

# Boundary Actuation Structure of Linearized Two-Phase Flow

S. Djordjevic, O.H. Bosgra, P.M.J. Van den Hof, D. Jeltsema

**Abstract**—In this paper, we introduce a two-phase flow model which describes motion of two incompressible fluids. The proposed model is governed by nonlinear partial differential algebraic equations (PDAEs) with open initial-boundary conditions. The well-posedness of the boundaries is analyzed using characteristic curves, which leads to development of a boundary actuation structure that is suitable for a control design. A particular emphasis is placed on a possible coordinate transformation that can reduce the problem to a set of partial differential equations (PDEs) without algebraic constraint. The analysis is presented using linearized model equations and tested on a numerical example for a quasi steady-state situation.

## I. INTRODUCTION

Two-phase flow is of large relevance for many industrial applications ranging from chemical industry to oil production, and nuclear engineering. It is generally understood as a simultaneous flow of two different phases which interact via an infinitesimal thin interface. In most cases, the phases are simply referred to as gas/vapor, liquid, or solid state. For a long time, the analysis of two-phase flow processes was limited to mostly empirical correlations, or to largely simplified engineering models [1], [2], [3]. In recent years, due to the wide range of applications, a large effort has been spent on analysis of fluid dynamics in two-phase systems, and on development of related numerical simulation methods [5], [6], [7]. The model equations for two-phase flow in the fluid dynamics approach are usually derived by averaging the equations describing the fluid dynamics of the single-phase flow and couple them via different interactive terms. The result of this approach is a set of equations having a similar structure as the single-phase flow (i.e., Navier-Stokes equations) from which they originated [4].

From a control perspective, the fluid models received considerable attention over the last few years [10]. As an initial study for the introduction of control theory into the fluid mechanical setting, the linearized Navier-Stokes equations have been considered [11], [12]. The linearized model equations govern small perturbations around laminar flow which can be influenced from the boundaries via different boundary control laws. In literature, several boundary control approaches have appeared [14], [15], [17] in which the manipulated boundary actuation is used to stabilize/destabilize

the flow. Although there is a number of complex issues underlying the fluid dynamics, the stabilization of laminar flow described by linearized Navier-Stokes equations shows a promising control design for practical applications of single-phase flow.

In this paper, our objective is to introduce the second phase in the control design of fluid mechanical systems, and to develop a boundary actuation structure for two-phase flow that guarantees well-posedness of the underlying equations. The problem of boundary actuation subject to input and state constraints is studied on the linearized model equations as a starting point for designing a boundary control law. The linearization technique is similar to the linearization technique presented in [17]. We also introduce different coordinate transformations which eventually lead to a PDE model with decoupled directional derivatives. The change of coordinates and boundary control laws for a class of quasi-linear PDEs is first introduced in [18], and recently extended for networks of hyperbolic PDEs [19]. First, we use a standard coordinate transformation to rewrite the original linearized PDAE problem as a linear PDE problem that involves only the dynamical part of the system equations. Second, a further modal decomposition is applied in order to decouple the directional derivatives of PDE model, and design a well-posed boundary structure. In order to demonstrate the ability of the boundary actuation in enforcing well-posedness of the problem statement, simulations are carried out using a spatially uniform steady-state solution (i.e., quasi steady-state regime) as an equilibrium point for the linearization.

The paper is organized as follows. First, the model equations of one-dimensional two-phase flow is presented. Thereafter, the well-posedness of the model equations is studied. Further, the linearized model equations are presented, and finally the boundary actuation structure is introduced using different coordinate transformations.

