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Extended Ho–Kalman algorithm for systems represented in generalized orthonormal bases☆

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Abstract

This paper considers the construction of minimal state space models of linear time-invariant systems on the basis of system representations in terms of generalized orthogonal basis function expansions. Starting from the classical Ho–Kalman algorithm that solves the problem using Markov parameter expansions, a generalization is obtained by analysing the matrix representations of the Hankel operators in generalized orthonormal bases. Using the so-called *Hambo*-domain techniques an efficient algorithm is given to implement the proposed method. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The study of rational basis functions dates back to Malmquist (1925) and was taken up by Walsh (1935) in the context of complex rational approximation theory. The idea of decomposing representations of linear time-invariant dynamical systems and related input/output signals in terms of orthogonal components other than the standard Fourier series, dates back to the work of Lee and Wiener in the 1930s, as reviewed in Lee (1960). Laguerre functions have been very popular in this respect, mainly because of the fact that their frequency response is rational. In an attempt to find more general classes of orthogonal basis functions with this same property, Kautz (1954) formulated a general class of functions, composed of damped exponentials, to be used for signal decomposition. In Wahlberg (1991, 1994a,b) Laguerre functions and the so-called two-parameter Kautz functions have been used in the identification of the expansion coefficients of approximate models by simple linear regression methods. Extending this work further, Heuberger (1991) has developed a theory on the construction of orthogonal basis functions, based on balanced realizations of inner (all-pass) transfer functions, see Heuberger, Van den Hof and Bosgra (1995). A further generalization of this situation is presented in Ninness and Gustafsson (1997), where concatenations of freely chosen all-pass sections are considered as basis-generators. Ward and Partington (1996) use rational wavelets on the disc algebra for robust identification.

These recently developed basis functions have been shown to have attractive properties in several respects. First the use of them as linear model parametrizations in system identification problems has been shown to be attractive; this is due to the fact that smartly chosen basis functions can provide a fast rate of convergence of the corresponding series expansion, thus leading to linear model parametrizations with a limited number of parameters. Statistical properties of time domain (least-squares) identification methods have been analysed in Van den Hof, Heuberger and Bokor (1995) and Ninness et al. (1999), and frequency-domain methods in Ninness

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and Gómez (1996), Schipp, Gianone, Bokor and Szabó (1996), Szabó, Schipp and Bokor (1999) and De Vries and Van den Hof (1998). Additionally, the linear model parametrizations are indispensable in explicitly quantifying (probabilistic and/or worst-case) model error bounds, see e.g. Hakvoort and Van den Hof (1997). Besides the use of these functions for identification purposes, also the problem of system approximation has been addressed, see particularly Wahlberg and Mäkilä (1996), Oliveira e Silva (1995, 1996), Den Brinker, Benders and Oliveira e Silva (1996) for approximation in the $\mathscr{H}_2$ norm and Bokor (1997) and Schipp and Bokor (1998), where approximation in the $\mathscr{H}_\infty$ norm was discussed.

One of the basic features in identification with orthogonal basis functions is the fact that only a relatively small number of coefficients needs to be estimated for arriving at highly accurate models. However, if one intends to use the identified models for further analysis or, e.g. for model-based control design, a more compact representation of their dynamics in terms of minimal state-space representations is desirable.

The classical example of a minimal realization algorithm using expansion coefficients is provided by the Ho–Kalman algorithm (Ho & Kalman, 1966) where a realization is obtained from Markov parameters, i.e. from expansion coefficients in terms of the standard basis functions $\{z^{-k}\}_{k=1}^{\infty}$. This algorithm has become very popular, not only as an exact realization algorithm, but also by providing methods for approximate realization (see e.g. Kung, 1978), and by giving rise to interesting developments in the analysis of balancing properties of state space models (Moore, 1981). Additionally the respective realization algorithms have been a principal landmark for the development of the so-called subspace identification methods (see e.g. Van Overschee & de Moor, 1996).

Given the recent interest in generalized orthonormal basis functions, a natural question is to consider the generalization of the realization problem to the newly developed orthogonal basis function expansions. Besides the relevance of this problem from a system theoretic point of view, it has direct consequences in providing algorithmic tools for constructing state-space models from (identified) expansion coefficients.

Obtaining a minimal state-space representation for transfer functions parametrized in a generalized orthogonal basis is not a trivial task. A detailed study of the representations and properties of the Hankel operators between the signal spaces in generalized orthogonal bases leads to a method that gives the minimal state-space representation of the system by extending the celebrated Ho–Kalman algorithm. Preliminary results were published in Fischer (1997) for the Laguerre domain and in Szabó and Bokor (1997) for the generalized orthonormal case. This paper intends to give the realization theoretical background that can serve to generalize the Ho–Kalman algorithm. The contribution of the paper is to provide the matrix representation of Hankel operators in the generalized orthonormal bases and to show how these matrices can be constructed from the expansion coefficients. The paper proves the applicability of the Ho–Kalman algorithm for the generalized case too, and provides a computational method to construct a minimal representation using the expansion coefficients of the identified transfer function in the generalized orthogonal basis.

The structure of the paper is as follows. After stating some basic facts and notation, the considered basis functions will be specified and reviewed in Section 2. Simple constructions will be shown based on balanced realizations of all-pass functions. This is followed by a short review on the matrix representation of a given Hankel operator in certain bases in Section 3. A new signal and system transform will be presented in Section 4 that can be viewed as an extension of the *Hambo*-transform. After reviewing a method for obtaining a non-minimal representation in Section 5, in Section 6 the generalized Ho–Kalman algorithm will be presented. In Section 7 some computational questions are answered and numerical examples are provided in Section 8.

2. Generalized orthonormal basis functions

Let us denote by $\mathscr{L}_2$ the space of the square integrable functions on the unit circle. Here and through this paper $\mathscr{H}_2$ is the space of strictly proper square integrable functions on the unit circle that can be extended analytically outside the unit disc and $\mathscr{H}_{2\perp}$ is its orthogonal complement, so that $\mathscr{L}_2 = \mathscr{H}_2 \oplus \mathscr{H}_{2\perp}$. Unless otherwise mentioned all systems in this paper are scalar systems.

An isometry S on a Hilbert space $\mathscr{H}$ such that $\bigcap_0^\infty S^j\mathscr{H} = \{0\}$, is called a *shift operator*. We recall the following property of shift operators (for details see Rosenblum and Rovnyak (1985)):

Proposition 1. *If S is a shift operator on $\mathscr{H}$ and $\mathscr{K} = \ker S^*$, then $\mathscr{H} = \sum_{j=0}^\infty S^j\mathscr{K}$. Each $f \in \mathscr{H}$ has a unique representation $f = \sum_{j=0}^\infty S^j f_j$, with $f_j \in \mathscr{K}$* and *$f_j = P_0 S^{*j} f$, where $P_0 = \mathbb{I} - SS^*$. Moreover, $\|f\|^2 = \sum_{j=0}^\infty \|f_j\|^2$.*

When S is the canonical shift operator on $\mathscr{H}_2$, i.e, $(Sf)(z) = \bar{z}f(z)$, then $\mathscr{K} = \{c\bar{z} \mid c \in \mathbb{C}\}$ and the expansion coincides with the classical Fourier series expansion. It is known that this basis can be extended to $\mathscr{L}_2$ and the operator denoted by $S_+ f = zf$ is a shift operator on $\mathscr{H}_{2\perp}$. Let us denote by $\mathscr{Z} = \{z^{-k}\}_{k=1}^\infty$ the orthonormal basis of $\mathscr{H}_2$ and by $\mathscr{Z}_+ = \{z^k\}_{k=0}^\infty$ the orthonormal basis of $\mathscr{H}_{2\perp}$.

