Closed-loop identification of uncertainty models for robust control design: a set membership approach

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Abstract

The paper considers the problem of identifying uncertainty model sets, defined by an approximated model of the plant to be identified and a frequency domain bound on the modeling error. It is supposed that the measurements consist of time domain samples, collected in closed loop operations and corrupted by a power bounded noise. The model is supposed to be used for robust control design, whose performance is measured by a given closed loop H_∞ norm, and the "goodness" of the model is measured by the discrepancy between the closed loop performance predicted by the model and the one actually achieved on the plant. It is shown that identifying a model minimizing this discrepancy is equivalent to finding the best approximated model of the dual Youla parametrization of the plant in a suitably weighted H_∞ norm. Then, an optimal uncertainty model is derived for the dual Youla parametrized plant, from which an uncertainty model for the actual plant is obtained. Such uncertainty model is finally used for designing a robust controller and evaluating the closed loop performance that can be guaranteed when the designed controller is applied to the actual plant.

1 Introduction

In the past few years, a growing attention has been devoted to set membership methodologies for system identification (see e.g. [1, 2, 3, 4]), largely motivated by the important progress in robust control design realized in the 80's.

Robust control methodologies aim to design controllers guaranteeing to meet the specifications not for a single nominal model, but for all models obtained by given perturbations of the nominal model. Such model set, called *uncertainty model*, is introduced to take into account that models derived by any identification method are always affected by uncertainty. A quite popular class of uncertainty models is obtained by considering dynamic perturbations, bounded in the frequency domain. The simplest case is the additive uncertainty model $\mathcal{M}$ defined as the set:

$$\mathcal{M}(M,W_M)=\left\{M(z)+\Delta(z):\left\|W_M^{-1}(z)\,\Delta(z)\right\|_\infty<1\right\} \quad (1)$$

where $M(z)$ is the transfer function of the nominal model, $\Delta(z)$ is the transfer function of the perturbation and $W_M(z)$ is a known transfer function. A large body of literature is available for designing robust controllers for such uncertainty models. However, in most practical applications, such models are not directly available to the control designer and have to be identified from actual measurements on the unknown process P_0 to be controlled and from available prior information (or assumptions) on P_0 and on the noise corrupting the measurements.

Since the final goal is to guarantee high performance of the controlled plant, it is of relevance to provide an uncertainty model able to achieve this requirement. In [5, 6] it is shown how to derive tight uncertainty models and evaluate the performance that can be guaranteed in closed loop on the true plant P_0, using open loop experiments. The methods used in those papers need that the plant P_0 to be identified is asymptotically stable.

In this paper, a method is proposed to achieve the same goals using closed loop experiments, thus allowing to identify uncertainty models for unstable plants. An approach is followed that is closely related to the one in [7]. Here, in particular, the focus is on deriving tight uncertainty models for the case of measurements corrupted by power bounded noise.

Another interesting feature, shared with few others papers (e.g. [7, 8, 9]), is that the uncertainty models are tuned to the closed loop measure of performance that is underlying the control design.

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2 Problem formulation

As a general set-up, the linear time-invariant finite-dimensional feedback interconnection of Fig. 1 is considered, where u and y are the measurable input and output of the plant, r_1 and r_2 are reference signals and e is a disturbance signal.

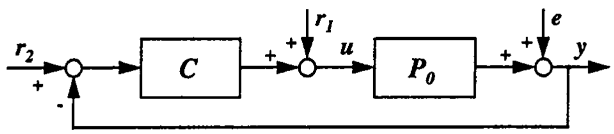

Figure 1: Feedback configuration.

A performance function of a closed loop configuration composed of plant P_0 and controller C is a system property, such as a step response, a sensitivity function, a complementary sensitivity function, etc. This control performance function can be formalized as an element $J(P_0, C)$ in some normed (Banach) space. The control performance cost is then measured by the norm $\|J(P_0, C)\|$, and a corresponding control design method will provide a controller that minimizes this cost. Many control design methods are based on the minimization of a particular performance cost. In the paper, the following ones are considered in detail [10]:

- *Mixed sensitivity optimization.* The mixed sensitivity design is reflected by the choice

$$J(P_0, C) = \begin{bmatrix} V_1 (I + P_0 C)^{-1} \\ V_2 P_0 C (I + P_0 C)^{-1} \end{bmatrix} \in \mathcal{R}H_\infty^{2\times 1} \tag{2}$$

  with weighting functions $V_1, V_2 \in \mathcal{R}H_\infty$, and the corresponding control performance cost is $\|J(P_0, C)\|_\infty$. In the sequel, $\mathcal{R}H_\infty$ denotes the set of real rational stable transfer functions.