## II. THE DYNAMICAL MODEL

Here, we consider a one-dimensional incompressible two-phase flow in a vertical pipe with an interfacial pressure and a drag force as the coupling terms between the two phases: gas and liquid phase [8]. The matrix form of the model equation is given as

$$\mathbf{E} \frac{\partial \Phi}{\partial t} + \mathbf{A}(\Phi) \frac{\partial \Phi}{\partial x} = \mathbf{c}(\Phi), \quad (1)$$

where  $\Phi = \begin{bmatrix} \alpha_g \\ v_g \\ v_l \end{bmatrix}$  is the vector of flowing variables,

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$\alpha_g$  is the volume fraction of the gas phase,  $v_g$  is the velocity of the gas phase, and  $v_l$  is the velocity of the liquid phase. The matrices  $\mathbf{E}$  and  $\mathbf{A}(\Phi)$  are the system matrices, and  $\mathbf{c}(\Phi)$  is the coupling force vector,

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho_g & -\rho_l \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}(\Phi) = \begin{bmatrix} v_g & \alpha_g & 0 \\ C_p \rho_l (v_g - v_l)^2 & \rho_g v_g & -\rho_l v_l \\ v_g - v_l & \alpha_g & 1 - \alpha_g \end{bmatrix},$$

$$\mathbf{c}(\Phi) = \begin{bmatrix} 0 \\ -(\rho_g - \rho_l)g - (v_g - v_l) \left( \frac{\beta}{\alpha_g} + \frac{\beta}{\alpha_l} \right) \\ 0 \end{bmatrix},$$

where  $\rho_g$  is the density of gas phase,  $\rho_l$  is the density of liquid phase,  $C_p$  is the interfacial pressure coefficient, and  $g$  is the gravitational acceleration. Since the matrix  $\mathbf{E}$  is a singular matrix, the proposed model for two-phase flow is a partial differential algebraic equations (PDAEs) model. For the model derivation, we refer to the appendix.

The drag force in  $\mathbf{c}(\Phi)$  is defined by the following closure equation

$$\beta = \frac{3C_d}{4d_b} \alpha_g \alpha_l \rho_l |v_g - v_l|,$$

where  $C_d$  is the drag coefficient, and  $d_b$  is the diameter of a single bubble (i.e., particle of a discrete phase). The sum of the volume fractions of gas and liquid phase in an observed volume equals one, i.e.,  $\alpha_g + \alpha_l = 1$ . Thus, the volume fraction of the gas phase can range between  $0 \leq \alpha_g \leq 1$ . If  $\alpha_g = 0$  then only the liquid phase is present, and if  $\alpha_g = 1$  then only the gas phase is present. In these situations, the model equations are in agreement with the equations for the single-phase flow.

#### A. Well-posedness of the model equations

The nature of the solutions to (1) is basically characterized by polynomial of the system matrices ( $\mathbf{A}(\Phi)$ ,  $\mathbf{E}$ ). For PDAE models, the characteristic polynomial is of lower order than the order of the system. This means that the eigenvalues corresponding to the algebraic part have infinite many solutions, and the eigenvalues with the finite solutions correspond to the dynamical part. For the model equations written as (1), the eigenvalue analysis shows that the system has one infinite eigenvalue and two finite eigenvalues which are solutions of the following characteristic polynomial

$$\det(\lambda \mathbf{E} - \mathbf{A}(\Phi)) = a_1 \lambda^2 + a_2 \lambda + a_3.$$

The polynomial coefficients can be grouped as

$$\begin{aligned} a_1 &= (-\alpha_g \rho_l - \rho_g + \rho_g \alpha_g), \\ a_2 &= (2 \rho_g v_g - 2 v_g \rho_g \alpha_g + 2 \alpha_g \rho_l v_l), \\ a_3 &= -C_p \rho_l (v_g - v_l)^2 \alpha_g^2 \\ &\quad + \left( \rho_g v_g^2 - \rho_l v_l^2 + C_p \rho_l (v_g - v_l)^2 \right) \alpha_g - \rho_g v_g^2. \end{aligned}$$