Let $\mathscr{B}$ be a finite Blaschke product of order n_b written in the form $\mathscr{B} = \prod_{j=1}^{n_b} b_{\alpha_j}$ where $b_{\alpha_j}(z) = (1 - \bar{\alpha}_j z)/(z - \alpha_j)$,

$|\alpha_j| < 1$, and each zero of $\mathscr{B}$ is repeated according to its multiplicity. Let us denote by $\mathbf{H}(\mathscr{B}) := \mathscr{H}_2 \ominus \mathscr{B}\mathscr{H}_2$, that is an n_b dimensional subspace of $\mathscr{H}_2$, see Sarason (1967). Let S_α be the multiplication operator by $\mathscr{B}$ on $\mathscr{H}_2$. Then S_α is a shift operator and $ker\, S_\alpha^* = \mathbf{H}(\mathscr{B})$, so $S_\alpha^* = \overline{\mathscr{B}(z)}(\mathbb{I} - P_{\mathbf{H}(\mathscr{B})})$, and one can consider an orthonormal set of basis functions $\{\phi_j \,|\, j = 1, \ldots, n_b\}$ in $\mathbf{H}(\mathscr{B})$, see Szabó et al. (1999). Applying Proposition 1 one can obtain the generalized orthonormal basis function expansions that were introduced in Heuberger et al. (1995), based on the preliminary work of Heuberger and Bosgra (1990) and Heuberger (1991).

Let $\mathscr{B}$ be a scalar inner function with McMillan degree $n_b > 0$, having a minimal balanced realization (A_b, B_b, C_b, D_b). Denoting by

$$V_k(z) := (zI - A_b)^{-1} B_b \mathscr{B}^{k-1}(z) \tag{1}$$

the components of the n_b-dimensional rational functions $V_k(z)$, i.e., the sequence

$$\mathscr{G} = \{e_i^{\mathrm{T}} V_k(z) \,|\, i = 1, \ldots, n_b,\ k = 1, 2, \ldots\} \tag{2}$$

will constitute an orthonormal basis for $\mathscr{H}_2$, where e_i is the ith Euclidean basis vector in $\mathbb{R}^n$. As for the classical situation this basis can also be extended to $\mathscr{L}_2$, i.e., the sequence

$$\{e_i^{\mathrm{T}} V_k(z) \,|\, i = 1, \ldots, n_b,\ k \in \mathbb{Z}\} \tag{3}$$

will constitute an orthonormal basis for $\mathscr{L}_2$. Let us denote by

$$\mathscr{G}_+ = \{e_i^{\mathrm{T}} V_{-k}(z) \,|\, i = 1, \ldots, n_b,\ k = 0, 1, 2, \ldots\} \tag{4}$$

the orthonormal basis of $\mathscr{H}_{2\perp}$. Let us mention here, that the multiplication operator by $\bar{\mathscr{B}}$ on $\mathscr{H}_{2\perp}$, denoted by S_α^+, is a shift operator on $\mathscr{H}_{2\perp}$.

Note that these basis functions exhibit the property that they can incorporate systems dynamics in a very general way. One can construct an inner function $\mathscr{B}$ from any given set of poles, and thus the resulting basis can incorporate dynamics of any complexity, combining, e.g. both fast and slow dynamics in damped and resonant modes.

3. Hankel operators

3.1. Hankel matrices in the Fourier basis

It is known that a stable linear time invariant system can be modeled through transfer functions $G \in \mathscr{H}_\infty$ using Hankel operators with symbol G defined as H_G: $\mathscr{H}_{2\perp} \to \mathscr{H}_2$, $H_G(f) = \mathbf{P}_{\mathscr{H}_2} Gf$, where $\mathbf{P}_{\mathscr{H}_2}$ denotes the orthogonal projection operator to $\mathscr{H}_2$. The matrix of this operator in the basis $\mathscr{Z}_+$, $\mathscr{Z}$ is the Hankel matrix

$$H_{\mathscr{Z}}^G = \begin{pmatrix} g_1 & g_2 & g_3 & \cdots \\ g_2 & g_3 & g_4 & \cdots \\ g_3 & g_4 & g_5 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \tag{5}$$

where $\{g_k\}$ are the Markov parameters of G. The index $\mathscr{Z}$ refers to the fact that the Hankel matrix is considered in the canonical basis. In the rest of the paper it will be assumed that the range of the Hankel operator is finite dimensional, i.e., G is a rational function.

3.2. Hankel matrices in the generalized orthonormal basis

To compute the matrix of the Hankel operator H_G in the basis $\mathscr{G}_+$, $\mathscr{G}$ formed by the generalized orthonormal basis functions one needs the following simple but fundamental lemma:

Lemma 2. *If* $a, b \in \mathbf{H}(\mathscr{B})$ *then* $ab \in \mathbf{H}(\mathscr{B}) \oplus \mathscr{B}\mathbf{H}(\mathscr{B})$, *moreover* $ab = A^{[1]}(b) + \mathscr{B}A^{[2]}(b)$, *where* $A^{[1]}, A^{[2]}$: $\mathbf{H}(\mathscr{B}) \to \mathbf{H}(\mathscr{B})$ *are linear operators.*

Proof. Suppose that the denominator of $\mathscr{B}$ is q. It is clear that for every $g \in \mathbf{H}(\mathscr{B})$, $g = p/q$ where $deg(p) < deg(q)$ and for every $g \in \mathbf{H}(\mathscr{B}^2)$, $g = p/q^2$ where $deg(p) < deg(q^2)$. For $a = p_1/q$, $b = p_2/q$ one has $ab = p_1 p_2/q^2$, so that $ab \in \mathbf{H}(\mathscr{B}^2)$. The assertion follows from the fact that $\mathbf{H}(\mathscr{B}^2) = \mathbf{H}(\mathscr{B}) \oplus \mathscr{B}\mathbf{H}(\mathscr{B})$, see, e.g. Frazho and Foias (1990). For the second part consider a basis $\Phi = \{\phi_j \,|\, j = 1, \ldots, n_b\}$ in $\mathbf{H}(\mathscr{B})$, and let us denote with a slight abuse of notation by $a = [a_l]_{l=1}^{n_b}$, $b = [b_l]_{l=1}^{n_b} \in \mathbb{C}^{n_b}$ the coordinates of $a(z)$ and $b(z)$ in the basis Φ. Let us consider $\phi_i \phi_j = \sum_{l=1}^{n_b} \alpha_{ij}^l \phi_l + \mathscr{B} \sum_{l=1}^{n_b} \beta_{ij}^l \phi_l$ and denote by $\alpha^l = [\alpha_{ij}^l]_{i,j=1}^{n_b}$ and by $\beta^l = [\beta_{ij}^l]_{i,j=1}^{n_b}$, then $ab = A^{[1]}(b) + \mathscr{B}A^{[2]}(b)$ where $A^{[1]}(b) = [a^{\mathrm{T}} \alpha^l] b$ and $A^{[2]}(b) = [a^{\mathrm{T}} \beta^l] b$. $\square$

