The Behavioral Approach to Linear Parameter-Varying systems

Roland Tóth, Jan C. Willems, Peter S. C. Heuberger and Paul M. J. Van den Hof

Abstract

Linear Parameter-Varying (LPV) systems are usually described in either state-space or input-output form. When analyzing system equivalence between different representations it appears that the time-shifted versions of the scheduling signal (dynamic dependence) need to be taken into account. Therefore, representations used previously to define and specify LPV systems are not equal in terms of dynamics. In order to construct a parametrization-free description of LPV systems that overcomes these difficulties a behavioral approach is introduced which serves as a basis for specifying system theoretic properties. LPV systems are defined as the collection of trajectories of system variables (like inputs and outputs) and scheduling variables. LPV kernel, input-output, and state-space system representations are introduced with appropriate equivalence transformations.

Index Terms

LPV, behavioral approach, dynamic dependence, equivalence.

I. Introduction

Many physical/chemical processes encountered in practice have non-stationary or nonlinear behavior and often their dynamics depend on external variables like space-coordinates, temperature, etc. For such processes, the theory of *Linear Parameter-Varying* (LPV) systems offers an attractive modeling framework [1]. This class of systems is particularly suited to deal with processes that operate in varying operating regimes. LPV systems can be seen as an extension of the class of *Linear Time-Invariant* (LTI) systems. In LPV systems, the signal relations are

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considered to be linear, but the parameters in the description of these relations are assumed to be functions of a time-varying signal, the so-called *scheduling variable* p. As a result of the parameter variation concept, the LPV system class can describe both time-varying and nonlinear phenomena. Practical use of this framework is stimulated by the fact that LPV control design is well developed, extending results of optimal and robust LTI control theory to nonlinear, time-varying plants [1], [2], [3].

In a discrete-time setting, LPV systems are commonly described in a *state-space* (SS) form:

$$x(k) = A(p(k))x(k) + B(p(k))u(k), \tag{1a}$$

$$y(k) = C(p(k))x(k) + D(p(k))u(k), \tag{1b}$$

where $u : \mathbb{Z} \mapsto \mathbb{R}^{n_\mathbb{U}}$ is the input, $y : \mathbb{Z} \mapsto \mathbb{R}^{n_\mathbb{Y}}$ is the output, $x : \mathbb{Z} \mapsto \mathbb{R}^{n_\mathbb{X}}$ is the state vector and the system matrices $\{A, B, C, D\}$ are functions of the scheduling signal $p : \mathbb{Z} \mapsto \mathbb{P}$, e.g. $A : \mathbb{P} \mapsto \mathbb{R}^{\cdot \times \cdot}$, where $\mathbb{P} \subseteq \mathbb{R}^{n_\mathbb{P}}$ is the scheduling space. It is assumed that p is an external signal of the system. In the identification literature, LPV systems are also described in the form of (filter-type) *input-output* (IO) representations [4], [5], [6], [7]:

$$y(k) = \sum_{i=1}^{n_\mathrm{a}} a_i(p(k))y(k-i) + \sum_{j=0}^{n_\mathrm{b}} b_j(p(k))u(k-j), \tag{2}$$

where $\{a_i, b_j\}$ are matrix functions of p. In Equations (1a-b) and (2), the coefficients depend on the instantaneous time value of p, which is called *static-dependence*. In analogy with the LTI system theory, it is commonly assumed that representations (1a-b) and (2) define the same class of LPV systems and that conversion between these representations follows similar rules as in the LTI case (see [8], [9], [10]). However, it has been observed recently that this assumption is invalid if attention is restricted to static-dependence [11].

Example 1: To illustrate the problem consider the following second-order SS representation:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & a_2(p(k)) \\ 1 & a_1(p(k)) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} b_2(p(k)) \\ b_1(p(k)) \end{bmatrix} u(k),$$

$$y(k) = x_2(k).$$

With simple manipulations this system can be written in an equivalent IO form:

$$y(k) = a_1(p(k-1))y(k-1) + a_2(p(k-2))y(k-2) + b_1(p(k-1))u(k-1) + b_2(p(k-2))u(k-2),$$

which is clearly not in the form defined by (2).

In order to obtain equivalence between the SS and IO representations, it is necessary to allow for a dynamic mapping between p and the coefficients, i.e. $\{A, B, C, D\}$ and $\{a_i, b_j\}$ should be allowed to depend on (finitely many) time-shifted instances of $p(k)$, i.e. $\{\ldots, p(k-1), p(k), p(k+1), \ldots\}$ [11]. We call such a dependence *dynamic* in the sequel. Currently, it is not well understood how to handle such dependencies in general, and how to formulate algorithms that provide transformations between the representation forms (an intermediate solution for the SISO case is given in [11]).

The necessity of dynamic dependence clearly indicates that representations (1a-b) and (2) used previously to define and specify LPV systems are not equal in terms of dynamics. Furthermore, the lack of realization/transformation theory associated with these representations hinders the use of many identification methods, based on IO models, like the extension of successful prediction error methods for the LTI case [5], [4], to provide state space models for control synthesis. The lack of understanding of similarity transformation for (1a-b) is also a source of many pitfalls [11]. Furthermore, the collection of transfer functions of (1a-b) and (2) for each value of $p(k)$, the so-called *frozen transfer functions*, does not specify the behavior of the system for non-constant trajectories of p, which is often overlooked in the literature, see [12], [13], [14]. As no global transfer function theory exists in the LPV case, definitions of input-output behavior of (1a-b) and (2) need to be considered in terms of solutions of these difference equations in the time-domain. These arguments indicate that the classical definitions of LPV systems and the "assumed" similarity transformation connected to them are inadequate, showing that the current LPV system theory is incomplete.

A parametrization-free definition of LPV systems and an algebraic framework where the previously considered representations and concepts of LPV systems are reestablished can be found by considering a behavioral approach to the problem. In this paper the behavioral framework, originally developed for LTI systems [15], is extended to discrete-time LPV systems. In this framework systems are described in terms of behaviors that corresponds to the collection of all valid signal trajectories. Our aim is to use the behavioral concept to establish well-defined LPV system representations as well as their interrelationships. Our further intention is to develop a unified LPV system theory that establishes connections between the available results.

The paper is organized as follows: In Section II LPV systems are defined from the behavioral point of view. In Section III, an algebraic structure of polynomials is introduced to define parameter-varying difference equations as representations of the system behavior. This is followed, in Section IV, by developing kernel, IO, and SS representations of LPV systems, together with the basic notions of IO partitions and state-variables. In Section V it is explored when two kernel, IO, or SS representation are equivalent. In Section VI equivalence transformations between SS and IO representations are worked out. Finally, in Section VII, the main conclusions are summarized. We only consider discrete-time systems, however analog results for the continuous-time case follow in a similar way (see [16]).