The discriminant of the given polynomial is defined by the given coefficients  $a_1$ ,  $a_2$ , and  $a_3$  as

$$D_c = a_2^2 - 4a_1 a_3.$$

If  $D_c > 0$  then the system of model equations is said to be hyperbolic. The eigenvalues of the hyperbolic equations are real and distinct. If  $D_c = 0$ , the system of model equations is parabolic with real repeated eigenvalues. If, however,  $D_c < 0$ , the system of model equations is elliptic with complex eigenvalues. The elliptic systems are ill-posed, whereas parabolic and hyperbolic systems are well-posed with a stable and unique solution [9]. For the system ( $\mathbf{A}(\Phi)$ ,  $\mathbf{E}$ ), the discriminant  $D_c$  can be rewritten as

$$D_c = D \left( 1 - \left( \frac{\rho_l \alpha_g + \alpha_l \rho_g}{\rho_g} \right) C_p \right) \quad (2)$$

where  $D = \alpha_g \rho_g \rho_l (v_g - v_l)^2 (-1 + \alpha_g)$ . Since  $\alpha_g < 1$ , for all values of  $v_g$  and  $v_l$   $D < 0$ . This means that the well-posedness of the model equations strongly depends on the interfacial coefficient  $C_p$ . The critical  $C_p$ , for which the discriminant  $D_c$  is equal to zero and the system is parabolic, can be obtained from the system parameters  $\rho_g$  and  $\rho_l$ , and volume fraction  $\alpha_g$

$$C_p = \frac{\rho_g}{\alpha_g \rho_l + \rho_g \alpha_l}.$$

The interfacial pressure constant  $C_p$  for the air/water system is reported to be between 0.25 and 0.5 [8] which assure the hyperbolic region for a wide range of gas fractions.

Many of the present advanced models use different interfacial coupling terms with complete physical background and closure equations [5]. Most of the closure relations are of empirical nature, or include some heuristic elements which can not be deduced completely from first principles. The correct formulation of the basic two-phase flow models and the appropriate form of the closure laws have been widely discussed during the past [9], [6], and, up to now, a commonly agreed approach has not been achieved. A specific concern has been that most models presently used in the large computer codes are based on governing equations which have complex eigenvalues and, therefore, do not represent a mathematically well-posed problem [7]. Nevertheless, there is a common agreement that the pure transport of the governing system of equations should be of hyperbolic nature with real distinct eigenvalues [8]. The finite eigenvalues for the proposed model are

$$\lambda_1(\Phi) = \frac{\alpha_g \rho_l v_l + \alpha_l \rho_g v_g}{\alpha_g \rho_l + \alpha_l \rho_g} + \sqrt{D_c}, \quad (3)$$

$$\lambda_2(\Phi) = \frac{\alpha_g \rho_l v_l + \alpha_l \rho_g v_g}{\alpha_g \rho_l + \alpha_l \rho_g} - \sqrt{D_c}, \quad (4)$$

which results in the characteristic velocities of the gas/liquid system and gives real distinct eigenvalues for  $0.25 < C_p < 0.5$ .

### III. LINEARIZED MODEL EQUATIONS

This section gives a short overview of the linearization of the model equations which are used for developing actuation structure of two-phase flow model. The same linearization technique is also used for deriving the linearized Navier-Stokes equations [17].

Suppose that  $\bar{\Phi}$  is a steady-state solution and  $\Phi'$  is a small perturbation around it, then the flow variable  $\Phi$  can be written as

$$\Phi = \bar{\Phi} + \Phi'. \quad (5)$$

Inserting (5) into the model equations (18)-(21), we obtain the linearized system of two-phase flow equations written in a matrix form as

$$\mathbf{E} \frac{\partial(\bar{\Phi} + \Phi')}{\partial t} + \mathbf{A}(\bar{\Phi} + \Phi') \frac{\partial(\bar{\Phi} + \Phi')}{\partial x} = \mathbf{c}(\bar{\Phi} + \Phi'). \quad (6)$$