- *H_∞ design based on robustness optimization.* This control design scheme proposed in [11] is reflected by the choice

$$J(P_0, C) = \begin{bmatrix} P_0 \\ I \end{bmatrix} (I + CP_0)^{-1} \begin{bmatrix} C & I \end{bmatrix} \in \mathcal{R}H_\infty^{2\times 2} \tag{3}$$

  and the corresponding control performance cost is $\|J(P_0, C)\|_\infty$.

For given model M and controller C designed on the basis of M, it holds that:

$$\begin{aligned} | \, \|J(M, C)\|_\infty - \|J(P_0, C) - J(M, C)\|_\infty \, | &\leq \\ \leq \; \|J(P_0, C)\|_\infty &\leq \\ \leq \; \|J(M, C)\|_\infty + \|J(P_0, C) - J(M, C)\|_\infty & \end{aligned} \tag{4}$$

The following terms can be distinguished:

$\|J(P_0, C)\|_\infty$ is the *achieved performance* when the compensator C is applied to the true plant P_0;

$\|J(M, C)\|_\infty$ is the *designed performance* when the compensator C is applied to the identified model M;

$\|J(P_0, C) - J(M, C)\|_\infty$ is the *performance degradation*, due to the fact that C has been designed from M rather than from P_0.

One aims at minimization of the upper bound of the performance cost in (4). However, this simultaneous optimization over both M and C is intractable by common identification and control design techniques, because they can optimize either the model or the controller, each while the other element is fixed. This has led to the introduction of several iterative schemes making use of separate stages of identification and control design, see e.g. [10, 12, 13] and the references therein. In the identification stage of the i-th iteration, a new model M_i is obtained by minimizing the performance degradation $\|J(P_0, C_{i-1}) - J(M, C_{i-1})\|_\infty$, where C_{i-1} is the controller designed in the previous iteration. In the control design stage, a new controller C_i is designed by minimizing the designed performance $\|J(M_i, C)\|_\infty$.

Indeed, a major motivation for iteration is due to the fact that a caution factor is introduced in control design based on model M_i only. This factor is used in order to prevent that the designed performance is high while the achieved performance may be poor and even the closed loop stability may be not achieved. The caution factor is progressively reduced as iterations go on and, hopefully, modeling error decreases.

In order to have a more systematic approach to deal with modeling errors, in this paper a method is proposed to derive, from measured data and suitable prior information, not only a model $\hat{M}$ but also a tight bounding function $W_{\hat{M}}$ on the modeling error $\Delta = P_0 - \hat{M}$. In this way, an uncertainty model $\mathcal{M}(\hat{M}, W_{\hat{M}})$ is obtained of the form (1) guaranteeing that $P_0 \in \mathcal{M}(\hat{M}, W_{\hat{M}})$. Such uncertainty model is suitable to be used by robust control techniques, giving a controller with guaranteed achieved performance.

3 Dual Youla parametrization approach

A closed loop identification approach is adopted, based on the (dual) Youla parametrization of all plants that are stabilized by a given known controller [10]. Given the feedback configuration in Fig. 2, it can be shown that, for given C stabilizing P_0 in closed loop, the unique value of R_0 that corresponds to the real plant P_0 is determined by

$$R_0 = D_c^{-1} (I + P_0 C)^{-1} (P_0 - P_x) D_x \tag{5}$$

In the scheme, C has right coprime factorization (rcf) $C = N_c D_c^{-1}$, P_x is any auxiliary system stabilized by C with rcf $P_x = N_x D_x^{-1}$ and

$$S = D_c^{-1} (I + P_0 C)^{-1} \tag{6}$$

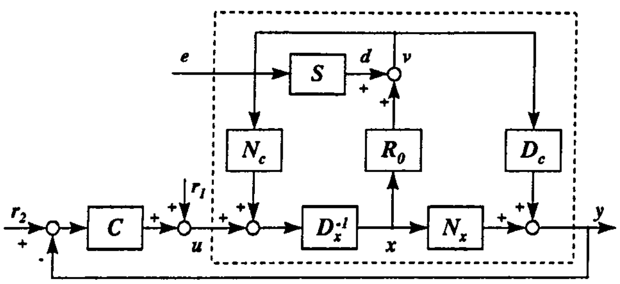

Figure 2: Dual Youla representation of the data generating system.