II. LPV SYSTEMS AND BEHAVIORS

In the general *Parameter-Varying* (PV) framework, the scheduling variable, commonly denoted by p, is an external[1], so-called free signal of the system, that governs the dynamical behavior. The variable p can be understood as another "time variable" that governs the change of signal relations. The trajectory of p is unknown in advance which distinguishes LPV systems from the *Linear Time-Varying* (LTV) system class, where the time variation is fixed and known. Based on this, the class of PV systems can be defined as follows:

Definition 1 (Parameter-varying dynamical system): A parameter-varying system $\mathcal{S}$ is defined as a quadruple $\mathcal{S} = (\mathbb{T}, \mathbb{P}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T} \subseteq \mathbb{R}$ is called the time axis, $\mathbb{P}$ denotes the scheduling space (i.e. $p(k) \in \mathbb{P}$), $\mathbb{W}$ is the signal space with dimension $n_\mathbb{W}$ and $\mathfrak{B} \subseteq (\mathbb{W} \times \mathbb{P})^\mathbb{T}$ is the *behavior* of the system ($\mathbb{X}^\mathbb{T}$ stands for all maps from $\mathbb{T}$ to $\mathbb{X}$).

The set $\mathbb{T}$ defines the time-axis of the system, describing *continuous-time* (CT), $\mathbb{T} = \mathbb{R}$, and *discrete-time* (DT), $\mathbb{T} = \mathbb{Z}$, systems alike, while $\mathbb{W}$ gives the range of the system signals. The behavior $\mathfrak{B} \subseteq (\mathbb{W} \times \mathbb{P})^\mathbb{T}$ is the space of all signal and scheduling trajectories that are compatible with the system. Note that there is no prior distinction between inputs and outputs in this setting.

The scheduling space $\mathbb{P}$ is usually a closed subset of a vector space. Often, the admissible trajectories of p are further restricted to bound their variations. This set of admissible scheduling trajectories is defined as the *projected scheduling behavior*:

$$\mathfrak{B}_\mathbb{P} = \pi_p \mathfrak{B} := \{p \in \mathbb{P}^\mathbb{T} \mid \exists w \in \mathbb{W}^\mathbb{T} \text{ s.t. } (w, p) \in \mathfrak{B}\}, \tag{3}$$

where π_p denotes projection onto $\mathbb{P}^\mathbb{T}$. $\mathfrak{B}_\mathbb{P}$ describes all possible scheduling trajectories of $\mathcal{S}$. For a given scheduling trajectory, $p \in \mathfrak{B}_\mathbb{P}$, we define the *projected behavior* as

[1]Note that systems where p is an internal variable (like output, input, or state) are called *quasi parameter-varying systems*. Still, such systems are commonly treated as a PV system with external scheduling variable.

$$\mathfrak{B}_p = \{w \in \mathbb{W}^{\mathbb{T}} \mid (w,p) \in \mathfrak{B}\}. \tag{4}$$

$\mathfrak{B}_p$ describes all possible signal trajectories compatible with p. In case of a constant scheduling trajectory, $p \in \mathfrak{B}_p$ with $p(t) = \bar{\mathrm{p}}$ for all $t \in \mathbb{T}$ where $\bar{\mathrm{p}} \in \mathbb{P}$, the projected behavior $\mathfrak{B}_p$ is called a *frozen behavior* and denoted as

$$\mathfrak{B}_{\bar{\mathrm{p}}} = \left\{w \in \mathbb{W}^{\mathbb{T}} \mid (w,p) \in \mathfrak{B} \text{ with } p(t) = \bar{\mathrm{p}},\ \forall t \in \mathbb{T}\right\}. \tag{5}$$

Definition 2 (Frozen system): Let $\mathcal{S} = (\mathbb{T}, \mathbb{P}, \mathbb{W}, \mathfrak{B})$ be a PV system and consider $\mathfrak{B}_{\bar{\mathrm{p}}}$ for a given $\bar{\mathrm{p}} \in \mathbb{P}$. The dynamical system $\mathcal{F}_{\bar{\mathrm{p}}} = (\mathbb{T}, \mathbb{W}, \mathfrak{B}_{\bar{\mathrm{p}}})$ is called a frozen system of $\mathcal{S}$.

Define q as the unit forward time-shift operator, e.g. $qw(t) = w(t+1)$. With the previously introduced concepts, we can define discrete-time LPV systems as follows:

Definition 3 (DT-LPV system): Let $\mathbb{T} = \mathbb{Z}$. The parameter-varying system $\mathcal{S}$ is called LPV, if

- $\mathbb{W}$ is a vector space and $\mathfrak{B}_p$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$ for all $p \in \mathfrak{B}_{\mathbb{P}}$ (linearity).
- For any $(w,p) \in \mathfrak{B}$ and any $\tau \in \mathbb{T}$, it holds that $(w(\cdot + \tau), p(\cdot + \tau)) \in \mathfrak{B}$, in other words $q^{\tau}\mathfrak{B} = \mathfrak{B}$ (time-invariance).

In terms of Definition 3, for a constant scheduling trajectory $p(k) \equiv \bar{\mathrm{p}}$, time-invariance of $\mathcal{S}$ implies time-invariance of $\mathcal{F}_{\bar{\mathrm{p}}}$. Based on this and the linearity condition of $\mathfrak{B}_p$, it holds for an LPV system that for each $\bar{\mathrm{p}} \in \mathbb{P}$ the associated frozen system $\mathcal{F}_{\bar{\mathrm{p}}}$ is an LTI system, which is in accordance with previous definitions of LPV systems [1]. In this way, the projected behaviors of a given $\mathcal{S}$ with respect to constant scheduling trajectories define a set of LTI systems:

Definition 4 (Frozen system set): Let $\mathcal{S} = (\mathbb{T}, \mathbb{P}, \mathbb{W}, \mathfrak{B})$ be an LPV system. The set of LTI systems

$$\mathscr{F}_{\mathcal{S}} = \{\mathcal{F} = (\mathbb{T}, \mathbb{W}, \mathfrak{B}') \mid \exists \bar{\mathrm{p}} \in \mathbb{P} \text{ s.t. } \mathfrak{B}' = \mathfrak{B}_{\bar{\mathrm{p}}}\} \tag{6}$$

is called the frozen system set of $\mathcal{S}$.

Naturally, the LPV system concept is advantageous compared to general nonlinear systems, as the relation of the signals is linear. Definition 3 also reveals the advantage of this system class over LTV systems: the variation of the system dynamics is not associated directly with time, but with the variation of a free signal. Thus, the LPV modeling concept, compared to LTV systems, is more suitable for non-stationary/coordinate-dependent physical systems as it describes the underlying phenomena directly.