To facilitate the notation Lemma 2 can be summarized as follows:

Corollary 3. *Using the definition of* $V_k(z)$ *given by* (1) *and denoting by*

$$P_j = [\alpha_{i,j}^l]_{i,l=1}^{n_b}, \qquad Q_j = [\beta_{i,j}^l]_{i,l=1}^{n_b} \tag{6}$$

it follows that

$$V_1(z) V_1^{\mathrm{T}}(z) = [P_1 V_1(z) \ldots P_{n_b} V_1(z)] + [Q_1 V_1(z) \ldots Q_{n_b} V_1(z)] \mathscr{B}(z). \tag{7}$$

The operators $A^{[1]}, A^{[2]}$ can be expressed using the matrices P_i, Q_i as follows:

$$A^{[1]} = \sum_{j=1}^{n_b} P_j^{\mathrm{T}} a_j, \qquad A^{[2]} = \sum_{j=1}^{n_b} Q_j^{\mathrm{T}} a_j. \tag{8}$$

As will be shown in (14) the matrices P_i, Q_i can be computed as solutions of certain Sylvester equations.

This result can now be applied to derive the Hankel operator of a system in the generalized orthonormal basis. Suppose that the transfer function is given in the form

$$G(z) = \sum_{k=0}^{\infty} \sum_{l=1}^{n_b} \gamma_l^k \phi_l(z) \mathscr{B}^k(z) := \sum_{k=0}^{\infty} G_k(z) \mathscr{B}^k(z),$$

where $\gamma_l^k \in \mathbb{R}^{p \times q}$, and $\{\phi_j \mid j = 1, \ldots, n_b\}$ is an orthonormal basis in $\mathbf{H}(\mathscr{B})$. Using the previous lemma one can show that:

Theorem 4. *The matrix of the Hankel operator in the generalized orthonormal basis has the following form*:

$$H_{\mathscr{G}}^G = \begin{pmatrix} G_0^{[2]} + G_1^{[1]} & G_1^{[2]} + G_2^{[1]} & G_2^{[2]} + G_3^{[1]} & \ldots \\ G_1^{[2]} + G_2^{[1]} & G_2^{[2]} + G_3^{[1]} & G_3^{[2]} + G_4^{[1]} & \ldots \\ G_2^{[2]} + G_3^{[1]} & G_3^{[2]} + G_4^{[1]} & G_4^{[2]} + G_5^{[1]} & \ldots \\ \ldots & & & \end{pmatrix}, \quad (9)$$

where $G_k^{[1]}$ and $G_k^{[2]}$ are the linear operators generated by the coefficients γ_l^k of $G_k(z)$ as

$$G_k^{[1]}(b)(z) = \sum_{l=1}^{n_b} \left[\sum_{p=1}^{n_b} \left(\sum_{q=1}^{n_b} \gamma_p^k \alpha_{pq}^l \right) b_p \right] \phi_l$$

and

$$G_k^{[2]}(b)(z) = \sum_{l=1}^{n_b} \left[\sum_{p=1}^{n_b} \left(\sum_{q=1}^{n_b} \gamma_p^k \beta_{pq}^l \right) b_p \right] \phi_l,$$

with α and β being defined according to Lemma 2.

Proof. The proof of this assertion resembles the same computational steps as the Markov parameters case. Let us denote by $\mathbf{P}_0$ the orthogonal projection to $\mathbf{H}(\mathscr{B})$. By definition $y = H_G(u) = \mathbf{P}_{\mathscr{H}_2} G u$, where

$$u = \sum_{k=1}^{\infty} \sum_{l=1}^{n_b} u_{l,k} \phi_l(z) \mathscr{B}^{-k}(z) := \sum_{k=1}^{\infty} u_k(z) \mathscr{B}^{-k}(z),$$

$$y = \sum_{k=0}^{\infty} \sum_{l=1}^{n_b} y_{l,k} \phi_l(z) \mathscr{B}^k(z) := \sum_{k=0}^{\infty} y_k(z) \mathscr{B}^k(z).$$

Now

$$y_l = \mathbf{P}_0 \mathscr{B}^{-l} \mathbf{P}_{\mathscr{H}_2} G u = \mathbf{P}_0 \mathscr{B}^{-l}(z) \mathbf{P}_{\mathscr{H}_2} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} G_j(z) u_k(z) \mathscr{B}^{j-k}(z),$$

i.e.

$$\begin{aligned} y_l &= \mathbf{P}_0 \mathscr{B}^{-l} \mathbf{P}_{\mathscr{H}_2} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} (G_j^{[1]}(u_k) + \mathscr{B} G_j^{[2]}(u_k)) \mathscr{B}^{j-k} \\ &= \mathbf{P}_0 \mathscr{B}^{-l} \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} (G_j^{[1]}(u_k) + G_{j-1}^{[2]}(u_k)) \mathscr{B}^{j-k} \\ &= \sum_{k=1}^{\infty} (G_{l+k}^{[1]} + G_{l+k-1}^{[2]})(u_k). \qquad \square \end{aligned}$$

As will be shown in Section 6, one can construct a state-space representation starting from the balanced realization of $\mathscr{B}$, and the expansion coefficients γ_l^k of a rational transfer function G in the generalized basis, but that representation is not minimal in general — even in the SISO case —, as an example in Section 8 shows. Therefore, one needs an algorithm to construct a minimal representation. The proposal of this paper to solve this problem is a generalization of the Ho–Kalman algorithm, see Ho and Kalman (1966).

4. Signal and system transforms induced by generalized orthonormal bases

The presented generalized orthonormal basis expansion on $\mathscr{H}_2$ induces a transformation of signals and systems to a transform domain. One specific example constructed starting from the ℓ_2 sequence spaces is referred to as the *Hambo*-transform (see Van den Hof et al., 1995). In this section a different and slightly extended definition of the *Hambo*-transform, in comparison with its introduction in the aforementioned paper, will be given. It will be shown, that the Hankel matrix constructed by using the Markov parameters of the transformed system is exactly the same as the Hankel matrix of the original system in the generalized orthonormal basis. Let us introduce a signal-transform on $\mathscr{L}_2$ by using the expansion in the generalized orthonormal basis defined by the inner function $\mathscr{B}$.