Now, defining the signals v, x as indicated in Fig. 2 and writing the node equations $x = D_x^{-1}(u + N_c v)$, $y = N_x x + D_c v$ and $u = r_1 + Cr_2 - Cy$, it follows that:

$$\begin{aligned} v &= (D_c + P_x N_c)^{-1}(y - P_x u) = \\ &= (D_c + P_x N_c)^{-1}[y - P_x(r_1 + Cr_2 - Cy)] \quad (7) \\ x &= (D_x + CN_x)^{-1}(u + Cy) = \\ &= (D_x + CN_x)^{-1}(r_1 + Cr_2) \quad (8) \end{aligned}$$

and

$$v = R_0 x + Se \quad (9)$$

Signals v and x can be reconstructed from closed loop data through known filters, provided the controller C is known. Thus, the identification of R_0 forms an open loop identification problem based on reconstructed measurements v, x.

The next propositions show that identifying a model $\hat{M}$ of P_0 minimizing $\|J(P_0, C) - J(M, C)\|_\infty$ is equivalent to finding an approximated model $\hat{R}$ of R_0 minimizing a suitably weighted H_∞ norm.

Proposition 1. For given M, let

$$R = D_c^{-1}(I + MC)^{-1}(M - P_x)D_x \quad (10)$$

Then,

- for the mixed sensitivity optimization:

$$J(P_0, C) - J(M, C) = \\ = \begin{bmatrix} -V_1 \\ V_2 \end{bmatrix} D_c(R_0 - R) D_x^{-1}(I + CP_x)^{-1} C \quad (11)$$

- for the H_∞ design based on robustness optimization:

$$J(P_0, C) - J(M, C) = \\ = \begin{bmatrix} I \\ -C \end{bmatrix} D_c(R_0 - R) D_x^{-1}(I + CP_x)^{-1} \begin{bmatrix} C & I \end{bmatrix} \quad (12)$$

Proof. See [14]. ∎

In the sequel of the paper, SISO case is considered and the identification criteria particularize according to the following proposition, where any term of type $F(\omega)$ stands for $F(j\omega)$ or $F(e^{j\omega})$ in continuous or discrete time domain respectively.

Proposition 2 (SISO case). For given M, let R as in (10). Then,

- for the mixed sensitivity optimization:

$$\|J(P_0, C) - J(M, C)\|_\infty = \\ = \sup_\omega \left| \frac{\sqrt{|V_1(\omega)|^2 + |V_2(\omega)|^2} N_c(\omega)}{D_x(\omega)[1 + C(\omega)P_x(\omega)]} [R_0(\omega) - R(\omega)] \right| \quad (13)$$

- for the H_∞ design based on robustness optimization:

$$\|J(P_0, C) - J(M, C)\|_\infty = \\ = \sup_\omega \left| \frac{D_c(\omega)[1 + |C(\omega)|^2]}{D_x(\omega)[1 + C(\omega)P_x(\omega)]} [R_0(\omega) - R(\omega)] \right| \quad (14)$$

Proof. The H_∞ norm of a stable transfer matrix G is defined as:

$$\|G\|_\infty = \sup_\omega \bar{\sigma}(G(\omega)) \quad (15)$$

where $\bar{\sigma}$ is the greatest singular value of G, i.e. the positive square root of the maximum eigenvalue of $G^H G$.

For the mixed sensitivity optimization, in the SISO case it results that, from (11):

$$J(P_0, C) - J(M, C) = \begin{bmatrix} -V_1 \\ V_2 \end{bmatrix} \frac{N_c}{D_x(1 + CP_x)} (R_0 - R)$$

and result (13) directly follows from definition (15), since for a rank-one matrix the H_∞ norm and the Frobenius norm coincide.

Analogously for the H_∞ design based on robustness optimization, in the SISO case it results, from (12):

$$J(P_0, C) - J(M, C) = \begin{bmatrix} C & 1 \\ -C^2 & -C \end{bmatrix} \frac{D_c}{D_x(1 + CP_x)} (R_0 - R)$$

Since the singular values of $\begin{bmatrix} C & 1 \\ -C^2 & -C \end{bmatrix}$ are $\sigma_1 = 0$ and $\sigma_2 = 1 + |C|^2$, result (14) follows from definition (15). ∎

If the compensator C is stable, a valid choice for the auxiliary system P_x stabilized by C is $N_x = 0$ and $D_x = 1$, and in this case $N_c = C$ and $D_c = 1$ can be chosen as rcf of the compensator. Then the R.H.S. of (13) and (14) can be rewritten as $\|R_0 - R\|_\infty^{W_C} = \sup_\omega W_C^{-1}(\omega) |R_0(\omega) - R(\omega)|$, where

- for the mixed sensitivity optimization:

$$W_C^{-1}(\omega) = \sqrt{|V_1(\omega)|^2 + |V_2(\omega)|^2} |C(\omega)| \quad (16)$$

- for the H_∞ design based on robustness optimization:

$$W_C^{-1}(\omega) = 1 + |C(\omega)|^2 \quad (17)$$

Note that this particular choice of coprime factors leads to the situation that $R_0 = \frac{P_0}{1 + P_0 C}$, and identification of R_0 actually reduces to the indirect method of closed loop identification.

