & Van den Hof, 2009).

The general advice is that one should remove the "bad" data by filtering i.e. one should choose a frequency range of interest where the assumptions made in our model (e.g. system structure) hold (see Agüero, Yuz, Goodwin, & Delgado, 2010; Goodwin et al., 2008; McKelvey, 2002; Pintelon & Schoukens, 2006; Yuz & Goodwin, 2008).

In addition, Claim 16 shows that whenever there is no-under-modelling (i.e. no bias) then the best choice for the filters are, in general, given by direct identification (i.e. $F_1 = F_4 = 1$, and $F_2 = F_3 = 0$). Note that there exists a class of systems where direct identification and the VCL (with $\bar{C} = C$) provide estimates with the same accuracy (see Agüero & Goodwin, 2006, for an example). Of course, in the case of under-modelling, it seems natural to choose different values for the filters in order to deal with the usual bias-variance trade-off.

If a "good" linear model for the controller is available, but no model for G_o and H_o is available, then our general advice to design the filters $\{F_i\}_{i=1,\dots,4}$ is as follows:

- Choose a filter $L(q)$ that selects the frequency range of interest (where one believes that all the assumptions made hold).
- Choose filter $F_1(q) = L(q), F_2(q) = L(q)\bar{C}(q)$, where $\bar{C}(q)$ is a good linear approximation of the true controller.
- Choose $F_3(q) = 0$, and $F_4(q) = L(q)$.

If good models for $G_o(q)$ and $H_o(q)$ are available (G_x and H_x), then choose $F_3(q)$ and $F_4(q)$ to render $K_o(q) \approx 1$ and such that they contain the filter $L(q)$. If a model for $G_o(q)$ and a "good" linear model for the controller are available, then choose the filters as in the Dual-Youla approach (see Table 2). Of course, the design of the filters $\{F_i\}_{i=1,\dots,4}$ depends on the particular problem of interest, and one might design the filters based on different criteria.

8. A numerical example

Consider a system described by:

$$y_t = G_o(q)u_t + v_t \tag{63}$$

$$v_t = H_o(q)w_t \tag{64}$$

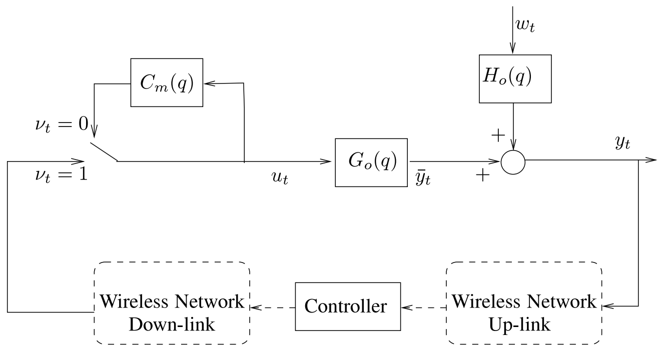

Fig. 4. System utilized in the numerical example.

where

$$G_o(q) = \frac{b_1^o q^{-1}}{1 + a_1^o q^{-1}} \tag{65}$$

$$H_o(q) = \frac{0.3814 + 0.3103q^{-1}}{1 - 0.8571q^{-1} + 0.5488q^{-2}} \tag{66}$$

and $a_1^o = -1.105$, $b_1^o = 0.3155$, and w_t is zero mean Gaussian white noise with variance σ_w^2.