Using the Taylor expansion for the time evolution of the model equations, the following general linearized model equations are obtained

$$\mathbf{E} \frac{\partial \Phi'}{\partial t} + \mathbf{A}(\bar{\Phi}) \frac{\partial \Phi'}{\partial x} + \mathbf{A}'(\bar{\Phi}) \frac{\partial \bar{\Phi}}{\partial x} = \mathbf{c}(\Phi') \quad (7)$$

where

$$\mathbf{A}'(\bar{\Phi}) \frac{\partial \bar{\Phi}}{\partial x} = \frac{\partial \mathbf{A}(\bar{\Phi})}{\partial \bar{\Phi}} \frac{\partial \bar{\Phi}}{\partial x} \Big|_{\bar{\Phi}} \frac{\partial \Phi'}{\partial x}, \quad \mathbf{c}'(\Phi') = \frac{\partial \mathbf{c}(\Phi)}{\partial \Phi} \Big|_{\bar{\Phi}} \Phi'$$

and

$$\mathbf{A}(\bar{\Phi}) = \begin{bmatrix} \bar{v}_g & \bar{\alpha}_g & 0 \\ C_p \rho_l (\bar{v}_g - \bar{v}_l)^2 & \rho_g \bar{v}_g & -\rho_l \bar{v}_l \\ \bar{v}_g - \bar{v}_l & \bar{\alpha}_g & 1 - \bar{\alpha}_g \end{bmatrix},$$

$$\mathbf{A}'(\bar{\Phi}) = \begin{bmatrix} v_g' & \alpha_g' & 0 \\ 2C_p \rho_l (\bar{v}_g - \bar{v}_l) (v_g' - v_l') & \rho_g v_g' & -\rho_l v_l' \\ v_g' - v_l' & \alpha_g' & -\alpha_g' \end{bmatrix}$$

$$\mathbf{c}'(\Phi') = \begin{bmatrix} 0 \\ 2\frac{3}{4} \frac{C_d}{d_b} \rho_l (\bar{v}_g - \bar{v}_l) (v_g' - v_l') \\ 0 \end{bmatrix}.$$

Note that the time evolution of  $\bar{\Phi}$  is

$$\mathbf{E} \frac{\partial \bar{\Phi}}{\partial t} + \mathbf{A}(\bar{\Phi}) \frac{\partial \bar{\Phi}}{\partial x} = \mathbf{c}(\bar{\Phi}), \quad (8)$$

and  $\frac{\partial \Phi'}{\partial x} \cdot \frac{\partial \Phi'}{\partial x}$  and  $\Phi' \cdot \frac{\partial \Phi'}{\partial x}$  are very small values in the vicinity of the steady-state solution. Equation (7) represents the linearized model where the steady-state solution can vary with respect to space according to  $\frac{\partial \bar{\Phi}}{\partial x}$ .

For the quasi steady-state regime where  $\bar{\Phi} = const$ , the linearized model reads as

$$\mathbf{E} \frac{\partial \Phi'}{\partial t} + \mathbf{A}(\bar{\Phi}) \frac{\partial \Phi'}{\partial x} = \mathbf{c}(\Phi'). \quad (9)$$

#### A. Boundary actuation

The behavior of the hyperbolic-like equations is usually described as a wave propagation with the speed defined by its eigenvalue. The fundamental idea associated with the wave propagation is the notion of characteristic curves in space-time domain along which these waves propagate. Using the linearized model equation (7), we can set

$$\mathbf{E} \frac{d\Phi'}{dt} = -\mathbf{A}(\bar{\Phi}) \frac{\partial \bar{\Phi}}{\partial x} + \frac{\partial \mathbf{c}(\Phi)}{\partial \Phi} \Big|_{\bar{\Phi}} \Phi',$$

along the curves defined by

$$\mathbf{E} \frac{dx}{dt} = \mathbf{A}(\bar{\Phi}).$$

As can be seen, the directional derivatives correspond to the eigenvalues of  $(\mathbf{A}(\bar{\Phi}), \mathbf{E})$  along the characteristic curves defined by two ordinary differential equations

$$\frac{dx_1}{dt} = \lambda_1(\bar{\Phi}), \quad \text{and} \quad \frac{dx_2}{dt} = \lambda_2(\bar{\Phi}).$$