4 Set membership identification

In the previous section it has been shown that the identification of P_0 is equivalent to the open loop identification of R_0, which is stable since it is supposed that C stabilizes the closed loop system. The SM approach developed in recent years for robust identification of SISO stable systems from open loop data can be used to identify an uncertainty model $\mathcal{R}(\hat{R}, W_{\hat{R}})$ of R_0 defined as the set:

$$\mathcal{R}(\hat{R}, W_{\hat{R}}) = \left\{ \hat{R} + \Delta : |\Delta(\omega)| \leq W_{\hat{R}}(\omega), \forall \omega \right\} \quad (18)$$

Then, an uncertainty model $\mathcal{M}(\hat{M}, W_{\hat{M}})$ of the plant P_0 can be determined on the basis of $\mathcal{R}(\hat{R}, W_{\hat{R}})$.

Methods for identifying uncertainty models have been developed for various specific cases, according to the type of experimental information (e.g. time or frequency domain data) and the noise assumptions, see e.g. [4, 10, 15] and the references therein. Here the case of identification of SISO, linear time-invariant, discrete-time systems using time domain data corrupted by power bounded noise is worked out in some detail. It is supposed that R_0 is a causal, BIBO stable, SISO, linear time-invariant, discrete-time system with impulse response $h^{R_0} = \{h_0^{R_0}, h_1^{R_0}, \ldots\}$, that controller C is known and stable, that known sequences r_1, r_2 are applied and that N output samples $y_0, \ldots, y_{N-1}$ are measured.

In view of (9), the experimental measurements give the following information on the impulse response h^{R_0} of R_0:

$$v_\ell = \sum_{k=0}^{\ell} h_k^{R_0} x_{\ell-k} + d_\ell , \quad \text{for } \ell = 0, \ldots, N-1 \quad (19)$$

where v_ℓ and x_ℓ, for $\ell = 0, \ldots, N-1$, are known, derived from measurements $r_{1\ell}$, $r_{2\ell}$ and y_ℓ, for $\ell = 0, \ldots, N-1$, through (7)-(8), and $d_\ell = \sum_{k=0}^{\ell} h_k^S e_{\ell-k}$. For the sake of simplicity, zero initial conditions are considered, but extension to nonzero case is easy. The noise sequence e is supposed unknown but power bounded, that is:

$$e^N = [e_0 \ldots e_{N-1}]^T \in \mathcal{B}_e = \left\{ e^N \in \Re^N : \tfrac{1}{\sqrt{N}} \|e^N\|_2 \leq \varepsilon \right\} \quad (20)$$

Then sequence d is power bounded, since system S is stable and $\frac{1}{\sqrt{N}}\|d^N\|_2 \leq \sup_\omega |S(\omega)| \frac{1}{\sqrt{N}} \|e^N\|_2 \leq \sup_\omega |D_c^{-1}(\omega)\left[1 + P_0(\omega) C(\omega)\right]^{-1}| \varepsilon$. The transfer function $F(z) = \left[1 + P_0(z) C(z)\right]^{-1}$ is not known. However, $F(z)$ is the transfer function from r_2 to $f = r_2 - y$, N samples of which are known from measured data. From these samples some estimate $\hat{F}(z)$ can be derived and $\delta = \sup_\omega |D_c^{-1}(\omega) \hat{F}(\omega)| \varepsilon$ can be used as an estimate of the power bound on d.

Now the aim is to derive an uncertainty model for R_0, consisting of a nominal model and a measure of its modeling error. From equation (19) it follows that the experimental measurements give information on $h_k^{R_0}$ only for $k < N$. Thus, from measurements only it is not possible to derive a finite bound on modeling error. To this end, some prior information on R_0 is needed. To make a minimal use of prior information, a ***residual*** type is assumed, i.e. giving constraints on the tail of h^{R_0} only. In particular it is assumed that $R_0 \in K_R$ where:

$$K_R = \{R : |h_k^R| \leq L\rho^k, \ \forall k \geq N\} \quad (21)$$

with known $L \geq 0$ and $0 < \rho < 1$. For a discussion of such type of prior information, see e.g. [16].

The Feasible Systems Set, i.e. the set of all systems consistent with prior information and available measurements, is then given by:

$$FSS = \{h^R \in K_R : \tfrac{1}{\sqrt{N}} \|v^N - X_N T_N h^R\|_2 \leq \delta\} \quad (22)$$

where $v^N = [v_0 \ \ldots \ v_{N-1}]^T$, T_N is the truncation operator defined as $T_N h^R = \left[h_0^R \ \ldots \ h_{N-1}^R\right]^T$ and X_N is the lower triangular $N \times N$ Toeplitz matrix formed by the sequence $x^N = [x_0 \ldots x_{N-1}]^T$.