Definition 5. Consider the expansion of a function $F \in \mathscr{L}_2$ in the generalized orthonormal basis, i.e., $F = \sum_{-\infty}^{\infty} F_k^{\mathrm{T}} V_1(z) \mathscr{B}^{k-1}(z)$, $F_k \in \mathbb{R}^{n_b}$, where the coefficient vectors are given by $F_k = [\langle F, e_i^{\mathrm{T}} V_1 \mathscr{B}^{k-1} \rangle]_{i=1}^{n_b}$, and $\langle , \rangle$ denotes the scalar inner product on $\mathscr{L}_2$. Then the signal-transform $\mathscr{H}: \mathscr{L}_2 \to \mathscr{L}_2^{n_b}$, will be defined as

$$\mathscr{H} F(\lambda) = \tilde{F}(\lambda) = \sum_{k=-\infty}^{\infty} F_k \lambda^{-k}. \quad (10)$$

It is clear that this transform is a unitary transform between the two function spaces. If one has an operator T: $\mathscr{L}_2 \to \mathscr{L}_2$, $y = Tu$, then the image of the operator that acts on the transform domain will be $\mathscr{H} T \mathscr{H}^*$. It is also clear that the images of the orthogonal projection operators $\mathbf{P}_-: \mathscr{L}_2 \to \mathscr{H}_2$ and $\mathbf{P}_+: \mathscr{L}_2 \to \mathscr{H}_{2\perp}$ will be the same type of projection operators $\mathbb{P}_-: \mathscr{L}_2^{n_b} \to \mathscr{H}_2^{n_b}$ and $\mathbb{P}_+: \mathscr{L}_2^{n_b} \to \mathscr{H}_{2\perp}^{n_b}$ where $\mathbb{P}_- = \mathscr{H}\mathbf{P}_-\mathscr{H}^*$ and $\mathbb{P}_+ = \mathscr{H}\mathbf{P}_+\mathscr{H}^*$.

Now consider a transfer function $G \in \mathscr{H}_\infty$ and Hankel operator with symbol G defined as $H_G: \mathscr{H}_{2\perp} \to \mathscr{H}_2$, $H_G(f) = \mathbf{P}_{\mathscr{H}_2} Gf$. Since $\mathscr{H} H_G \mathscr{H}^* = \mathbb{P}_- \mathscr{H} G \mathscr{H}^*|_{\mathscr{H}_{2\perp}^{n_b}}$, it follows that the image of the the Hankel operator in the transformed-domain is also a Hankel operator with symbol $\mathscr{H} G \mathscr{H}^*(\lambda) := \tilde{G}(\lambda) \in \mathscr{L}_\infty^{n_b \times n_b}$.

Definition 6. Consider the function $G \in \mathscr{L}_\infty$. Then its system-transform $\mathscr{H}_s: \mathscr{L}_\infty \to \mathscr{L}_\infty^{n_b \times n_b}$, will be defined as

$$\mathscr{H}_s G(\lambda) = \tilde{G}(\lambda) := \mathscr{H} G \mathscr{H}^*(\lambda). \tag{11}$$

Theorem 7. *Consider a function $G \in \mathscr{L}_\infty$ with the expansion*

$$G(z) = \sum_{k=-\infty}^{\infty} G_k^{\mathrm{T}} V_1(z) \mathscr{B}(z)^{k-1}, \quad G_k \in \mathbb{R}^{n_b}.$$

Then its system-transform is given by

$$\tilde{G}(\lambda) = \sum_{k=-\infty}^{\infty} M_k \lambda^{-k}$$

with

$$M_k = \sum_{j=1}^{n_b} e_j^{\mathrm{T}} G_{k+1} P_j^{\mathrm{T}} + e_j^{\mathrm{T}} G_k Q_j^{\mathrm{T}}. \tag{12}$$

Proof. The ith column of $\tilde{G}$ is given by $\mathscr{H} G \mathscr{H}^* e_i$, i.e.

$$\begin{aligned}
\tilde{G}(\lambda) e_i &= \mathscr{H} G(z) e_i^{\mathrm{T}} V_1(z) \mathscr{B}^{-1}(z) \\
&= \sum_{k=-\infty}^{\infty} G_k^{\mathrm{T}} V_1(z) e_i^{\mathrm{T}} V_1(z) \mathscr{B}(z)^{k-2} \\
&= \mathscr{H} \sum_{k=-\infty}^{\infty} (G_{k+1}^{\mathrm{T}} P_i V_1(z) + G_k^{\mathrm{T}} Q_i V_1(z)) \mathscr{B}(z)^{k-1} \\
&= \sum_{k=-\infty}^{\infty} (P_i^{\mathrm{T}} G_{k+1} + Q_i^{\mathrm{T}} G_k) \lambda^{-k}.
\end{aligned}$$

By using the fact that $e_j^{\mathrm{T}} P_i = e_i^{\mathrm{T}} P_j$ and $e_j^{\mathrm{T}} Q_i = e_i^{\mathrm{T}} Q_j$ the theorem follows. □

By adding an inverse Fourier transform to the signal transform introduced above one can obtain a transformation that acts on sequence spaces. The restriction of this transformation to ℓ_2 is exactly the *Hambo*-signal transform introduced by Van den Hof et al. (1995). Therefore, this new transform can be viewed as an extension of the *Hambo*-transform and by an abuse of definition one can refer to it as the *Hambo*-transform.

Remark 8. By applying the theorem above for the basis functions $\phi_i(z) = e_i^{\mathrm{T}} V_1(z)$ one can obtain

$$\tilde{\phi}_i(\lambda) = P_i^{\mathrm{T}} + Q_i^{\mathrm{T}} \lambda^{-1}. \tag{13}$$

By computing the Hambo-system-transform of the function z^{-1} one can obtain

$$\mathscr{H}_s z^{-1}(\lambda) := N(\lambda) = A_b + \frac{B_b C_b}{\lambda - D_b} \tag{14}$$

for the proof and details see Van den Hof et al. (1995). By consulting (9), (8) and (12) one has the following assertion:

Proposition 9. *$H_{\mathscr{G}}^{G} = H_{\Lambda}^{\tilde{G}}$, where $H_{\Lambda}^{\tilde{G}}$ is the Hankel matrix of $\tilde{G}$ in the canonical basis, i.e., the Hankel matrix obtained by using the Markov parameters M_k of $\tilde{G}$.*

5. Non-minimal state-space realization

In this section a method will be presented to give a non-minimal state-space realization for a MIMO transfer function represented in a generalized orthonormal basis. First let us recall a result from Roberts and Mullis (1987) and Heuberger et al. (1995). Given a balanced realization of $\mathscr{B}$ one can directly construct a balanced realization of $\mathscr{B}^k$ for any $k > 0$:

Proposition 10. *Let $\mathscr{B}$ be a square inner transfer function with minimal balanced realization (A_b, B_b, C_b, D_b) having state dimension $n_b > 0$. Then for any $k > 1$ the realization (A_k, B_k, C_k, D_k) with*

$$A_k = \begin{bmatrix} A_b & 0 & \cdots & \cdot & 0 \\ B_b C_b & A_b & 0 & \cdot & 0 \\ B_b D_b C_b & B_b C_b & \cdot & \cdot & 0 \\ \vdots & \vdots & \cdot & \ddots & 0 \\ B_b D_b^{k-2} C_b & B_b D_b^{k-3} C_b & \cdots & B_b C_b & A_b \end{bmatrix},$$

$$B_k^{\mathrm{T}} = [B_b^{\mathrm{T}} \quad D_b^{\mathrm{T}} B_b^{\mathrm{T}} \quad D_b^{\mathrm{T}2} B_b^{\mathrm{T}} \quad \cdots \quad D_b^{\mathrm{T}(k-1)} B_b^{\mathrm{T}}],$$

$$C_k = [D_b^{k-1} C_b \quad D_b^{k-2} C_b \quad \cdots \quad D_b C_b \quad C_b],$$

$$D_k = D_b^k \tag{15}$$

is a minimal balanced realization of $\mathscr{B}^k$ with state dimension $n_b \cdot k$.