Example 2: To emphasize the advantage of LPV systems, we investigate the modeling of the motion of a varying mass connected to a spring (see Figure 1). This problem is one of

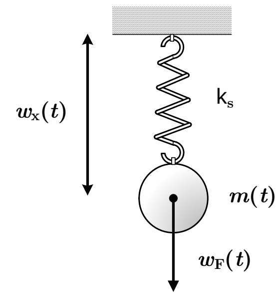

Fig. 1. Varying-mass connected to a spring.

the typical phenomena occurring in systems with time-varying masses like in motion control (robotics, rotating crankshafts, rockets, etc.).Denote by w_{x} the position of the varying mass m. Let $\mathtt{k}_{\mathrm{s}} > 0$ be the spring constant, introduce w_{F} as the force acting on the mass, and assume that there is no damping. By Newton's second law of motion, the following equation holds:

$$\frac{d}{dt}\left(m\frac{d}{dt}w_{\mathrm{x}}\right) = w_{\mathrm{F}} - \mathtt{k}_{\mathrm{s}}w_{\mathrm{x}}. \tag{7}$$

Using an Euler type of discretization with step size $\mathtt{T}_{\mathrm{d}} > 0$, a DT approximation of (7) is

$$(\mathtt{T}_{\mathrm{d}}^2\mathtt{k}_{\mathrm{s}} + m(k))w_{\mathrm{x}}(k) - (m(k+1) + m(k))w_{\mathrm{x}}(k+1) + m(k+1)w_{\mathrm{x}}(k+2) = \mathtt{T}_{\mathrm{d}}^2 w_{\mathrm{F}}(k), \tag{8}$$

It is immediate that by taking m as a scheduling variable, the behavior of this process can be described as an LPV system, preserving the physical insight of Newton's second law. On the other hand, viewing m as a time-varying parameter, whose trajectory is fixed and known in time, results in a LTV system. Such a system would explain the behavior of the process for only a fixed trajectory of the mass.

In the sequel, we restrict our attention to DT systems with $\mathbb{W} = \mathbb{R}^{n_{\mathbb{W}}}$ and with $\mathbb{P}$ a closed subset of $\mathbb{R}^{n_{\mathbb{P}}}$. In fact, we consider LPV systems described by finite order linear difference equations with parameter-varying effects in the coefficients.

III. Algebraic preliminaries

In order to re-establish the concept of LPV-IO and SS representations, we introduce difference equations with varying coefficients as the representation of the behavior $\mathfrak{B}$. These difference equations are described by polynomials of an algebraic ring where equivalence of representations and other system theoretic concepts can be characterized by simple algebraic manipulations.

A. Coefficient functions

First, we define the set of functional dependencies considered in the sequel:

Definition 5 (Real-meromorphic function [17]): A real-meromorphic function $f : \mathbb{R}^n \mapsto \mathbb{R}$, is a function $f = \frac{g}{h}$, where $g, h : \mathbb{R}^n \mapsto \mathbb{R}$ are holomorphic (analytic) functions and $h \neq 0$. Meromorphic functions consist of all rational, polynomial, trigonometric expressions, rational exponential functions etc. Thus, this class contains the common functional dependencies that result during LPV modeling of physical systems.

Next we establish an algebraic field $\mathcal{R}$ of a wide class of multivariable real-meromorphic functions from which the p-dependent coefficients of the representations will follow. Variables of these functions will be associated with the elements of the scheduling variable and their time-shifts in order to represent dynamic dependencies. However to uniquely define these dependencies (to establish a field) it must be ensured that in terms of an ordering, the "last" variable have a role in the considered functions. For instance $f(\mathtt{x}_1, \mathtt{x}_2) = \mathtt{x}_1$ should be excluded from the considered set as only $\hat{f}(\mathtt{x}_1) = \mathtt{x}_1$ is need to express this functional dependence. To ensure this property, we introduce operators $\mho_j$ and $\mho_*$ to exclude non-unique functional dependencies in the construction of $\mathcal{R}$.

Let $\mathcal{R}_n$ denote the field of real-meromorphic functions with n variables. Denote the variables of a $r \in \mathcal{R}_n$ as $\zeta_1, \ldots \zeta_n$. Also define an operator $\mho_j$ on $\mathcal{R}_n$ with $1 \leq j \leq n$ such that

$$\mho_j(r(\zeta_1, \ldots, \zeta_n)) := r(\zeta_1, \ldots, \zeta_j, 0, \ldots, 0). \tag{9}$$

Note that $\mho_j$ projects a meromorphic function to a lower dimensional domain. Introduce

$$\bar{\mathcal{R}}_n = \{r \in \mathcal{R}_n \mid \mho_{n-1}(r) \neq r\}. \tag{10}$$

It is clear that $\bar{\mathcal{R}}_n$ consist of all functions $\mathcal{R}_n$ in which the variable ζ_n has a nonzero contribution, i.e. it plays a role in the function. Also define the operator $\mho_* : (\cup_{i\geq 0}\mathcal{R}_i) \mapsto (\cup_{i\geq 0}\bar{\mathcal{R}}_i)$, which associates a given $r \in \mathcal{R}_n$ with a $r' \in \bar{\mathcal{R}}_{n'}$, $n \geq n'$, i.e. $\mho_*(r) = r'$, such that $r'(\zeta_1, \ldots, \zeta_{n'}) = r(\zeta_1, \ldots, \zeta_{n'}, 0, \ldots, 0)$ for all $\zeta_1, \ldots, \zeta_{n'} \in \mathbb{R}$, $\mho_{n'}(r) = r$ and n' is minimal. In this way, $\mho_*$ reduces the variables of a function till $\zeta_{n'}$ can not be left out from the expression because it has a nonzero contribution to the value of the function. Now define the collection of all real-meromorphic functions with finite many variables as follows:

$$\mathcal{R} = \bigcup_{i\geq 0} \bar{\mathcal{R}}_i, \quad \text{with} \quad \bar{\mathcal{R}}_0 = \mathbb{R}. \tag{11}$$

The function class $\mathcal{R}$ will be used as the collection of coefficient functions (like $\{A, \ldots, D\}$ and $\{a_i, b_j\}$ in (1a-b) and (2)) for the representations, giving the basic building block of PV difference equations. These functions are not only used to express dependence over multidimensional p but also to enable a distinction between dynamic scheduling dependence of the coefficients and the dynamic relation between the signals of the system. The following lemma is important:

Lemma 1 (Field property of $\mathcal{R}$): The set $\mathcal{R}$ is a field.

The proof of this lemma is straightforward and can be found in [16].

B. Representing scheduling dependence

The next step is to associate the variables of the coefficient functions with elements of p and its time-shifts, which will provide the characterization of dynamic dependencies in the representations. Naturally, this association is dependent on the dimension of the scheduling space considered.