The FSS is the smallest set of systems that, on the basis of assumed prior information and available measurements, is guaranteed to include R_0, thus representing the "best" possible uncertainty model for R_0. However, this set is not in a suitable form to be used by available robust design techniques. Then, the smallest uncertainty model of the form (18) is looked for, such that $FSS \subseteq \mathcal{R}(\hat{R}, W_{\hat{R}})$. This is obtained by computing $\hat{R}$ as a central estimate, i.e. the center of the minimal ball in the $\|\cdot\|_\infty^{W_C}$ norm including FSS with radius:

$$r = \sup_{R \in FSS} \|\hat{R} - R\|_\infty^{W_C} = \inf_{\tilde{R}} \sup_{R \in FSS} \|\tilde{R} - R\|_\infty^{W_C} \quad (23)$$

The quantity r is called (local) radius of information in SM identification literature and represents the minimal error that can be guaranteed on the basis of the given prior information and measurements. For this reason, $\hat{R}$ is called (locally) optimal estimate of R_0.

Then the bounding function $W_{\hat{R}}(\omega)$ is obtained by evaluating $\sup_{R \in FSS} |\hat{R}(\omega) - R(\omega)|$. The next proposition provides the solution to this problem.

Proposition 3.

i) The central estimate $\hat{R}$ is the FIR_N system with impulse response $h^{\hat{R}} = [h_0^{\hat{R}}, \ldots, h_{N-1}^{\hat{R}}, 0, 0, \ldots]$ such that:

$$T_N h^{\hat{R}} = X_N^{-1} v^N \quad (24)$$

ii) For any $\omega \in [0, 2\pi]$, it results:

$$\sqrt{N}\delta\bar{\sigma}(\Sigma(\omega)) - \frac{L\rho^N}{1-\rho} \leq \sup_{R \in FSS} |\hat{R}(\omega) - R(\omega)| \leq$$
$$\leq \sqrt{N}\delta\bar{\sigma}(\Sigma(\omega)) + \frac{L\rho^N}{1-\rho} \quad (25)$$

where $\bar{\sigma}(\Sigma(\omega))$ is the maximal singular value of $\Sigma(\omega) = \Omega_N(\omega) X_N^{-1}$, $\Omega_N(\omega) = [\Re e(\Psi_N(\omega))^T \quad \Im m(\Psi_N(\omega))^T]^T$ and $\Psi_N(\omega) = \left[1\ e^{-j\omega}\ e^{-j2\omega}\ \ldots\ e^{-j(N-1)\omega}\right]$.

Proof. From definition (22) of FSS, it follows that $(I - T_N)FSS = (I - T_N)K_R$ and $T_N FSS = \{h^N \in \Re^N : (h^N - X_N^{-1}v^N)^T X_N^T X_N (h^N - X_N^{-1}v^N) \le N\delta^2\}$ which is an ellipsoid with center in $X_N^{-1}v^N$. Then it follows that $\hat{R}$ is a center of symmetry of FSS. Thus, i) follows from the well known result that a center of symmetry of a set is its Chebiceff center in any norm, see e.g. [17].

Let $R^N(\omega)$ be the z-transform of $T_N h^R$. Then:

$$\sup_{R\in FSS} |\hat{R}(\omega) - R^N(\omega)| - \frac{L\rho^N}{1-\rho} \le \sup_{R\in FSS} |\hat{R}(\omega) - R(\omega)| \le$$
$$\le \sup_{R\in FSS} |\hat{R}(\omega) - R^N(\omega)| + \frac{L\rho^N}{1-\rho}$$

Since $\hat{R}(\omega) - R^N(\omega) = \Psi_N(\omega)T_N(h^{\hat{R}} - h^R)$ it results $|\hat{R}(\omega) - R^N(\omega)| = \|\Omega_N(\omega)T_N(h^{\hat{R}} - h^R)\|_2$. Then:

$$\begin{aligned}\sup_{R\in FSS} |\hat{R}(\omega) - R^N(\omega)| &= \\ &= \sup_{h^R:\|v^N - X_N T_N h^R\|_2 \le \sqrt{N}\delta} \|\Omega_N(\omega)(X_N^{-1}v^N - T_N h^R)\|_2 = \\ &= \sup_{d^N:\|d^N\|_2 \le \sqrt{N}\delta} \|\Omega_N(\omega) X_N^{-1} d^N\|_2\end{aligned} \quad (26)$$

The R.H.S. of (26) is $\sqrt{N}\delta$ times the induced ℓ_2 norm of matrix $\Sigma(\omega) = \Omega_N(\omega) X_N^{-1}$, which is well known to be $\bar{\sigma}(\Sigma(\omega))$, thus proving ii).