Since the system is unstable, we perform the identification in closed loop using the following nominal control law:

$$u_t = C(q)[r_t - y_t] \tag{67}$$

where the reference signal is zero mean Gaussian white noise with variance $\sigma_r^2 = 1$ and $C(q) = \frac{0.3q^{-1}}{1-0.5q^{-1}}$. However, we will assume that $C(q)$ is implemented over a communication network. Our motivation for this choice is the observation that control over networks has become a topic of considerable research interest in the last few years (see e.g. Schenato, Sinopoli, Franceschetti, Poolla, & Sastry, 2007). In this area, the controller receives and sends signals through a network (see Fig. 4). This means that the controller and the process can be located at distant points. A common drawback of this approach is that some data in the up-link (from plant to controller) or in the down-link (from controller to plant) may be missing due to packet loss. In this case, it is necessary to have "smarter" controllers and actuators. We will capture the idea of data dropouts in our control law. In the case that a control signal u_t is missed, it is common to hold the previous value for the input signal. For illustrative purposes, we use instead an actuation given by the average of the previous 10 control actions. We assume that the missing data satisfies a Bernoulli distribution with probability of loss data P.

We test the PEM-direct, RIV and the VCL identification techniques for two scenarios:

- Nominal case: No-under-modelling in G_0, and there is no missing data ($\nu_t = 1, \forall t$).
- Non-nominal case: Under-modelling in G_0 i.e. the data is generated by a system given by $G_o(q)D(q)$ where $D(q) = \frac{0.5609+0.3038q^{-1}}{1-0.1353q^{-1}}$ and there is missing data $P = 0.5$.

In order to illustrate the impact of under-modelling of H_o we use the simplest noise model for the different techniques under study. For VCL we use an output error model (i.e. $K = 1$ for VCL), and for direct identification and RIV we use an ARX model (since OE models are not suitable for unstable processes). We use a model for G with the same structure as G_o (see Fig. 5).

For PEM-direct we use the following algorithm:

$$\hat{\rho} = \arg\min_{\rho} \sum_{t=1}^{N} (\varepsilon_t^F)^2 \tag{68}$$

where

$$\varepsilon_t^F = L(q)\varepsilon_t \tag{69}$$

$$\varepsilon_t = A(q, \rho)y_t - B(q, \rho)u_t \tag{70}$$

and $A(q, \rho) = 1 + a_1 q^{-1}$, $B(q, \rho) = b_1 q^{-1}$.

For the RIV method we use the following iterative procedure to estimate ρ_o (Gilson et al., 2009):

(1) Choose an initial value for $\hat{\rho}$.
(2) Calculate the transfer function $G(q, \hat{\rho})$.

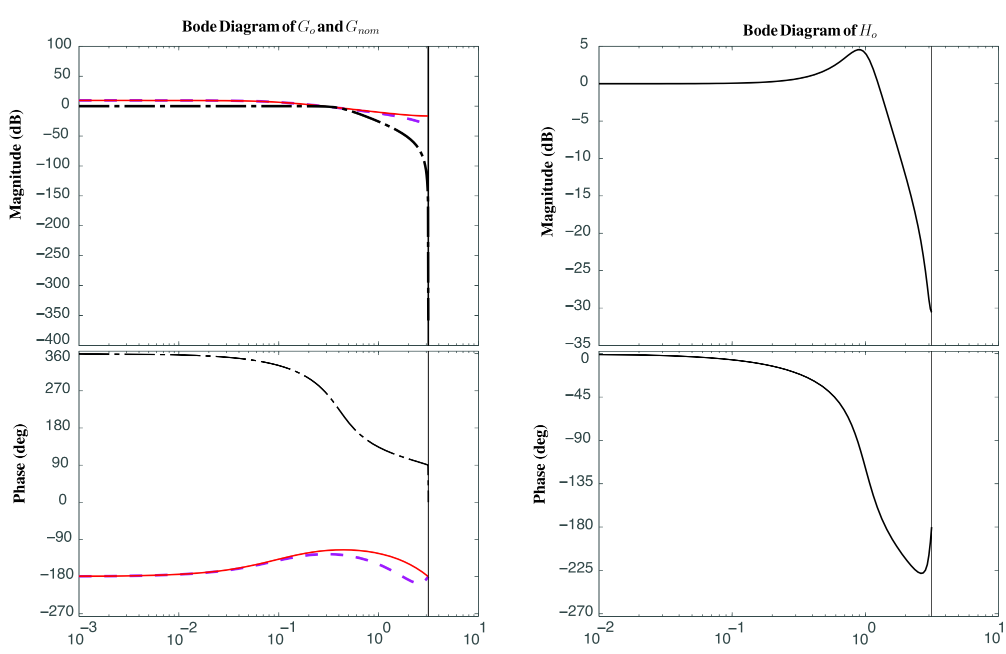

Fig. 5. Left side: Bode diagram of G_o (red-solid line) and G_oD (magenta-dashed line), and the filter $L(q)$ (black-dash–dotted line). Right side: Bode diagram of Ho. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