By evaluating the realization it follows that it can be constructed by the following recursive mechanism

$$A_k = \begin{bmatrix} A_{k-1} & 0 \\ B_b C_{k-1} & A_b \end{bmatrix}, \qquad B_k = \begin{bmatrix} B_{k-1} \\ B_b D_b^{k-1} \end{bmatrix},$$

$$C_k = [D_b^{k-1} C_b \quad C_{k-1}], \qquad D_k = D_b \cdot D_{k-1}.$$

Given a transfer function in the form $G(z)=\sum_{k=0}^{N-1} G_k^T V_1(z)\mathscr{B}^k(z)$, it follows that $G(z)\in \mathbf{H}(\mathscr{B}^N)$, i.e., if one considers the inner function $\mathscr{B}^N$ with an input balanced realization (A_N, B_N, C_N, D_N) then $(A_N, B_N, C, 0)$ is a realization for $G(z)$, where $C=[G_0^T\ G_1^T\ \dots\ G_{N-1}^T]$.

A slightly different non-minimal realization based on a Horner scheme was presented in Szabó and Bokor (1997) or based on a mixed partial fraction expansion and Horner scheme in Nalbantoglu, Bokor and Balas (1997).

6. Minimal state-space realization

In this section a method will be presented to give a minimal state-space realization for a MIMO transfer function represented in a generalized orthonormal basis. The Ho–Kalman algorithm (Ho & Kalman, 1966) can be summarized as follows. Consider a system with q inputs and p outputs and denote:

$$H_r := \begin{pmatrix} g_1 & g_2 & \cdots & g_r \\ g_2 & g_3 & \cdots & g_{r+1} \\ \cdots & & & \\ g_r & g_{r+1} & \cdots & g_{2r} \end{pmatrix}$$

and by

$$\tau(H_r) := \begin{pmatrix} g_2 & g_3 & \cdots & g_{r+1} \\ g_3 & g_4 & \cdots & g_{r+2} \\ \cdots & & & \\ g_{r+1} & g_{r+2} & \cdots & g_{2r+1} \end{pmatrix},$$

where $\{g_k\}$ is the set of Markov parameters of the transfer function G, i.e. $G(z)=\sum_{k=1}^{\infty} g_k z^{-k}$, and r is greater than or equal to the McMillan degree of the system. The operator τ corresponds to the adjoint shift operator on $\mathscr{H}_{2\perp}$, i.e. multiplication by z. There exist matrices P and Q such that

$$PH_rQ = \begin{bmatrix} \mathbb{I}_s & 0 \\ 0 & 0 \end{bmatrix} = J.$$

Let us denote by

$$U_s = [\mathbb{I}_s \quad 0], \qquad E_k = [\mathbb{I}_k \quad 0_k \quad \dots \quad 0_k],$$

where the dimension of the matrix U_s may vary according to the dimensions of the expressions in which it appears. Then a minimal state-space realization (A, B, C) is given by

$$A = U_s JP\tau(H_r)QJU_s^*,$$

$$B = U_s JPH_r E_q^*, \qquad C = E_p H_r QJU_s^*.$$

Let us denote by T the matrix that corresponds to the change of basis from $\mathscr{Z}=\{z^{-k}\,|\,k\in\mathbb{N}\}$ to $\mathscr{G}=\{\phi_l m^k\,|\,l=1,\dots,n_b,\ k\in\mathbb{N}o\}$, by T_+ the matrix that corresponds to the change of basis from $\mathscr{Z}_+=\{z^{-k}\,|\,k\in\mathbb{Z}\backslash\mathbb{N}\}$ to $\mathscr{G}_+=\{\phi_l m^k\,|\,l=1,\dots,n_b,\ k\in\mathbb{Z}\backslash\mathbb{N}o\}$, and by $H_{\mathscr{Z}}$ and $H_{\mathscr{G}}$ the matrices of the input–output map in terms of these bases. The elements of T are the Markov parameters of the basis functions from $\mathscr{G}$, i.e.

$$\phi_l \mathscr{B}^k = \sum_{i=1}^{\infty} t_{l+n_b k, i} z^{-i}.$$

Since both bases are orthonormal, the matrices $T_{(+)}$ are unitary, i.e. $T_{(+)}^* = T_{(+)}^{-1}$. If one supposes that the transfer function is an element of $\mathbf{H}(\mathscr{B}^N)$, i.e., that $r=n_bN$, then the Hankel matrix $H_{\mathscr{G}}$ will contain only one nonzero block, say $H_{\mathscr{G}}^r$. As in the previous section, let us suppose that the transfer function is given in the form

$$G(z) = \sum_{k=0}^{N} \sum_{l=1}^{n_b} \gamma_l^k \phi_l(z)\mathscr{B}^k(z),$$

where $\gamma_l^k \in \mathbb{R}^{p\times q}$. With this notation one has the following minimal representation theorem:

Theorem 11. *If for a rational transfer function G one has* $PH_{\mathscr{G}}^r Q = U_s U_s^* = J$ *then*

$$A = U_s JP\tau(H_{\mathscr{G}}^r)QJU_s^*, \qquad B = U_s JPH_{\mathscr{G}}^r T_{(+)}^{11} E_q^*,$$

$$C = E_p (T^{11})^* H_{\mathscr{G}}^r QJU_s^*$$

gives a minimal representation, where $r=(N+1)n_b$ *and* $T_{(+)}^{11} = U_r T_{(+)} U_r^*$, *i.e. the first* $r\times r$ *block of* $T_{(+)}$.