In case of a scalar p, i.e. $n_{\mathbb{P}} = 1$, we can associate each variable $\{\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_2, \ldots\}$ of a given $r \in \mathcal{R}$ with $\{p, qp, q^{-1}, q^2 p, \ldots\}$ in order to express a given dynamic coefficient dependency. For example, the dependence $2p \cdot \sin(q^{-1}p)$ can be expressed in this way by a unique $r \in \mathcal{R}$ given as $r(\mathrm{x}_1 \mathrm{x}_2, \mathrm{x}_3) = 2\mathrm{x}_1 \sin(\mathrm{x}_3)$.

Now we can consider the general case. For a given $\mathbb{P}$ with dimension $n_{\mathbb{P}}$ and $r \in \bar{\mathcal{R}}_n$ label the variables of r according to the following ordering:

$$r(\zeta_{0,1}, \ldots, \zeta_{0,n_{\mathbb{P}}}, \zeta_{1,1}, \ldots, \zeta_{1,n_{\mathbb{P}}}, \zeta_{-1,1}, \ldots, \zeta_{-1,n_{\mathbb{P}}}, \zeta_{2,1}, \ldots).$$

For a given scheduling signal p, associate the variable ζ_{ij} with $q^i p_j$. For this association we introduce the operator

$$\diamond : (\mathcal{R}, \mathfrak{B}_{\mathbb{P}}) \to \mathbb{R}^{\mathbb{Z}}, \quad \text{defined by} \quad r \diamond p = r\left(p, qp, q^{-1}p, \ldots\right).$$

The value of a (p-dependent) coefficient in an LPV system representation is now given by an operation $(r \diamond p)(k)$.

Example 3 (Coefficient function): Let $\mathbb{P} = \mathbb{R}^{n_{\mathbb{P}}}$ with $n_{\mathbb{P}} = 2$. Consider the real-meromorphic coefficient function $r : \mathbb{R}^3 \mapsto \mathbb{R}$, defined as $r(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) = \frac{1+\mathrm{x}_3}{1-\mathrm{x}_2}$. Then for a scheduling signal $p : \mathbb{Z} \mapsto \mathbb{R}^2$, $(r \diamond p)(k) = r(p_1, p_2, qp_1)(k) = \frac{1+p_1(k+1)}{1-p_2(k)}$. On the other hand, if $n_{\mathbb{P}} = 3$, then $(r \diamond p)(k) = r(p_1, p_2, p_3)(k) = \frac{1+p_3(k)}{1-p_2(k)}$.

In the sequel the (time-varying) coefficient sequence $(r \diamond p)$ will be used to operate on a signal w (like $a_i(p)$ in (2)), giving the varying coefficient sequence of the representations. In this respect

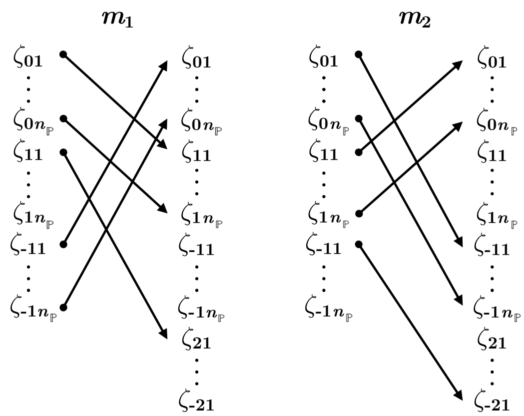

Fig. 2. Variable assignment by the functions m_1 and m_2 in Definition 6.

an important property is that multiplication of the $\diamond$ operation with the shift operator q is not commutative, in other words $q(r \diamond p) \neq (r \diamond p)q$. To handle this multiplication, for $r \in \mathcal{R}$ we define the shift operations $\overrightarrow{r}, \overleftarrow{r}$.

Definition 6 (Shift operators): Let $r \in \bar{\mathcal{R}}_n$. For a given scheduling dimension $n_{\mathbb{P}}$, denote the variables of r as $\{\zeta_{ij}\}$ based on the previously introduced labeling. The forward-shift and backward-shift operators on $\mathcal{R}$ are defined as

$$\overrightarrow{r} := \mho_*(r \circ m_1), \quad \overleftarrow{r} := \mho_*(r \circ m_2), \tag{12}$$

where $\circ$ denotes function concatenation, $m_1, m_2 \in \mathcal{R}^n_{n+2n_{\mathbb{P}}}$, and m_1 assigns each variable ζ_{ij} to $\zeta_{(i+1)j}$, while m_2 assigns each ζ_{ij} to $\zeta_{(i-1)j}$ as depicted in Figure 2.

In other words, if $r \diamond p$ is dependent on p and qp, then $\overrightarrow{r}$ is the "same" function (disregarding the number of variables) except $\overrightarrow{r} \diamond p$ is dependent on qp and q^2p. With these notions we can write $qr = \overrightarrow{r}q$ and $q^{-1}r = \overleftarrow{r}q^{-1}$ which corresponds to

$$q(r \diamond p)w = (\overrightarrow{r} \diamond p)qw \quad \text{and} \quad q^{-1}(r \diamond p)w = (\overleftarrow{r} \diamond p)q^{-1}w$$

on the signal level.

Example 4: Consider the coefficient function r given in Example 3 with $n_{\mathbb{P}} = 2$. Then $\overrightarrow{r}$ is a function $\mathbb{R}^5 \mapsto \mathbb{R}$, given by $\overrightarrow{r}(\zeta_{01}, \zeta_{02}, \zeta_{11}, \zeta_{12}, \zeta_{-11}, \zeta_{-12}, \zeta_{21}) = \frac{1+\zeta_{21}}{1-\zeta_{12}}$. For a scheduling trajectory $p : \mathbb{Z} \mapsto \mathbb{R}^2$, it holds that $(\overrightarrow{r} \diamond p)(k) = (r \diamond (qp))(k) = \frac{1+p_1(k+2)}{1-p_2(k+1)}$.

The considered operator $\diamond$ can straightforwardly be extended to matrix functions $r \in \mathcal{R}^{n_r \times n_{\mathbb{W}}}$ where the operation $\diamond$ is applied to each scalar entry of the matrix.