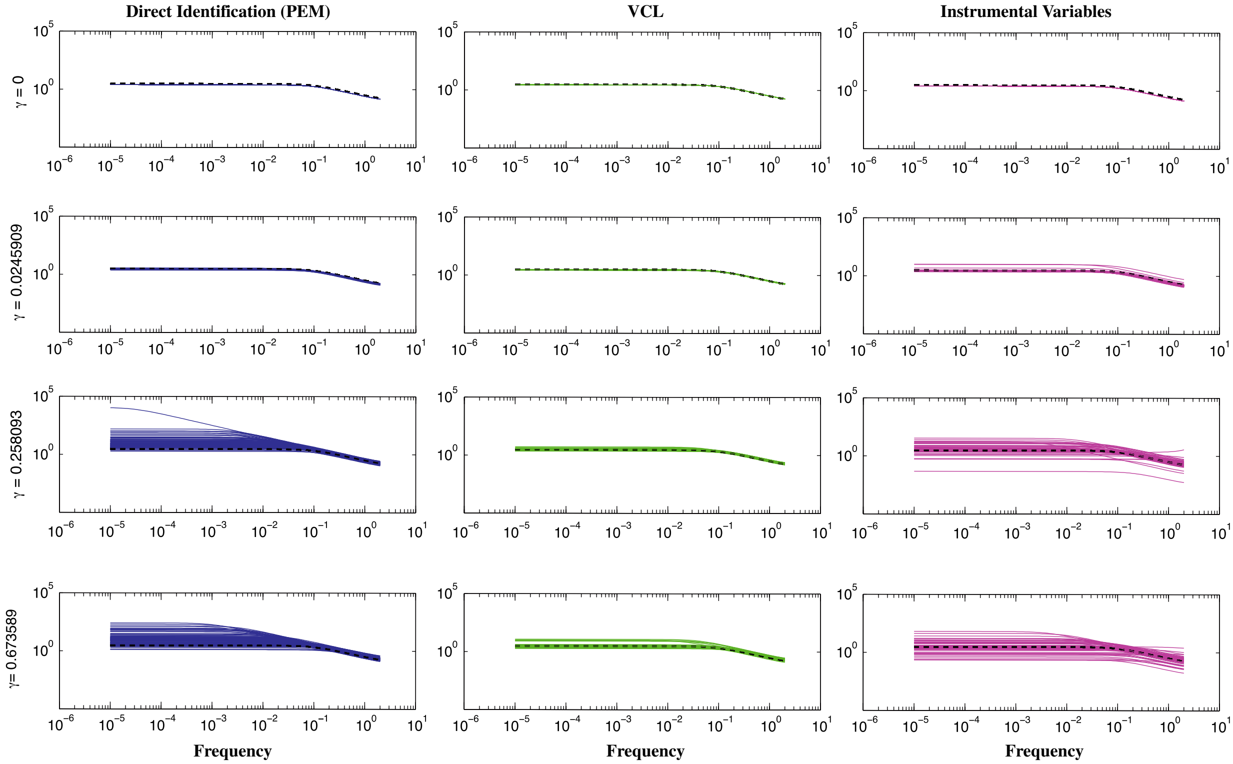

Fig. 6. Bode-magnitude plot of G_o (black-dashed line) and $\hat{G}$ for different Monte Carlo simulations for PEM-direct (left-hand plot, blue line), VCL (plot in the middle, green line), and RIV (right-hand plot, magenta) in non-nominal conditions using $N = 100$ data points. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