Proof. It is clear that for $\mathscr{G}=\mathscr{Z}$ one can obtain the Ho–Kalman algorithm. One can also derive that $H_{\mathscr{Z}} = T^* H_{\mathscr{G}} T_{(+)}$ and

$$H_r = U_r H_{\mathscr{Z}} U_r^* = U_r T^* H_{\mathscr{G}} T_{(+)} U_r^* = (T^{11})^* H_{\mathscr{G}}^r T_{(+)}^{11}.$$

If $G\in\mathbf{H}(\mathscr{B}^N)$, one has $zG = c + \tilde{G}(z)$, where $\tilde{G}\in\mathbf{H}(\mathscr{B}^N)$, and $c\in\mathbb{R}$. By definition

$$H_{zG}(u) = \mathbf{P}_{\mathscr{H}_2} zGu = \mathbf{P}_{\mathscr{H}_2}(cu + \tilde{G}u)$$
$$= \mathbf{P}_{\mathscr{H}_2} \tilde{G}u = H_{\tilde{G}}(u)$$

since $u\in\mathscr{H}_{2\perp}$, so one can obtain that $\tau(H_{\mathscr{Z}}) = T^*\tau(H_{\mathscr{G}})T_{(+)}$ therefore,

$$\tau(H_r) = U_r \tau(H_{\mathscr{Z}}) U_r^*$$
$$= U_r T^* \tau(H_{\mathscr{G}}) T_{(+)} U_r^* = (T^{11})^* \tau(H_{\mathscr{G}}^r) T_{(+)}^{11},$$

where $\tau(H_{\mathscr{G}}^r)$ is the matrix of the Hankel operator that corresponds to $\tilde{G}$. The desired result follows by substitution of these formulas into the original Ho–Kalman algorithm. □

As an observation, it has to be stated here that the realization algorithm presented above gives the desired result in the full information case. The partial realization problem using *Hambo*-domain techniques is a more delicate question and it is beyond the scope of this paper to give an answer to that problem.

7. Computational aspects

The elements of the transformation matrices $T_{i,j}$, $(T_{(+)})_{i,j}$ can be easily computed using the following proposition, see Heuberger and Van den Hof (1995b):

Proposition 12. *Let $\mathscr{B}$ be a scalar inner function with McMillan degree n_b having a minimal balanced realization (A_b, B_b, C_b, D_b). Let us consider*

$$\varphi_1(t) = A_b^{t-1}B_b, \qquad \varphi_{j+1} = \mathscr{B}(q)\mathbb{I}_{n_b}\varphi_j(t) \tag{16}$$

and

$$\psi_1(t) = (A_b^{\mathrm{T}})^{t-1}C_b^{\mathrm{T}}, \qquad \psi_{j+1} = \mathscr{B}(q)\mathbb{I}_{n_b}\psi_j(t), \tag{17}$$

where the shift-operator q operates on the time sequences $\varphi_k(t)$, $\psi_k(t)$ and $\varphi_k(t) = 0$, $\psi_k(t)$, $t \le 0$. Then

$$T_{i,j} = \varphi_i(j), \qquad (T_{(+)})_{i,j} = \psi_i(j). \tag{18}$$

The matrices P_i, Q_i needed to compute the Hankel matrices in the generalized orthonormal basis can be calculated efficiently using Sylvester equations. To show this we need the following technical Lemma:

Lemma 13. *The matrices $\{P_i, Q_i,\ i = 1,\dots,n_b\}$ satisfy the following relations*:

$$A_bP_i^{\mathrm{T}} = P_i^{\mathrm{T}}A_b, \tag{19}$$

$$A_bQ_i^{\mathrm{T}} - Q_i^{\mathrm{T}}A_b = B_bC_bP_i^{\mathrm{T}} - P_i^{\mathrm{T}}B_bC_b, \tag{20}$$

$$B_b^{\mathrm{T}}P_i = e_i^{\mathrm{T}}A_b^{\mathrm{T}}, \tag{21}$$

$$B_b^{\mathrm{T}}Q_i^{\mathrm{T}} = e_i^{\mathrm{T}}(B_bC_b - A_bD_b)^{\mathrm{T}}. \tag{22}$$

Proof. To save space, only the idea of the proof will be given. Relation (19) and (20) follow by the fact that $z^{-1}\phi_i(z) = \phi_i(z)z^{-1}$ so one has $\mathscr{H}_s z^{-1}(\lambda)\tilde{\phi}_i(\lambda) = \tilde{\phi}_i(\lambda)\mathscr{H}_s z^{-1}(\lambda)$. The rest of the relations follow by a comparison of the coefficients of λ^{-1} in the expansion of $\mathscr{H}_s z^{-1}$ given by (14) and those given by (12).

Proposition 14. *The matrices $\{P_i,\ Q_i\,|\, i = 1,\dots,n_b\}$ are the solutions of the following set of Sylvester equations*:

$$A_bP_iA_b^{\mathrm{T}} + B_be_i^{\mathrm{T}}A_b^{\mathrm{T}} = P_i, \tag{23}$$

$$A_bQ_iA_b^{\mathrm{T}} + B_be_i^{\mathrm{T}}C_b^{\mathrm{T}}B_b^{\mathrm{T}} + A_bP_iC_b^{\mathrm{T}}B_b^{\mathrm{T}} = Q_i. \tag{24}$$

Proof. Since $\mathscr{B}$ is an inner function, one has

$$A_bA_b^{\mathrm{T}} + B_bB_b^{\mathrm{T}} = \mathbb{I}, \qquad B_bD_b + A_bC_b^{\mathrm{T}} = \mathbb{O}.$$

By multiplying from left the first identity by P_i and using (19) one can obtain (23). By multiplying from right by A the transpose of (20) one has

$$A_bA_b^{\mathrm{T}}Q_i - A_bQ_iA_b^{\mathrm{T}} = A_bP_iC_b^{\mathrm{T}}B_b^{\mathrm{T}} - A_bC_b^{\mathrm{T}}B_b^{\mathrm{T}}P_i.$$

By multiplying from right by B_b (22) and using the first identity one can obtain

$$Q_i - A_bA_b^{\mathrm{T}}Q_i = B_be_i^{\mathrm{T}}C_b^{\mathrm{T}}B_b^{\mathrm{T}} - B_be_i^{\mathrm{T}}A_b^{\mathrm{T}}D_b.$$

Putting these two relations together and considering the second identity results in (24). □

The expansion coefficients of zG in the generalized orthonormal basis can be computed using the fact that

$$z(z\mathbb{I} - A)^{-1} = \mathbb{I} + A(z\mathbb{I} - A)^{-1}.$$

Using these theorems one can build up the Hankel matrices $H_{\mathscr{G}}^r$ and $\tau(H_{\mathscr{G}}^r)$ in a computational efficient way by starting from the elements of the minimal balanced realization of the inner function $\mathscr{B}$ and the coefficients of the expansion in the generalized orthonormal basis. The transformation matrices $T_{(+)}^{11}$ can be computed by a fast algorithm as well.

8. Examples

Let us consider now a SISO system with transfer function

$$f(z) = \frac{1}{z - 0.9} + \frac{0.1}{z^2 - 1.8z + 0.81} + \frac{z + 0.5}{z^2 - 1.6z + 0.8}$$

and poles $a = [0.9\ \ 0.8 + 0.4i\ \ 0.8 - 0.4i]$. Consider the inner function $\mathscr{B}$ with these poles and its balanced realization:

$$A_0 = \begin{bmatrix} 0.9640 & 0.2640 & -0.0322 \\ -0.2540 & 0.8782 & -0.4053 \\ -0.0567 & 0.2872 & 0.6578 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.6940 \end{bmatrix},$$

$C_0 = [-0.0546\ \ 0.2768\ \ 0.6340]$ and $D_0 = -0.72$.