The physical interpretation of the characteristic curves analysis is rather straightforward. The wave speeds of the propagation correspond to the characteristic velocities  $\lambda_1(\bar{\Phi})$  and  $\lambda_2(\bar{\Phi})$  and carry the void fraction of phases injected at the boundaries. When we inject the gas/liquid phase at inlet/outlet, the phases will propagate upwards or downwards according to the sign of  $\xi_1$  and  $\xi_2$  Riemann invariant. If we have  $0 < \lambda_1(\bar{\Phi}) < \lambda_2(\bar{\Phi})$ , then both characteristic families of curves propagate as  $\xi_1 = x_1 - \lambda_1(\bar{\Phi})t$  and  $\xi_2 = x_2 - \lambda_2(\bar{\Phi})t$  in  $(x, t)$  plane. This means that the boundary conditions must be specified at  $\Phi'(0, t)$ . However, if the eigenvalues are negative the characteristic curves are defined as  $\xi_1 = x_1 + \lambda_1(\bar{\Phi})t$  and  $\xi_2 = x_2 + \lambda_2(\bar{\Phi})t$ , and the boundary conditions must be defined at  $\Phi'(L, t)$ . It is rather common to have positive and negative eigenvalues, where for the positive eigenvalues the boundary conditions have to be specified at  $\Phi'(0, t)$  whereas for the negative eigenvalues the boundary conditions should be specified at  $\Phi'(L, t)$ . This type of flow is known as a *fluvial*, or *subcritical* flow. Since we deal with PDAE model, i.e., descriptor system along the characteristic curves, we will introduce similarity transformations which can eliminate the algebraic part of the model equations and decouple the wave velocities.

#### B. Coordinate transformations

In order to develop a boundary actuation structure, we utilize coordinate transformations which can transform the system from one coordinate to another. First, the system matrix  $\mathbf{E}$  is diagonalized, so that in the new coordinate system the linearized model can be reduced for the algebraic equation

$$\mathbf{E}^D = \mathbf{E}T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The coordinate transformations  $T$  that can map two spaces  $\Psi'$  and  $\Phi'$  can be regarded as  $\Phi' = T\Psi'$ , where

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho_g^{-1} & \rho_l \\ 0 & 0 & \rho_g \end{bmatrix}, \quad \Psi' = \begin{bmatrix} \alpha'_g \\ \rho_g v'_g - \rho_l v'_l \\ \frac{v'_l}{\rho_g} \end{bmatrix}.$$

This transformation allows the elimination of the algebraic part of PDAEs model which leads to a set of PDEs. The algebraic equation in the new coordinate system corresponds to  $\frac{\partial \Psi}{\partial t} = 0$  according to the matrix  $\mathbf{E}^D$ , and the model equations in the new coordinate system can be written as

$$\mathbf{E}T \frac{\partial \Psi'}{\partial t} + \mathbf{A}(\bar{\Psi})T \frac{\partial \Psi'}{\partial x} + \mathbf{A}(\Psi')T \frac{\partial \bar{\Psi}}{\partial x} = FT\Psi' \quad (10)$$

After the elimination of the algebraic equation in (10), the directional derivatives of the resulting set of PDEs can be further decoupled by diagonalizing  $\mathbf{A}(\bar{\Psi})T$  in (10)

$$\begin{aligned} \frac{\partial}{\partial t} \begin{bmatrix} W'_1 \\ W'_2 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} W'_1 \\ W'_2 \end{bmatrix} \\ = \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix} \begin{bmatrix} W'_1 \\ W'_2 \end{bmatrix}. \end{aligned} \quad (11)$$

Equation (11) describes a linearized system of PDEs with decoupled wave propagation. Now, the boundary conditions can be fully determined by the sign of the eigenvalues

$$\frac{\partial W'_i}{\partial t} + \lambda \frac{W'_i - W'_{i-1}}{\Delta x} = cW'_i \quad \text{for } \lambda < 0, \quad (12)$$

$$\frac{\partial W'_i}{\partial t} + \lambda \frac{W'_{i+1} - W'_i}{\Delta x} = cW'_i \quad \text{for } \lambda > 0, \quad (13)$$

where index  $i$  denotes the point in space. The top boundary actuation  $W'(t, L)$  of the linearized model is possible for  $\lambda > 0$ , whereas the bottom boundary actuation  $W'(t, 0)$  is possible for  $\lambda < 0$ . The two solutions  $W'_1(t, x)$  and  $W'_2(t, x)$  of the system (11) are constant along the characteristic lines  $\lambda_1$  and  $\lambda_2$ . The equations are coupled according to the right hand side of (11).