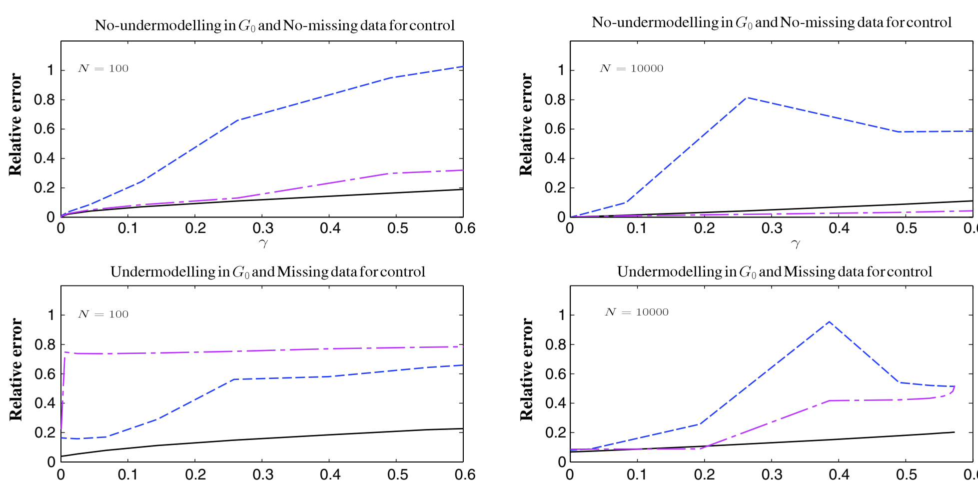

Fig. 7. Relative error R_e for different values of γ under nominal conditions ($N = 100$: top-left plot, $N = 10000$: top-right plot) and non-nominal conditions ($N = 100$: bottom-left plot, $N = 10000$: bottom-right plot) conditions. VCL (black-solid line), PEM-direct (blue-dashed line), RIV (magenta-dash–dotted line). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

(3) Calculate the transfer functions $T(q) = \frac{GC}{1+GC}$ and $S(q) = 1 - T(q)$.
(4) Calculate the signals $y_t^r = T(q)r_t$ and $u_t^r = S(q)r_t$.
(5) Calculate the vectors $\varphi_t = [L(q)u_{t-1} - L(q)y_{t-1}]^T$ and $z_t^k = [L(q)u_{t-1}^r - L(q)y_{t-1}^r]^T$.
(6) $\hat{\rho}_k = [\sum z_t^k \varphi_t^T]^{-1} \sum z_t^k L(q)y_t$.
(7) Set $\rho = \hat{\rho}_k$ go to step (2) until convergence.

For the VCL method we choose $F_1(q) = L(q)$, $F_2 = 0.6L(q)$, $F_3(q) = 0$, $F_4(q) = L(q)$. Thus, we have that $\bar{C} = 0.6$ which is the dc gain of the linear controller in the loop when there is no missing data.

For all of the techniques we choose the filter $L(q)$ as a Butterworth filter of third order and having a cut-off frequency $0.4/\pi\,[1/s]$. We analyse the performance of all algorithms for $N = 100$ and $N = 10000$ data points by using 100 Monte Carlo experiments for different values of γ given by

$$\gamma = \frac{\hat{\sigma}_v^2}{\hat{\sigma}_{\bar{y}}^2} \tag{71}$$

where $\hat{\sigma}_v^2$ and $\sigma_{\bar{y}}^2$ are estimates of the noise and noise-free-output ($\bar{y}_t = y_t - v_t$) variance obtained from the data.

Fig. 6 shows the Bode-magnitude diagram for the estimates of 100 Monte Carlo runs for different values of γ obtained in the non-nominal case using $N = 100$. We see that the VCL method provides the most accurate estimates.