The expansion coefficients of the transfer function in the basis generated by these settings are:

$$\begin{bmatrix} -3.5573 & -0.7004 \\ -5.6557 & -0.3148 \\ 3.3282 & 0.6199 \end{bmatrix}.$$

Therefore, if one uses the algorithm presented in Section 5, the dimension of the state-space realization will be 6 instead of 4, the degree of the minimal realization. This shows that even in the SISO case one can obtain non-minimal state-space representations using this algorithm:

$$A_1 = \begin{bmatrix} 0.964 & 0.264 & -0.032 & 0 & 0 & 0 \\ -0.254 & 0.878 & -0.405 & 0 & 0 & 0 \\ -0.056 & 0.287 & 0.657 & 0 & 0 & 0 \\ 0.000 & 0.000 & 0.000 & 0.964 & 0.264 & -0.032 \\ 0.000 & 0.000 & 0.000 & -0.254 & 0.878 & -0.403 \\ 0.037 & 0.192 & 0.440 & -0.056 & 0.287 & 0.658 \end{bmatrix},$$

$B_1^{\mathrm{T}} = [0.000 \quad 0.000 \quad 0.694 \quad 0 \quad 0 \quad -0.499]$,

$C_1 = [-3.557 \quad -5.655 \quad 3.328 \quad -0.700 \quad -0.314 \quad 0.619]$.

Using the generalized Ho–Kalman algorithm one can compute the Hankel matrices

$$H = \begin{pmatrix} 4.418 & 0.719 & -6.339 & 1.324 & -0.419 & -0.807 \\ 0.621 & -7.112 & -6.119 & 0.595 & -0.188 & -0.362 \\ -8.942 & 7.177 & 2.049 & -1.172 & 0.371 & 0.714 \\ 1.324 & -0.419 & -0.807 & 0 & 0 & 0 \\ 0.595 & -0.188 & -0.362 & 0 & 0 & 0 \\ -1.172 & 0.371 & 0.714 & 0 & 0 & 0 \end{pmatrix},$$

$$\tau(H) = \begin{pmatrix} 4.652 & 2.078 & -4.700 & 1.192 & -0.377 & -0.726 \\ -1.081 & -3.923 & -6.321 & 0.535 & -0.169 & -0.326 \\ -6.792 & 7.743 & 4.346 & -1.054 & 0.334 & 0.643 \\ 1.192 & -0.377 & -0.726 & 0 & 0 & 0 \\ 0.535 & -0.169 & -0.326 & 0 & 0 & 0 \\ -1.054 & 0.334 & 0.643 & 0 & 0 & 0 \end{pmatrix}.$$

The transformation matrices are

$$T = \begin{pmatrix} 0.000 & -0.022 & -0.110 & -0.226 & -0.333 & -0.407 \\ 0.000 & -0.281 & -0.426 & -0.435 & -0.337 & -0.173 \\ 0.694 & 0.456 & 0.220 & 0.029 & -0.093 & -0.139 \\ 0.000 & 0.016 & 0.069 & 0.109 & 0.116 & 0.097 \\ 0.000 & 0.202 & 0.183 & 0.066 & -0.047 & -0.102 \\ -0.499 & -0.023 & 0.189 & 0.192 & 0.077 & -0.062 \end{pmatrix},$$

$$T_+ = \begin{pmatrix} -0.054 & -0.158 & -0.274 & -0.370 & -0.425 & -0.430 \\ 0.276 & 0.410 & 0.406 & 0.296 & 0.125 & -0.057 \\ 0.634 & 0.306 & 0.040 & -0.129 & -0.193 & -0.164 \\ 0.039 & 0.090 & 0.116 & 0.110 & 0.085 & 0.058 \\ -0.199 & -0.173 & -0.053 & -0.060 & 0.113 & 0.097 \\ -0.456 & 0.058 & 0.240 & 0.194 & 0.042 & -0.105 \end{pmatrix}.$$

The system given by the generalized Ho–Kalman algorithm is

$$A = \begin{bmatrix} 0.8639 & 0.2620 & 0.0433 & -0.0272 \\ -0.2620 & 0.7371 & 0.3207 & 0.0708 \\ 0.0433 & -0.3207 & 0.9405 & -0.0166 \\ 0.0272 & 0.0708 & 0.0166 & 0.8586 \end{bmatrix},$$

$B^{\mathrm{T}} = [-1.1933 \quad -1.3967 \quad -0.3804 \quad 0.2185]$,

$C = [-1.8153 \quad 1.1195 \quad 0.1475 \quad -0.2526]$.

A comparison between the coefficients and poles of the resulting transfer function for the non-minimal and minimal algorithm is given below:

$$G_1 = \frac{2.0z^5 - 6.90z^4 + 9.51z^3 - 6.379z^2 + 1.968z - 0.188}{z^6 - 5.0z^5 + 10.73z^4 - 12.64z^3 + 8.6176z^2 - 3.2256z + 0.5184},$$

$$r_1 = [0.8434 \pm 0.0913i, \quad 0.1633, \quad 0.8 \pm 0.4i],$$

$$p_1 = [0.8 \pm 0.4i, \quad 0.9, \quad 0.9, \quad 0.8 \pm 0.4i] \text{ and}$$

$$G_{\text{Ho-Kalm}} = \frac{2.0z^3 - 3.70z^2 + 1.99z - 0.235}{z^4 - 3.40z^3 + 4.49z^2 - 2.736z + 0.648},$$

$$r = [0.8434 \pm 0.0913i, \quad 0.1633], p = [0.80 \pm 0.40i, \quad 0.90, \quad 0.90].$$

Note the zero-pole cancellation that occurs at the non-minimal representation.

9. Conclusion

Given the expansion coefficients of a rational transfer function G in a generalized orthonormal basis generated by an inner function $\mathscr{B}$, one can construct a state-space representation starting from the balanced realization of $\mathscr{B}$, and following the rules known for the composition of the state-space representation of the systems, but that representation is not minimal even in the SISO case, as was shown by an example, in general. However, if pole-zero cancellation does not occur for the transfer function obtained forming the common denominator and doing all the computations, that representation is minimal for the SISO case, but not for MIMO systems, in general. Therefore, one needs an algorithm to construct a minimal representation. This paper gives a generalization of exact realization theory for expansions using generalized orthonormal basis functions. The resulting realization theory will be the same as the one obtained by application of the Ho–Kalman algorithm for the standard Fourier expansion. In the standard Markov parameter case the algorithm that solves the partial realization problem is basically the same as for the infinite data case. The generalized algorithm can be applied in an approximative fashion as it is the case with the standard Ho–Kalman algorithm but the consequences of the method when compared to the standard partial realization algorithm still have to be further explored. Results in this direction can be found in Heuberger, Szabó, de Hoog, Van den Hof and Bokor (1999).