### C. Operational regimes

In general, a steady-state solution is obtained by setting time derivatives to zero. For the proposed model, the steady-state solution can be obtained from the following equation

$$\mathbf{A}(\Phi) \frac{\partial \Phi}{\partial x} = \mathbf{c}(\Phi).$$

In this heterogeneous regime, the steady state solution can vary with respect to space. For the homogeneous regime, the gas distribution can be assumed to be a constant value in the entire space  $\bar{\Phi} = \text{const}$ . According to the model equations, the steady-state solution of the homogeneous regime can be obtained from  $\mathbf{c}(\Phi) = 0$ . This means that in the homogeneous steadiness the gravity force and the drag force are in balance. If we compare this result with (16) and (17) (see Appendix), we can recover the pressure in the steadiness as

$$\frac{dp}{dx} = -(\alpha_g \rho_g + \alpha_l \rho_l)g.$$

As can be expected, the pressure in the homogeneous steadiness is equal to hydrostatic pressure which gives a physical interpretation of the quasi steady-state solution.

In order to predict the velocities in the quasi steady-state, we introduce a slip velocity  $v_{slip}$  as a difference between the velocity of the gas phase and the velocity of the liquid phase

$$v_{slip} = v_g - v_l = \sqrt{\frac{4(\rho_l - \rho_g)gd_b}{3C_d \rho_l}}.$$

Due to the compensating volumetric fluxes across an observed volume  $\alpha_g v_g + \alpha_l v_l = 0$ , the velocities in the homogeneous steady-state equals  $v_g = \alpha_l v_{slip}$ , and  $v_l = -\alpha_g v_{slip}$ .

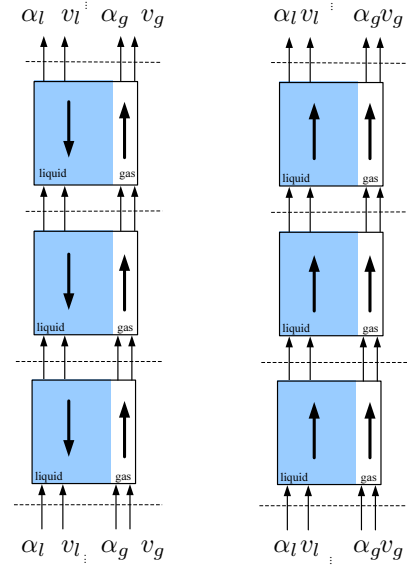


Fig. 1. Operational regimes: counter-current flow (left) and co-current flow (right).

Depending on the flow direction, the liquid velocity can have a positive or a negative sign, which leads to different flow regimes. Fig. 1 illustrates two extreme regimes: counter-current, and co-current regime in a vertical column divided on few macroscopic observation volumes. In principle for a real flow, there is a certain amount of fluids moving downwards, and certain amount that moves upwards. This circulation phenomena for the one-dimensional flow can be observed as a sum of the upwards and downwards flow. Which flow is dominant depends on the gas and liquid injection, and boundary actuation. In the following section, we describe the use of these equations for a control purpose.

## IV. NUMERICAL EXAMPLE

In this section, we consider a numerical example for the linearized model equations around the previously described quasi steady-state solution. The fluid properties and the parameter values are given in Table I. The flow variables in the quasi steady-state are obtained from the constant gas distribution in the entire space divided into  $N = 10$  observed volumes. For the constant gas fraction  $\bar{\alpha}_g = 0.1$ , the slip

velocity is  $v_{slip} = 0.17$ , the gas velocity is  $\bar{v}_g = 0.155$ , and the liquid velocity  $\bar{v}_l = -0.017$ . According to the sign of the steady velocities, the phases move in the opposite direction (see Fig.1 counter-current flow).

TABLE I  
FLUID PROPERTIES AND SYSTEM PARAMETERS.