We calculate the average 2-norm of the relative error for the models obtained given by:

$$R_e = \frac{1}{2\pi N_s} \sum_{k=1}^{N_s} \int_{-\pi}^{\pi} \left| \frac{\hat{G}_k - G_o}{G_o} \right|^2 \mathrm{d}\omega \quad (72)$$

where N_s is the number of Monte Carlo simulations, and $\hat{G}_k$ is the corresponding model obtained from the data in each experiment. This is plotted in Fig. 7 as a function of γ.

We see from Fig. 7 that PEM-direct method provides good estimates when the noise is small. However, the quality of the estimates deteriorates when the noise is large.

We see that, under nominal conditions and large data-length, the RIV technique provides good estimates. However, in the non-nominal case and for short data-length the estimates are not satisfactory.

On the other hand, the estimates obtained by VCL are good in both scenarios irrespective of data-length.

9. Conclusions

In this paper we have presented a general method (the Virtual Closed Loop method) to perform identification of systems operating in closed loop. The method is sufficiently general to be applied to different types of systems. We have presented a correlation analysis of the signals of interest, analysed asymptotic bias due to feedback and noise model mismatching, and also the impact on the variance of $\hat{G}$ at different frequencies. We have shown that the new parametrization generalizes known methods for closed loop identification and also offers additional flexibility. A numerical example has confirmed the claimed merits of the approach.

References

Agüero, J. C. (2005). System identification methodologies incorporating constraints. *Ph.D. thesis*. School of Electrical Engineering and Computer Science. The University of Newcastle. Australia.

Agüero, J. C., & Goodwin, G. C. (2004). Virtual closed loop identification: a subspace approach. In *Proceedings of the 43rd IEEE conference on decision and control. CDC. Atlantis, Paradise Island, Bahamas* (pp. 364–369).

Agüero, J. C., & Goodwin, G. C. (2006). On the optimality of open and closed loop experiments in system identification. In *45th IEEE conference on decision and control. CDC. San Diego, California*.

Agüero, J. C., & Goodwin, G. C. (2007). Choosing between open and closed loop experiments in linear system identification. *IEEE Transactions on Automatic Control*, *52*(8), 1475–1480.

Agüero, J. C., Goodwin, G. C., & Van den Hof, P. M. J. (2008). Virtual closed loop identification: a generalized tool for identification in closed loop. In *Proceedings of the 47th IEEE conference on decision and control. CDC*.

Agüero, J. C., Yuz, J. I., Goodwin, G. C., & Delgado, R. A. (2010). On the equivalence of time and frequency domain maximum likelihood estimation. *Automatica*, *46*(2), 260–270.

Forssell, U. (1999). Closed-loop identification. Methods, theory, and applications. *Ph.D. thesis*. Linköping University. Linköping.

Forssell, U., & Ljung, L. (1999). Closed loop identification revisited. *Automatica*, *35*, 1215–1241.

Forssell, U., & Ljung, L. (2000). A projection method for closed loop identification. *IEEE Transactions on Automatic Control*, *45*(11), 2101–2106.

Gevers, M. (2005). Identification for control: from the early achievements to the revival of experiment design. *European Journal of Control*, *11*(4–5), 335–352.

Gevers, M., Ljung, L., & Van den Hof, P. M. J. (2001). Asymptotic variance expressions for closed-loop identification. *Automatica*, *37*(5), 781–786.

Gilson, M., Garnier, H., Young, P., & Van den Hof, P. M. J. (2009). Refined instrumental variable method for closed-loop system identification. In *15th IFAC symposium on system identification. Saint Malo, France*.

Gilson, M., & Van den Hof, P. M. J. (2005). Instrumental variable methods for closed-loop system identification. *Automatica*, *41*(2), 241–249.

Goodwin, G. C., Agüero, J. C., Welsh, J. S., Yuz, J. I., Adams, G. J., & Rojas, C. R. (2008). Robust identification of process models from plant data. *Journal of Process Control*, *18*, 810–820.