■

An uncertainty model $\mathcal{R}(\hat{R}, W_{\hat{R}})$ can be obtained by taking $\hat{R}$ as given by i) of proposition 3 and

$$W_{\hat{R}}(\omega) = \sqrt{N}\delta\bar{\sigma}(\Sigma(\omega)) + \frac{L\rho^N}{1-\rho} \quad (27)$$

Note that L and ρ represent some information about the "memory" of the closed loop system. If the duration of the experiment is not shorter than the "memory" of the closed loop system, as needed for obtaining acceptable identification errors, then the term $\frac{L\rho^N}{1-\rho}$ is typically negligible with respect to $\sqrt{N}\delta\bar{\sigma}(\Sigma(\omega))$. If this is the case, the derived uncertainty model is close to be the smallest uncertainty model of the form (18) guaranteed to include R_0.

Given the uncertainty model $\mathcal{R}(\hat{R}, W_{\hat{R}})$ of R_0, the corresponding uncertainty model $\mathcal{M}(\hat{M}, W_{\hat{M}})$ of the plant P_0 is then given by the following proposition.

Proposition 4. If C is stable then, with the choice $N_x = 0$, $D_x = 1$, $N_c = C$ and $D_c = 1$:

$$R_0 \in \mathcal{R}(\hat{R}, W_{\hat{R}}) \Leftrightarrow P_0 \in \mathcal{M}(\hat{M}, W_{\hat{M}}) \quad (28)$$

where

$$\hat{M}(\omega) = \frac{1}{C}\left(\frac{(1 - C\hat{R})^*}{|1 - C\hat{R}|^2 - |C|^2 W_{\hat{R}}^2} - 1\right) \quad (29)$$

$$W_{\hat{M}}(\omega) = \frac{W_{\hat{R}}}{|1 - C\hat{R}|^2 - |C|^2 W_{\hat{R}}^2} \quad (30)$$

Proof. See [14].

■

Note that $R_0 = \dfrac{P_0}{1 + P_0 C}$, but $\hat{R} \ne \dfrac{\hat{M}}{1 + \hat{M}C}$.

Making use of such uncertainty model, a new compensator can be designed, using robust design methods. For example, H_∞ design techniques allow one to compute a controller $C_{\mathcal{M}}$ such that

$$C_{\mathcal{M}} = \arg\min_{C\in\mathcal{C}_{rs}} \|J(\hat{M}, C)\|_\infty \quad (31)$$

where $\mathcal{C}_{rs}$ is the set of all controllers guaranteeing robust stability with respect to any system in the uncertainty model $\mathcal{M}(\hat{M}, W_{\hat{M}})$.

Standard H_∞ design techniques require that model $\hat{M}$ and model perturbation bound $W_{\hat{M}}$ have rational transfer functions. Then, $W_{\hat{R}}$ has to be chosen as a rational transfer function overbounding (27), by using e.g. the method in [18]. Its order has to be kept low because it affects the order of $\hat{M}$ and of $W_{\hat{M}}$, which in turn affects the order of $C_{\mathcal{M}}$. Indeed, even if the order of $W_{\hat{R}}$ is kept low, the order of $\hat{M}$ is large, greater than N, since $\hat{R}$ has transfer function of order N. If a low order model is desired, order reduction techniques can be used to derive from $\hat{R}$ an approximated model $\hat{R}_n$ of order $n < N$. In particular, the closed loop approximation method [19] may be appropriate here. Estimate $\hat{R}_n$ is no more optimal, giving the identification error:

$$E(\hat{R}_n) = \sup_{R\in FSS} \|\hat{R}_n - R\|_\infty^{W_C} = \alpha r \quad (32)$$

where $\alpha > 1$ measures the degradation in the identification error with respect to the radius of information, which is the minimal guaranteed error. Straightforward computation gives:

$$\alpha \le 1 + \frac{\|\hat{R}_n - \hat{R}\|_\infty^{W_C}}{\sqrt{N}\delta \sup_\omega W_C^{-1}(\omega)\bar{\sigma}(\Sigma(\omega)) - \frac{L\rho^N}{1-\rho}} \quad (33)$$

As $n \to N$, $\hat{R}_n$ is close to be optimal, i.e. $\alpha \to 1$. Indeed, typically it results that yet for moderate values of n, $\|\hat{R}_n - \hat{R}\|_\infty^{W_C}$ is small with respect to r and then $\alpha \approx 1$.