Symbol	Value	Unit
$\rho_g$	1	$[kg/m^3]$
$\rho_l$	1000	$[kg/m^3]$
$C_d$	0.440	$[-]$
$d_b$	0.001	$[m]$
$C_p$	1/4	$[-]$
$g$	9.81	$[m/s^2]$

For the given quasi steady-state situation, the eigenvalues are  $\lambda_1 = -0.0954$ , and  $\lambda_2 = 0.064$ . According to (12), the boundary condition for  $\lambda_1$  has to be defined at the inlet, i.e.,  $W'_1(t, 0)$ , and the boundary condition for  $\lambda_2 = 0.064$  has to be defined at the outlet, i.e.,  $W'_2(t, L)$ . Fig. 2 and Fig. 3 illustrate the wave propagation of characteristic solutions from the boundary conditions without force vector  $\mathbf{c}_1 W'_1 = c_{1,1} W'_1 + c_{1,2} W'_2 = 0$  for  $\lambda_1$  and  $\mathbf{c}_2 W'_2 = c_{2,1} W'_1 + c_{2,2} W'_2 = 0$  for  $\lambda_2$ , respectively. The simulation results clearly show the wave propagations, and the boundary structure of the linearized model equations.

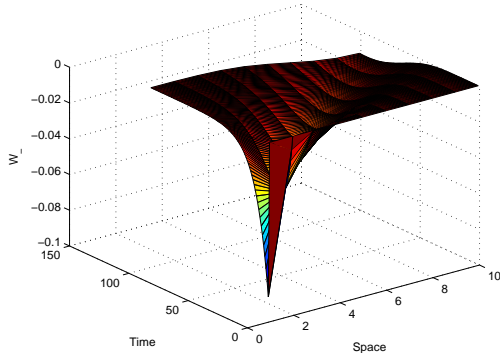


Fig. 2. Wave propagation of the state  $W'_1(t, x)$  with speed  $\lambda_1$  without forcing term  $\mathbf{c}_1 W'_1$ .

The influence of the linearized force vector is presented in Fig. 4 and Fig. 5. As can be seen, the force vector has a strong influence on the system dynamics, and, consequently, on the behavior at the boundaries. Thus by introducing the boundary controller at the inlet/outlet, the system can be stabilized using the proper boundary structure.

## V. CONCLUSIONS AND FUTURE WORKS

This paper introduces a linearized two-phase flow model and structural analysis of the boundary actuation. The actuation strategy is based on decoupling the wave propagation of the linearized model equation using coordinate transformation. The well-posed actuation structure can be used for

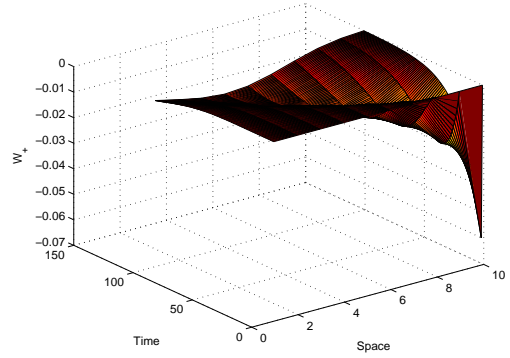


Fig. 3. Wave propagation of the state  $W'_2(t, x)$  with speed  $\lambda_2$  without forcing term  $\mathbf{c}W$ .

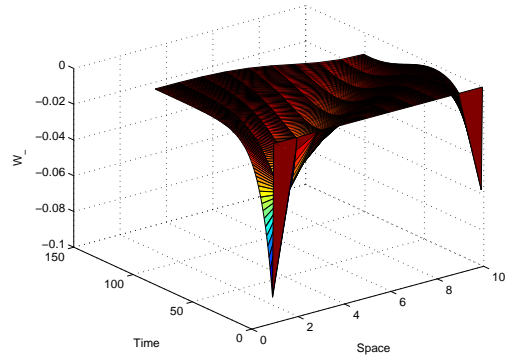


Fig. 4. Wave propagation of the state  $W'_1(t, x)$  with speed  $\lambda_1$  forced by  $\mathbf{c}_1 W'_1$ .

designing a control law which can stabilize the flow around the quasi steady-state solution. The same methodology can be applied to other PDAEs models with the same structure.