Goodwin, G. C., Graebe, S. F., & Salgado, M. E. (2001). *Control system design*. Upper Saddle River, NJ: Prentice Hall.

Goodwin, G. C., & Payne, R. (1977). *Dynamic system identification: experiment design and data analysis*. Academic Press.

Hansen, F., Franklin, G., & Kosut, R. (1989). Closed loop identification via the fractional representation: experiment design. In *Proceedings of the American control conference* (pp. 1422–1427).

Hassibi, B., Sayed, A. H., & Kailath, T. (1999). *Indefinite quadratic estimation and control. A unified approach to H^2 and H^∞ theories*. SIAM.

Heath, W. P. (2001). Bias of indirect non-parametric transfer function estimates for plants in closed loop. *Automatica*, *37*(10), 1529–1540.

Hjalmarsson, H. (2005). From experiment design to closed-loop control. *Automatica*, *41*(3), 393–438.

Hjalmarsson, H., Gevers, M., & De Bruyne, F. (1996). For model-based control design, closed-loop identification gives better performance. *Automatica*, *32*(12), 1659–1673.

Ljung, L. (1978). Convergence analysis of parametric identification methods. *IEEE Transactions on Automatic Control*, *AC-23*, 770–783.

Ljung, L. (1999). *System identification: theory for the user* (2nd ed.) Prentice Hall.

Ljung, L. (2003). Aspects and experiences of user choices in subspace identification methods. In *13th IFAC symposium on system identification. Rotterdam, The Netherlands* (pp. 1802–1807).

McKelvey, T. (2002). Frequency domain identification methods. *Circuits, Systems and Signal Processing*, *21*(1), 39–55.

Pintelon, R., & Schoukens, J. (2006). Box-jenkins identification revisited—part I: theory. *Automatica*, *42*(1), 63–75.

Schenato, L., Sinopoli, B., Franceschetti, M., Poolla, K., & Sastry, S. (2007). Foundations of control and estimation over lossy networks. *Proceedings of the IEEE*, *95*(1), 163–187.

Schrama, R. J. P. (1991). An open-loop solution to the approximate closed-loop identification problem. In *Identification and system parameter estimation 1991. Budapest, Hungary* (pp. 761–766).

Söderström, T. (2002). *Discrete-time stochastic systems: estimation and control* (second ed.) Prentice Hall.

Söderström, T., & Stoica, P. (1989). *System identification*. Prentice-Hall International.

Söderström, T., Stoica, P., & Trulsson, E. (1987). Instrumental variable methods for closed-loop systems. In *Proc. 10th IFAC world congress. Munich, Germany* (pp. 363–368).

Stuart, A., Ord, J. K., & Arnold, S. (1999). *Kendall's advanced theory of statistics*: *vol. 2A*. Edward Arnold.

Van den Hof, P. M. J. (1998). Closed-loop issues in system identification. *Annual Reviews in Control*, *22*, 173–186.

Van den Hof, P. M. J., & Schrama, R. J. P. (1995). Identification and control-closed loop issues. *Automatica*, *31*(12), 1751–1770.

Van Donkelaar, E. T., & Van den Hof, P. M. J. (2000). Analysis of closed-loop identification with a tailor-made parameterization. *European Journal of Control*, *6*(1), 54–62.

Welsh, J. S., & Goodwin, G. C. (2002). Finite sample properties of indirect nonparametric closed-loop identification. *IEEE Transactions on Automatic Control*, *47*(8), 1277–1292.

White, H. (1996). *Estimation, inference and specification analysis*. Cambridge University Press.

Yuz, J. I., & Goodwin, G. C. (2008) Robust identification of continuous-time systems from sampled data. In H. Garnier, & L. Wang (Eds.), *Continuous-time model identification from sampled data* (pp. 67–89). Springer, (Chapter 3).

Zheng, W. X., & Feng, C. B. (1995). A bias-correction method for indirect identification of closed-loop systems. *Automatica*, *31*(7), 1019–1024.

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