In order to derive an uncertainty model of the form $\mathcal{R}(\hat{R}_n, W_{\hat{R}_n})$, a bound on $|\hat{R}_n(\omega) - R_0(\omega)|$ is needed. The following result directly follows from proposition 3.

Proposition 5. For any $\omega \in [0, 2\pi]$, it results:

$$\begin{aligned}\sqrt{N}\delta\bar{\sigma}(\Sigma(\omega)) - \tfrac{L\rho^N}{1-\rho} - |\hat{R}_n(\omega) - \hat{R}(\omega)| &\le \\ \le \sup_{R\in FSS} |\hat{R}_n(\omega) - R(\omega)| &\le \\ \le \sqrt{N}\delta\bar{\sigma}(\Sigma(\omega)) + \tfrac{L\rho^N}{1-\rho} + |\hat{R}_n(\omega) - \hat{R}(\omega)|&\end{aligned} \quad (34)$$

■

Typically, the above bounds are sufficiently tight for practical purposes. If needed, tighter bounds can be derived by use of theorem 2 of [16].

By use of proposition 4, a "reduced order" uncertainty model $\mathcal{M}(\hat{M}_n, W_{\hat{M}_n})$ for P_0 can be derived from the "reduced order" uncertainty model $\mathcal{R}(\hat{R}_n, W_{\hat{R}_n})$ for R_0, where $W_{\hat{R}_n}(\omega)$ is a rational transfer function over-bounding the R.H.S. of (34). A "reduced order" robust controller can be derived using in (31) the uncertainty model $\mathcal{M}(\hat{M}_n, W_{\hat{M}_n})$. Since it may be convenient to choose n so that $\|\hat{R}_n - \hat{R}\|_\infty^{W_C}$ is sufficiently small to ensure that $\alpha \approx 1$ and the bounds of proposition 5 are reasonably tight, the complexity of the obtained controller may be not as low as desirable. Then, order reduction techniques can be used to derive a controller $C^r_{\mathcal{M}}$ of further reduced order. The performance degradation due to the use of the reduced order controller instead of the full order one $C_{\mathcal{M}}$ can be evaluated by considering the *robust performance* $\overline{J}(C^r_{\mathcal{M}})$ achievable by $C^r_{\mathcal{M}}$ defined as:

$$\overline{J}(C^r_{\mathcal{M}}) = \sup_{M \in \mathcal{M}(\hat{M}, W_{\hat{M}})} \|J(M, C^r_{\mathcal{M}})\|_\infty \qquad (35)$$

Robust performance $\overline{J}(C^r_{\mathcal{M}})$ is the minimal performance that it can be guaranteed, using the available information, when controller $C^r_{\mathcal{M}}$ is applied to the unknown plant P_0. A method for the computation of $\overline{J}(C^r_{\mathcal{M}})$ is proposed in [9], requiring a sequence of μ-tests which may be computationally demanding. The following proposition gives bounds on $\overline{J}(C^r_{\mathcal{M}})$ that can be easily computed. Alternative bounds can be found in [7].

Proposition 6.

$$\|J(\hat{M}, C^r_{\mathcal{M}})\|_\infty - \sup_\omega W_C^{-1}(\omega)\left[\sqrt{N}\delta\bar{\sigma}(\Sigma(\omega)) + \frac{L\rho^N}{1-\rho}\right] \leq$$

$$\leq \overline{J}(C^r_{\mathcal{M}}) \leq$$

$$\|J(\hat{M}, C^r_{\mathcal{M}})\|_\infty + \sup_\omega W_C^{-1}(\omega)\left[\sqrt{N}\delta\bar{\sigma}(\Sigma(\omega)) + \frac{L\rho^N}{1-\rho}\right] \qquad (36)$$

Proof. From propositions 2 and 4 and from (27), the next inequalities directly follow:

$$\begin{aligned}
\overline{J}(C^r_{\mathcal{M}}) &= \sup_{M \in \mathcal{M}(\hat{M}, W_{\hat{M}})} \|J(M, C^r_{\mathcal{M}})\|_\infty \leq \\
&\leq \|J(\hat{M}, C^r_{\mathcal{M}})\|_\infty + \sup_{M \in \mathcal{M}(\hat{M}, W_{\hat{M}})} \|J(\hat{M}, C^r_{\mathcal{M}}) - J(M, C^r_{\mathcal{M}})\|_\infty \\
&= \|J(\hat{M}, C^r_{\mathcal{M}})\|_\infty + \sup_{R \in \mathcal{R}(\hat{R}, W_{\hat{R}})} \sup_\omega W_C^{-1}(\omega)\, |\hat{R}(\omega) - R(\omega)| \\
&= \|J(\hat{M}, C^r_{\mathcal{M}})\|_\infty + \sup_\omega W_C^{-1}(\omega) \sup_{R \in \mathcal{R}(\hat{R}, W_{\hat{R}})} |\hat{R}(\omega) - R(\omega)| \\
&\leq \|J(\hat{M}, C^r_{\mathcal{M}})\|_\infty + \sup_\omega W_C^{-1}(\omega)\left[\sqrt{N}\delta\bar{\sigma}(\Sigma(\omega)) + \frac{L\rho^N}{1-\rho}\right]
\end{aligned}$$

Since analogous inequalities hold for the lower bound of $\overline{J}(C^r_{\mathcal{M}})$, the claim (36) is proved.