C. Polynomials over $\mathcal{R}$

Next we define the algebraic structure of the representations we use to describe LPV systems. Introduce $\mathcal{R}[\xi]$ as all polynomials in the indeterminant ξ and with coefficients in $\mathcal{R}$. $\mathcal{R}[\xi]$ is a ring as it is a general property of polynomial spaces over a field, that they define a ring. Also introduce $\mathcal{R}[\xi]^{\cdot\times\cdot}$, the set of matrix polynomial functions with elements in $\mathcal{R}[\xi]$. Using $\mathcal{R}[\xi]$ and the operator $\diamond$, we are now able to define a PV difference equation:

Definition 7 (PV difference equation): Consider $R(\xi) = \sum_{i=0}^{n_\xi} r_i \xi^i \in \mathcal{R}[\xi]^{n_\mathrm{r}\times n_\mathbb{W}}$ and $(w,p) \in (\mathbb{R}^{n_\mathbb{W}} \times \mathbb{R}^{n_\mathbb{P}})^\mathbb{Z}$.

$$(R(q) \diamond p)w := \sum_{i=0}^{n_\xi} (r_i \diamond p) q^i w = 0 \tag{13}$$

is called a PV difference equation with order $n_\xi = \deg(R)$.

In this notation the shift operator q operates on the signal w, while the operation $\diamond$ takes care of the time/schedule-dependent coefficient sequence. Since the indeterminant ξ is associated with q, multiplication with ξ is noncommutative on $\mathcal{R}[\xi]^{n_\mathrm{r}\times n_\mathbb{W}}$, i.e. $\xi r = \overrightarrow{r}\xi$ and $r\xi = \xi\overleftarrow{r}$.

In the following we only consider scheduling trajectories for which the coefficients of $R(\xi)\diamond p$ are bounded, so the set of solutions associated with R is well defined. PV difference equations in the form of (13) are used to define the class of DT-LPV systems we consider in this paper. It will be shown that this class contains all the popular definitions of LPV-SS and IO models.

Example 5 (PV difference equation): Consider Example 2. Let $p = m$ with scheduling space $\mathbb{P} = [1, 2]$ and let $w = [w_\mathrm{x} \quad w_\mathrm{F}]^\top$. Then the difference equation (8), which defines the possible signal trajectories of the DT approximation of the mass-spring system, can be written in the form of (13) with $n_\mathbb{W} = 2$, $n_\xi = 1$, $n_\mathbb{P} = 1$:

$$(R(q) \diamond p)w = (r_0 \diamond p)w + (r_1 \diamond p)qw + (r_2 \diamond p)q^2 w = 0 \tag{14}$$

where $r_0 \diamond p = [\mathrm{T}_\mathrm{d}^2 \mathrm{k}_\mathrm{s} + p \quad -\mathrm{T}_\mathrm{d}^2]$, $r_1 \diamond p = [-qp - p \quad 0]$, $r_2 \diamond p = [qp \quad 0]$.

Due to its algebraic structure, it easily follows that $\mathcal{R}[\xi]$ is a domain, i.e. for all $R_1, R_2 \in \mathcal{R}[\xi]$ it holds that $R_1(\xi)R_2(\xi) = 0 \quad\Rightarrow\quad R_1(\xi) = 0 \;$ or $\; R_2(\xi) = 0$. Then with the above defined noncommutative multiplicative rules $\mathcal{R}[\xi]$ defines an Ore algebra [18] and it is a left and right Euclidian domain [19]. The latter implies that there exists division by remainder. This means, that if $R_1, R_2 \in \mathcal{R}[\xi]$ with $\deg(R_1) \geq \deg(R_2)$ and $R_2 \neq 0$, then there exist unique polynomials $R', R'' \in \mathcal{R}[\xi]$ such that $R_1(\xi) = R_2(\xi)R'(\xi) + R''(\xi)$ where $\deg(R_2) > \deg(R'')$. The notion of unimodular matrices, essential to characterize equivalent representations, is also introduced:

Definition 8 (Unimodular matrix): Let $M \in \mathcal{R}[\xi]^{n\times n}$. M is called unimodular if there exists a $M^\dagger \in \mathcal{R}[\xi]^{n\times n}$ such that $M^\dagger(\xi)M(\xi) = I$ and $M(\xi)M^\dagger(\xi) = I$.

Any unimodular matrix operator in $\mathcal{R}[\xi]^{\cdot\times\cdot}$ is equivalent to the product of finite many elementary row and column operations:

1) Interchange row (column) i and row (column) j.
2) Multiply row (column) i by a $r \in \mathcal{R}$, $r \neq 0$.
3) For $i \neq j$, add to row (column) i row (column) j multiplied by ξ^n, $n > 0$.

Example 6 (Unimodular matrix): The matrix polynomials $M, M^\dagger \in \mathcal{R}[\xi]^{2\times 2}$, defined as

$$M(\xi)=\begin{bmatrix} r_2 & r_2\xi \\ r_1\xi & r_1\xi^2+r_1 \end{bmatrix},\ M^\dagger(\xi)=\begin{bmatrix} r_1+\xi^2 r_1 & -\xi r_2 \\ -\xi r_1 & r_2 \end{bmatrix}\frac{1}{r_1 r_2},$$

are unimodular as $M(\xi)M^\dagger(\xi) = M^\dagger(\xi)M(\xi) = I$. Note that $\xi r_1 \neq r_1\xi$ due to the non-commutativity of the multiplication by ξ on $\mathcal{R}[\xi]$.

Another important property of $\mathcal{R}[\xi]^{\cdot\times\cdot}$ is the existence of a Jacobson form (generalization of the Smith form):

Theorem 1 (Jacobson form [19]): Let $R \in \mathcal{R}[\xi]^{n_{\mathrm{r}}\times n_{\mathbb{W}}}$ with $n = \mathrm{rank}(R)$. Then there exist unimodular matrices $M_1 \in \mathcal{R}[\xi]^{n_{\mathrm{r}}\times n_{\mathrm{r}}}$ and $M_2 \in \mathcal{R}[\xi]^{n_{\mathbb{W}}\times n_{\mathbb{W}}}$ such that

$$M_1(\xi)R(\xi)M_2(\xi) = \begin{bmatrix} Q(\xi) & 0 \\ 0 & 0 \end{bmatrix}, \tag{15}$$

where $Q = \mathrm{diag}(r'_1, \ldots, r'_n) \in \mathcal{R}[\xi]^{n\times n}$ with monic non-zero $r'_i \in \mathcal{R}[\xi]$. Furthermore, there exist $g'_i \in \mathcal{R}[\xi]$ such that $r'_{i+1} = g'_i r'_i$ for $i = 1, \ldots, n-1$.

Due to the algebraic structure of $\mathcal{R}[\xi]^{\cdot\times\cdot}$, the proof of Theorem 1 similarly follows as in [19].