## VI. APPENDIX

The two phase flow is described by instantaneous changes of mass and momentum of each phases involved. For the one-dimensional flow, the equations read as

$$\frac{\partial \alpha_g}{\partial t} + \frac{\partial \alpha_g}{\partial x} v_g + \alpha_g \frac{\partial v_g}{\partial x} = 0, \quad (14)$$

$$\frac{\partial \alpha_l}{\partial t} + \frac{\partial \alpha_l}{\partial x} v_l + \alpha_l \frac{\partial v_l}{\partial x} = 0, \quad (15)$$

$$\alpha_g \rho_g \frac{\partial v_g}{\partial t} + \alpha_g v_g \rho_g \frac{\partial v_g}{\partial x} + \alpha_g \frac{dp}{dx} + \Delta p_g \frac{\partial \alpha_g}{\partial x} = -\alpha_g \rho_g g - \beta (v_g - v_l), \quad (16)$$

$$\alpha_l \rho_l \frac{\partial v_l}{\partial t} + \alpha_l v_l \rho_l \frac{\partial v_l}{\partial x} + \alpha_l \frac{dp}{dx} + \Delta p_l \frac{\partial \alpha_g}{\partial x} = -\alpha_l \rho_l g + \beta (v_g - v_l), \quad (17)$$

In (16) and (17), the pressure effect is modelled as a sum of two pressures: bulk pressure and interfacial pressure. For

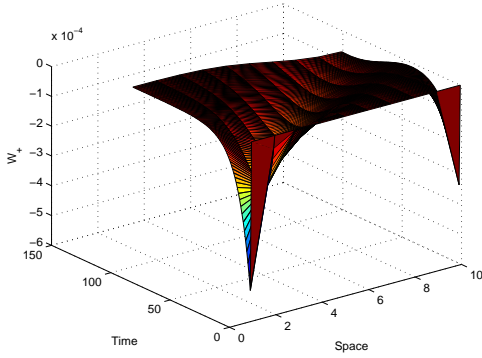


Fig. 5. Wave propagation of the state  $W_2'(t, x)$  with speed  $\lambda_2$  forced by  $c_2 W_2'$ .

the gas/liquid systems, the pressure difference within the gas phase is extremely small due to the low density of the gas phase,  $\Delta p_g = p - p_{gi} = 0$ , whereas the pressure difference within the liquid phase can be estimated  $\Delta p_l = p - p_{li} = C_p \alpha_l \rho_l (v_g - v_l)^2$  [8], [5], [9]. In order to eliminate the pressure gradient, we divide (16) and (17) by  $\alpha_g$  and  $\alpha_l$  respectively, and subtract one from the other. This step simplifies the model equations with respect to the implicitly modeled pressure. Furthermore, the model can be rewritten with respect to the gas volume fraction only by using the following algebraic constraint equation,  $\alpha_g + \alpha_l = 1$ . After these few simplification steps, the final form of the model equations can be written as follows

$$\frac{\partial \alpha_g}{\partial t} + \frac{\partial \alpha_g}{\partial x} v_g + \alpha_g \frac{\partial v_g}{\partial x} = 0, \quad (18)$$

$$\rho_g \frac{\partial v_g}{\partial t} - \rho_l \frac{\partial v_l}{\partial t} + \rho_g v_g \frac{\partial v_g}{\partial x} - \rho_l v_l \frac{\partial v_l}{\partial x} \quad (19)$$

$$\begin{aligned} &+ C_p \rho_l (v_g - v_l)^2 \frac{\partial \alpha_g}{\partial x} = \\ &- (\rho_g - \rho_l) g - (v_g - v_l) \left( \frac{\beta}{\alpha_g} + \frac{\beta}{\alpha_l} \right), \\ &\frac{\partial \alpha_g}{\partial x} (v_g - v_l) + \frac{\partial v_g}{\partial x} \alpha_g + (1 - \alpha_g) \frac{\partial v_l}{\partial x} = 0. \end{aligned} \quad (20)$$

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