■

References

[1] M. Milanese, R. Tempo and A. Vicino, eds., *Robustness in Identification and Control.* New York: Plenum Press, 1989.

[2] A.B. Kurzhanski and V.M. Veliov, eds., *Modeling Techniques for Uncertain Systems.* Boston: Birkhäuser, 1994.

[3] R.S. Smith and M. Dahleh, eds., *The Modeling of Uncertainty in Control Systems.* London: Springer-Verlag, 1994.

[4] M. Milanese, J. Norton, H. Piet-Lahanier and É. Walter, eds., *Bounding Approaches to System Identification.* New York: Plenum Press, 1996.

[5] M. Canale, S.A. Malan and M. Milanese, "Model quality evaluation in identification for H_∞ control," *IEEE Transactions on Automatic Control*, vol. AC-42, no. 11, 1997.

[6] M. Canale, S. Malan, M. Milanese and M. Taragna, "Model structure selection in identification for control," in *Proc. of the 13th IFAC World Congress*, (San Francisco, CA), pp. 109–114, vol. I, 1996.

[7] P.M.J. Van den Hof, E.T. van Donkelaar, H.G.M. Dötsch and R.A. de Callafon, "Control-relevant uncertainty modelling directed towards performance robustness," in *Proc. of the 13th IFAC World Congress*, (San Francisco, CA), pp. 103–108, vol. I, 1996.

[8] R.G. Hakvoort and P.M.J. Van den Hof, "A practically feasible procedure for identification of $\mathcal{H}_\infty$ error bounds," in *Proc. of the 3rd European Control Conference*, (Rome, Italy), pp. 925–930, 1995.

[9] R.A. de Callafon and P.M.J. Van den Hof, "Suboptimal feedback control by a scheme of iterative identification and control design," *Mathematical Modelling of Systems*, vol. 3, no. 1, pp. 77–101, 1997.

[10] P.M.J. Van den Hof and R.J.P. Schrama, "Identification and control – closed-loop issues," *Automatica*, vol. 31, no. 12, pp. 1751–1770, 1995.

[11] D. McFarlane and K. Glover, *Robust Controller Design Using Normalized Coprime Factor Plant Descriptions.* Berlin: Springer, 1990.

[12] Z. Zang, R.R. Bitmead and M. Gevers, "Iterative weighted least-squares identification and weighted LQG-control," *Automatica*, vol. 31, pp. 1577–1594, 1995.

[13] R.R. Bitmead, M. Gevers and A.G. Partanen, "Introducing caution in iterative controller design," in *Proc. of IFAC SYSID '97*, (Kitakyushu, Japan), pp. 1701–1706, 1997.

[14] M. Milanese, M. Taragna and P.M.J. Van den Hof, "Closed-loop identification of uncertainty models for robust control design: a set membership approach," Tech. Rep. MiTV97, Politecnico di Torino - TU Delft, August 1997.

[15] G. Fiorio, S. Malan, M. Milanese and M. Taragna, "Robust design of low order controllers via uncertainty model identification," in *Proc. of the 2nd IFAC Symposium on Robust Control Design*, (Budapest, Hungary), pp. 49-54, 1997.

[16] L. Giarré, M. Milanese and M. Taragna, "H_∞ identification and model quality evaluation," *IEEE Transactions on Automatic Control*, vol. AC-42, no. 2, pp. 188–199, 1997.

[17] B.Z. Kacewicz, M. Milanese, R. Tempo and A. Vicino, "Optimality of central and projection algorithms for bounded uncertainty," *Systems & Control Letters*, vol. 8, pp. 161–171, 1986.

[18] R.E. Scheid, D.S. Bayard and Y. Yam, "A linear programming approach to characterizing norm bounded uncertainty from experimental data," in *Proc. of the American Control Conference*, (Boston, MA), pp. 1956–1958, 1991.

[19] C. Ceton, P.M.R. Wortelboer and O.H. Bosgra, "Frequency weighted closed-loop balanced reduction," in *Proc. of the 2nd European Control Conference*, (Groningen, the Netherlands), pp. 697–701, 1993.