References

Bokor, J. (1997). Approximate identification for robust control. *Preprints robust control design 2nd IFAC symposium*, Budapest, Hungary, pp. 25–36.

Den Brinker, A. C., Benders, F. P. A., & Oliveira e Silva, T. A. M. (1996). Optimality conditions for truncated Kautz series. *IEEE Transactions on Circuits and Systems II*, *43*, 117–122.

Fischer, B. (1997). *Two topics in system identification*. Licentiate thesis, control engineering group, Department of Computer Science and Electrical Engineering, Lulea University of Technology, Sweden.

Frazho, A. E., & Foias, C. (1990). *The commutant lifting approach to interpolation problems*. Basel: Birkhauser.

Hakvoort, R. G., & Van den Hof, P. M. J. (1997). Identification of probabilistic uncertainty regions by explicit evaluation of bias and variance errors. *IEEE Transactions on Automatic Control*, *42*, 1516–1528.

Heuberger, P.S.C. (1991). *On approximate system identification with system based orthonormal functions*. Dr. dissertation, Delft University of Technology, The Netherlands, 1991.

Heuberger, P.S.C., & Bosgra, O.H. (1990). Approximate system identification using system based orthonormal functions. *Proceedings of the 29th IEEE Conference in Decision and Control*. Honolulu, HI (pp. 1086–1092).

Heuberger, P.S.C., Szabó, Z., de Hoog, T.J., Van den Hof, P.M.J., & Bokor, J. (1999). Realization algorithms for expansions in generalized orthonormal basis functions. *Proceedings 14th IFAC world congress*, vol. H. Beijing, P.R. China (pp. 385–390).

Heuberger, P. S. C., & Van den Hof, P. M. J (1995b). The Hambo transform: a signal and system transform induced by generalized orthonormal basis functions. *Selected Topics in Identification Modelling and Control*, *8*, 85–94.

Heuberger, P. S. C., Van den Hof, P. M. J., & Bosgra, O. H. (1995). A generalized orthonormal basis for linear dynamical systems. *IEEE Transactions on Automatic Control*, *40*, 451–465.

Ho, B. L., & Kalman, R. E. (1966). Effective construction of linear state-variable models from input/output functions. *Regelungstechnik*, *14*(12), 545–592.

Kautz, W. H. (1954). Transient synthesis in the time domain. *IRE Transactions on Circuit Theory*, *CT-1*, 29–39.

Kung, S. (1978). A new identification and model reduction algorithm via singular value decompositions. *Proceedings 12th asilomar conference on circuits systems and computers*. Pacific Grove, CA (pp. 705–714).

Lee, Y. W. (1960). *Statistical theory of communication*. New York: Wiley.

Malmquist, F. (1925). Sur la détermination d'une classe de fonctions analytiques par leurs valeurs dans un ensemble donné de points. Comptes Rendus du Sixiémme Congréss des mathématiciens scandinaves. Kopenhagen, Denmark (pp. 253–259).

Moore, B. C. (1981). Principal component analysis in linear systems: controllability, observability and model reduction. *IEEE Transactions on Automataic Control*, *26*, 17–32.

Nalbantoglu, V., Bokor, J., & Balas, G. (1997). System identification with orthonormal basis functions: an application to flexible structures. *Preprints of the 2nd IFAC symposium on robust control design*. Budapest, Hungary (pp. 423–428).

Ninness, B., & Gómez, J.C. (1996). Asymptotic analysis of mimo system estimates by the use of orthonormal basis functions. *Preprints 13th triennial IFAC world congress*. San Francisco, CA (pp. 363–368).

Ninness, B. M., & Gustafsson, F. (1997). A unifying construction of orthonormal bases for system identification. *IEEE Transactions on Automatic Control*, *42*, 515–521.

Ninness, B., Hjalmarsson, H., & Gustafsson, F. (1999). The fundamental role of general orthonormal bases in system identification. *IEEE Transactions on Automatic Control*, *44*, 1384–1407.

Oliveira e Silva, T. (1995). Optimality conditions for truncated Kautz networks with two periodically repeating complex conjugate poles. *IEEE Transactions on Automatic Control*, *40*, 342–346.

Oliveira e Silva, T. (1996). A *N*-width result for the generalized orthonormal basis function model. *Preprints 13th triennial IFAC world congress*. San Francisco, CA (pp. 375–380).

Roberts, R. A., & Mullis, C. T. (1987). *Digital signal processing*. Reading, MA: Addison Wesley Publ. Comp.

Rosenblum, M., & Rovnyak, J. (1985). *Hardy classes and operator theory*. New York: Oxford University Press.

Sarason, D. (1967). Generalized interpolation in $\mathscr{H}_\infty$. *Transactions of the American Mathematical Society*, *127*, 179–203.

Schipp, F., & Bokor, J. (1998). Approximate identification in Laguerre and Kautz Bases. *Automatica*, *34*(4), 463–468.

Schipp, F., Gianone, L., Bokor, J., & Szabó, Z. (1996). Identification in generalized orthogonal basis — a frequency domain approach. *Preprints 13th triennial IFAC world congress*. San Francisco, CA (pp. 387–392).

Szabó, Z., & Bokor, J. (1997). Minimal state space realization for transfer functions represented by coefficients using generalized orthonormal basis. *Proceedings of 36th IEEE CDC*. San Diego, CA (pp. 169–175).

Szabó, Z., Schipp, F., & Bokor, J. (1999). Identification of rational approximate models in H^∞ using generalized orthonormal basis. *IEEE Transactions on Automatic Control*, *44*, 153–158.

Van den Hof, P. M. J., Heuberger, P. S. C., & Bokor, J. (1995). System identification with generalized orthonormal basis functions. *Automatica*, *31*, 1821–1834.

Van Overschee, P., & de Moor, B. (1996). *Subspace identification for linear systems*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

De Vries, D. K., & Van den Hof, P. M. J. (1998). Frequency domain identification with generalized orthonormal basis functions. *IEEE Transactions on Automatic Control*, *43*, March 1998.

Wahlberg, B. (1991). System identification using Laguerre models. *IEEE Transactions on Automatic Control*, *36*, 551–562.

Wahlberg, B. (1994a). System identification using Kautz models. *IEEE Transactions on Automatic Control*, *39*, 1276–1282.

Wahlberg, B. (1994b). Laguerre and Kautz models. *Preprints 10th IFAC symposium system identification*. Copenhagen, Denmark (pp. 1–12).

Wahlberg, B., & Mäkilä, P. M. (1996). On approximation of stable linear dynamical systems using Laguerre and Kautz functions. *Automatica*, *32*, 693–708.

Walsh, J. (1935). *Interpolation and approximation by rational functions in the complex domain*. New York: American Mathematical Society.

Ward, N. F., & Partington, J. R. (1996). Robust identification in the disc algebra using rational wavelets and orthonormal basis functions. *International Journal of Control*, *64*, 409–423.

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