Example 7 (Jacobson form): Consider

$$R(\xi) = \begin{bmatrix} r+\xi & -1 & -1 \\ -r & 1+\xi & -\overrightarrow{r} \end{bmatrix} \in \mathcal{R}[\xi]^{2\times 3},$$

where r is a meromorphic function and $\xi = q$. Then the Jacobson form of R is

$$M_1(\xi)R(\xi)M_2(\xi)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & r+\overrightarrow{r}+\xi & 0 \end{bmatrix},\ M_1(\xi)=\begin{bmatrix} 1 & 0 \\ -\overrightarrow{r} & 1 \end{bmatrix},\ M_2(\xi)=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & r \\ -1 & -1 & \xi \end{bmatrix}.$$

Now it is possible to show that there exists a duality between the solution spaces of PV difference equations and the polynomial modules in $\mathcal{R}[\xi]^{\cdot\times\cdot}$ associated with them, which is implied by a so-called *injective cogenerator* property. This property makes it possible to use the developed algebraic structure to characterize behaviors and manipulations on them. Originally the injective cogenerator property has been shown for the solution spaces of the polynomial ring over $\mathcal{R}_1$ in [20]. In the Appendix this proof is extended to $\mathcal{R}[\xi]$.

IV. SYSTEM REPRESENTATIONS

A. Kernel representation

Using the developed concepts, we introduce *kernel representation* (KR) of an LPV system in the form of (13).

Definition 9 (DT-KR-LPV representation): The parameter varying difference equation (13) is called a discrete-time kernel representation, denoted by $\mathfrak{R}_{\mathrm{K}}(\mathcal{S})$, of the LPV dynamical system $\mathcal{S} = (\mathbb{Z}, \mathbb{R}^{n_{\mathbb{P}}}, \mathbb{R}^{n_{\mathbb{W}}}, \mathfrak{B})$ with scheduling vector p and signals w, if

$$\mathfrak{B} = \{(w,p) \in (\mathbb{R}^{n_{\mathbb{W}}} \times \mathbb{R}^{n_{\mathbb{P}}})^{\mathbb{Z}} \mid (R(q) \diamond p)\, w = 0\}. \tag{16}$$

It is also important, that the allowed trajectories of p in term of (16) are not restricted by (13) (only those $p \in (\mathbb{R}^{n_{\mathbb{P}}})^{\mathbb{Z}}$ are excluded for which a coefficient $r_i \diamond p$ is unbounded). This is in accordance with the classical concept of p being an external variable of the system. One can also include further restrictions on $\mathfrak{B}_{\mathbb{P}} = \pi_p \mathfrak{B}$, like description of the admissible scheduling trajectories as solutions of a differential equation, etc. However, to preserve the generality of the developed framework, we do not consider the latter case. An important concept to be established is the full row rank KR representation. Denote by $\mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(R)$ and $\mathrm{span}_{\mathcal{R}}^{\mathrm{col}}(R)$ the subspace spanned by the rows (columns) of $R \in \mathcal{R}[\xi]^{\cdot \times \cdot}$, viewed as a linear space of polynomial vector functions with coefficients in $\mathcal{R}^{\cdot \times \cdot}$. Then it can be shown that

$$\mathrm{rank}(R) = \dim(\mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(R)) = \dim(\mathrm{span}_{\mathcal{R}}^{\mathrm{col}}(R)). \tag{17}$$

Based on the concept of rank, the following theorem holds:

Theorem 2 (Full row rank KR representation): Let $\mathfrak{B}$ be given with a KR representation (13). Then, $\mathfrak{B}$ can also be represented by a $R' \in \mathcal{R}[\xi]^{\cdot \times n_{\mathbb{W}}}$ with full row rank.
The proof of this theorem is given in the Appendix.

B. IO representation

Partitioning of the signals w into input signals $u \in (\mathbb{R}^{n_{\mathbb{U}}})^{\mathbb{Z}}$ and output signals $y \in (\mathbb{R}^{n_{\mathbb{Y}}})^{\mathbb{Z}}$, i.e. $w = \mathrm{col}(u, y)$, is often considered convenient. Such a partitioning is called an IO partition [15].

Definition 10 (IO partition of a LPV system): Let $\mathcal{S} = (\mathbb{Z}, \mathbb{R}^{n_\mathbb{P}}, \mathbb{R}^{n_\mathbb{W}}, \mathfrak{B})$ be an LPV system. The partitioning of the signal space as $\mathbb{R}^{n_\mathbb{W}} = \mathbb{U} \times \mathbb{Y} = \mathbb{R}^{n_\mathbb{U}} \times \mathbb{R}^{n_\mathbb{Y}}$ and partitioning of $w \in (\mathbb{R}^{n_\mathbb{W}})^\mathbb{Z}$ correspondingly with $u \in (\mathbb{R}^{n_\mathbb{U}})^\mathbb{Z}$ and $y \in (\mathbb{R}^{n_\mathbb{Y}})^\mathbb{Z}$ is called an IO partition of $\mathcal{S}$, if

1) u is free, i.e. for all $u \in (\mathbb{R}^{n_\mathbb{U}})^\mathbb{Z}$ and $p \in \mathfrak{B}_\mathbb{P}$, there exists a $y \in (\mathbb{R}^{n_\mathbb{Y}})^\mathbb{Z}$ such that $(\mathrm{col}(u, y), p) \in \mathfrak{B}$.
2) y does not contain any further free component, i.e. given u, none of the components of y can be chosen freely for every $p \in \mathfrak{B}_\mathbb{P}$ (maximally free).

An IO partition implies the existence of matrix-polynomial functions $R_\mathrm{y} \in \mathcal{R}[\xi]^{n_\mathbb{Y} \times n_\mathbb{Y}}$ and $R_\mathrm{u} \in \mathcal{R}[\xi]^{n_\mathbb{Y} \times n_\mathbb{U}}$ with R_y full row rank, such that (13) can be written as

$$(R_\mathrm{y}(q) \diamond p)\, y = (R_\mathrm{u}(q) \diamond p)\, u, \tag{18}$$

with $n_\mathbb{W} = n_\mathbb{U} + n_\mathbb{Y}$ and the corresponding behavior $\mathfrak{B}$ is

$$\left\{(u, y, p) \in (\mathbb{U} \times \mathbb{Y} \times \mathbb{P})^\mathbb{Z} \mid (R_\mathrm{y}(q) \diamond p) y = (R_\mathrm{u}(q) \diamond p) u\right\},$$

with $\mathbb{U} = \mathbb{R}^{n_\mathbb{U}}$ and $\mathbb{Y} = \mathbb{R}^{n_\mathbb{Y}}$. An IO partition defines a causal mapping in case the solutions of (18) are restricted to have left compact support. Otherwise, initial conditions also matter [21]. Similar to the LTI case, LPV systems with no IO partition are called autonomous[2]. Now it is possible to introduce IO representations of DT-LPV systems:

Definition 11 (LPV-IO representation): The discrete-time IO representation of $\mathcal{S} = (\mathbb{Z}, \mathbb{P} \subseteq \mathbb{R}^{n_\mathbb{P}}, \mathbb{R}^{n_\mathbb{U} + n_\mathbb{Y}}, \mathfrak{B})$ with IO partition (u, y) and scheduling vector p is denoted by $\mathfrak{R}_\mathrm{IO}(\mathcal{S})$ and defined as a parameter-varying difference-equation system with order n_a:

$$\sum_{i=0}^{n_\mathrm{a}} (a_i \diamond p)\, q^i y = \sum_{j=0}^{n_\mathrm{b}} (b_j \diamond p)\, q^j u. \tag{19}$$

where $a_j \in \mathcal{R}^{n_\mathbb{Y} \times n_\mathbb{Y}}$ and $b_j \in \mathcal{R}^{n_\mathbb{Y} \times n_\mathbb{U}}$ with $a_{n_\mathrm{a}} \neq 0$ and $b_{n_\mathrm{b}} \neq 0$ are the meromorphic parameter-varying coefficients of the matrix polynomials $R_\mathrm{u}(\xi) = \sum_{j=0}^{n_\mathrm{b}} b_j \xi^j$ and full row rank $R_\mathrm{y}(\xi) = \sum_{i=0}^{n_\mathrm{a}} a_i \xi^i$ with $n_\mathrm{a} \geq n_\mathrm{b} \geq 0$ and $n_\mathrm{a} > 0$.

It is apparent that (19) is the "dynamic-dependent" counterpart of (2).

Example 8 (IO partition and representation): In Example 5, the sampled force variable w_x is a free variable as it represents the inhomogeneous part of difference equation (8). Thus the

[2] It is possible that the freedom of the components of w can change for specific scheduling trajectories. In this case, the autonomous part of the behavior is related to the scheduling dependent nature of the system.

choice of $w = [y \quad u]^\top = [w_\mathrm{x} \quad w_\mathrm{F}]^\top$ yields a valid IO partition. With m being the scheduling signal, the discrete-time PV behavior can be represented in the form of (19) with polynomials

$$R_\mathrm{y}(\xi) = a_0 + a_1\xi + a_2\xi^2, \quad R_\mathrm{u}(\xi) = b_0,$$

which have coefficients: $a_0 \diamond p = \mathtt{T}_\mathrm{d}^2\mathtt{k}_\mathrm{s} + p,\ a_1 \diamond p = -p - qp,\ a_2 \diamond p = qp,\ b_0 \diamond p = \mathtt{T}_\mathrm{d}^2$. Obviously, $R_\mathrm{y}(\xi)$ has full row rank. This implies that $R_\mathrm{y}(\xi)$ and $R_\mathrm{u}(\xi)$ define an IO representation of the model with coefficients as above.

For LPV systems, the notion of transfer function or frequency response has no meaningful[3] interpretation. The a viable formulation can be tackled via the formal series approach of [22], constructed for DT-SS systems of the LTV case. However, the direct extension of this approximative transfer function calculus to the class of systems considered here is not available yet.

C. State-space representation

In the modeling of dynamical systems, auxiliary variables (often called *latent variables*) are commonly used [21]. The natural counterpart of (13) to cope with such variables is

$$(R_\mathrm{w}(q) \diamond p)w = (R_\mathrm{L}(q) \diamond p)w_\mathrm{L}, \tag{20}$$

where $w_\mathrm{L} : \mathbb{Z} \mapsto \mathbb{R}^{n_\mathrm{L}}$ are the latent variables and $R_\mathrm{L} \in \mathcal{R}[\xi]^{n_\mathrm{r} \times n_\mathrm{L}}$. The set of equations (20) is called a *latent variable representation* of the LPV *latent variable system* $(\mathbb{Z}, \mathbb{R}^{n_\mathbb{P}}, \mathbb{R}^{n_\mathbb{W}} \times \mathbb{R}^{n_\mathrm{L}}, \mathfrak{B}_\mathrm{L})$, where the so-called *full behavior* $\mathfrak{B}_\mathrm{L}$ of this system is defined as

$$\mathfrak{B}_\mathrm{L} = \{(w, w_\mathrm{L}, p) \in (\mathbb{R}^{n_\mathbb{W}} \times \mathbb{R}^{n_\mathrm{L}} \times \mathbb{R}^{n_\mathbb{P}})^\mathbb{Z} \mid \text{(20) holds}\}.$$

Additionally, $\mathfrak{B} = \pi_{(w,p)}\mathfrak{B}_\mathrm{L}$ is introduced as the *manifest behavior* associated with $\mathfrak{B}_\mathrm{L}$.

Example 9 (Latent variable representation): By considering the DT system in Example 5 with scheduling $p = m$ and $\mathbb{P} = [1, 2]$, the following latent variable representation of the model has the same manifest behavior:

$$\begin{bmatrix} \mathtt{T}_\mathrm{d}^2\mathtt{k}_\mathrm{s} + p & -\mathtt{T}_\mathrm{d}^2 \\ (-p - q^{-1}p) & 0 \\ (-q^{-1}p) & 0 \end{bmatrix} \begin{bmatrix} w_\mathrm{x} \\ w_\mathrm{F} \end{bmatrix} = \begin{bmatrix} q & 0 \\ -1 & q \\ 0 & 1 \end{bmatrix} w_\mathrm{L}. \tag{21}$$

This can be proved by substituting the third row of (21) into the second row, giving

$$w_{\mathrm{L},1} = (p + q^{-1}p)w_\mathrm{x} - pqw_\mathrm{x}. \tag{22}$$

[3]Some authors [12], [13], [14] introduce LPV transfer functions with varying parameters. As they commonly refer only to the collection of transfer functions associated with $\mathscr{F}_\mathcal{S}$, this notion of the LPV transfer function is misleading.

Substitution of (22) into the first row of (21) gives a PV difference equation in the variables w_{x} and w_{F}, which is equal to (8).

Elimination of latent variables is always possible on $\mathcal{R}[\xi]^{\cdot\times\cdot}$.

Theorem 3 (Elimination property): Given a LPV latent variable system $(\mathbb{Z}, \mathbb{R}^{n_{\mathbb{P}}}, \mathbb{R}^{n_{\mathbb{W}}} \times \mathbb{R}^{n_{\mathrm{L}}}, \mathfrak{B}_{\mathrm{L}})$ with a signal variable w, a latent variable w_{L}, and scheduling variable p, there exists a $R' \in \mathcal{R}[\xi]^{\cdot\times n_{\mathbb{W}}}$ which defines a LPV-KR representation of $\mathfrak{B} = \pi_{(w,p)}\mathfrak{B}_{\mathrm{L}}$.

For a proof see the Appendix. Now it is possible to define the concept of state for LPV systems.

Definition 12 (Property of state): Let $(\mathbb{Z}, \mathbb{R}^{n_{\mathbb{P}}}, \mathbb{R}^{n_{\mathbb{W}}} \times \mathbb{R}^{n_{\mathrm{L}}}\ \mathfrak{B}_{\mathrm{L}})$ be a LPV latent variable system. Then the latent variable w_{L} is a state if for every $k_0 \in \mathbb{Z}$ and $(w_1, w_{\mathrm{L},1}, p)$, $(w_2, w_{\mathrm{L},2}, p) \in \mathfrak{B}_{\mathrm{L}}$ with $w_{\mathrm{L},1}(k_0) = w_{\mathrm{L},2}(k_0)$ it follows that the concatenation of these signals at k_0 satisfies

$$(w_1, w_{\mathrm{L},1}, p) \underset{k_0}{\wedge} (w_2, w_{\mathrm{L},2}, p) \in \mathfrak{B}_{\mathrm{L}}. \tag{23}$$

Then $\mathfrak{B}_{\mathrm{L}}$ is called a state-space behavior, and the latent variable w_{L} is called the state.

To decide whether a latent variable is a state, the following theorem is important:

Theorem 4 (State-kernel form): The latent variable w_{L} is a state, iff there exist matrices $r_{\mathrm{w}} \in \mathcal{R}^{n_{\mathrm{r}}\times n_{\mathbb{W}}}$ and $r_0, r_1 \in \mathcal{R}^{n_{\mathrm{r}}\times n_{\mathrm{L}}}$ such that the full behavior $\mathfrak{B}_{\mathrm{L}}$ has the kernel representation:

$$r_{\mathrm{w}} w + r_0 w_{\mathrm{L}} + r_1 q w_{\mathrm{L}} = 0. \tag{24}$$

The proof of this theorem is given in the Appendix. Now we formulate the DT state-space representation, based on an IO partition (u, y), as a first-order PV difference equation system.

Definition 13 (DT-LPV-SS representation): The discrete-time state-space representation of $\mathcal{S} = (\mathbb{Z}, \mathbb{P} \subseteq \mathbb{R}^{n_{\mathbb{P}}}, \mathbb{R}^{n_{\mathbb{U}}+n_{\mathbb{Y}}}, \mathfrak{B})$, with scheduling vector p is denoted by $\mathfrak{R}_{\mathrm{SS}}(\mathcal{S})$ and defined as a first-order parameter-varying difference equation system in the latent variable $x : \mathbb{Z} \mapsto \mathbb{X}$:

$$qx = (A \diamond p)x + (B \diamond p)u, \tag{25a}$$

$$y = (C \diamond p)x + (D \diamond p)u, \tag{25b}$$

where (u, y) is the IO partition of $\mathcal{S}$, x is the state-vector, $\mathbb{X} = \mathbb{R}^{n_{\mathbb{X}}}$ is the state-space,

$$\mathfrak{B}_{\mathrm{SS}} = \left\{(u, x, y, p) \in (\mathbb{U} \times \mathbb{X} \times \mathbb{Y} \times \mathbb{P})^{\mathbb{Z}} \mid \text{(25a-b) hold}\right\},$$

is the full behavior of (25a-b), $\mathfrak{B}$ is equal to the manifest behavior of (25a-b), i.e. $\mathfrak{B} = \pi_{u,y,p}\mathfrak{B}_{\mathrm{SS}}$, and

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \left[\begin{array}{c|c} \mathcal{R}^{n_{\mathbb{X}}\times n_{\mathbb{X}}} & \mathcal{R}^{n_{\mathbb{X}}\times n_{\mathbb{U}}} \\ \hline \mathcal{R}^{n_{\mathbb{Y}}\times n_{\mathbb{X}}} & \mathcal{R}^{n_{\mathbb{Y}}\times n_{\mathbb{U}}} \end{array}\right].$$

Note that in $\mathfrak{B}_{\mathrm{SS}}$, the latent variable x trivially fulfills the state property. It is apparent that (25a-b) are the "dynamic-dependent" counterparts of (1a-b).

Example 10 (SS representation): Continuing Example 9, the LPV-SS representation of the model follows by taking $[y \quad u]^\top = [w_{\mathrm{x}} \quad w_{\mathrm{F}}]^\top$ as the IO partition and $x = w_{\mathrm{L}}$ as the state:

$$qx = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} \mathtt{T}_{\mathrm{d}}^2 \mathtt{k}_{\mathrm{s}} + p & -\mathtt{T}_{\mathrm{d}}^2 \\ -p - q^{-1}p & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \quad y = \begin{bmatrix} 0 & \frac{1}{-q^{-1}p} \end{bmatrix} x.$$

By substitution of the second equation into the first one, the state equation in the form of (25a) results, while the second equation gives the output equation in the form of (25b). Thus, the corresponding SS representation is

$$\left[\begin{array}{c|c} A \diamond p & B \diamond p \\ \hline C \diamond p & D \diamond p \end{array}\right] = \left[\begin{array}{cc|c} 0 & -\frac{p + \mathtt{T}_{\mathrm{d}}^2 \mathtt{k}_{\mathrm{s}}}{q^{-1}p} & -\mathtt{T}_{\mathrm{d}}^2 \\ 1 & 1 + \frac{p}{q^{-1}p} & 0 \\ \hline 0 & \frac{-1}{q^{-1}p} & 0 \end{array}\right].$$

V. Equivalence relations

Using the behavioral framework, it is possible to consider equivalence of kernel representations, IO representations and state-space forms via equality of the represented behaviors.

A. Equivalent kernel forms

In the LTI case, two DT kernel representations are equivalent, i.e. they define the same system, if their associated behaviors are equal. Similar to the LTI framework, $R_1, R_2 \in \mathcal{R}[\xi]$ are expected to define an equal behavior if they are equivalent up to multiplication by a $r \in \mathcal{R}$, $r \neq 0$. However, r can be a rational function for which $(r \diamond p)(k) = \infty$ for some $p \in \mathbb{P}^{\mathbb{T}}$ and $k \in \mathbb{Z}$. The associated behavior of a kernel representation in terms of (16) is defined to contain only those trajectories of p for which a solution exists. The latter is guaranteed by the boundedness of $r \diamond p$. In this way, the behavior of R_1 is equal to the behavior of $R_2(\xi) = rR_1(\xi)$ except for those trajectories for which $r \diamond p$ is unbounded.

To consider equality of LPV-KR representations with this phenomenon of singularity in mind, we define the restriction of $\mathfrak{B}$ to $\bar{\mathfrak{B}}_{\mathbb{P}} \subseteq \mathfrak{B}_{\mathbb{P}}$ as

$$\mathfrak{B}\,|_{\bar{\mathfrak{B}}_{\mathbb{P}}} = \left\{ (w,p) \in \mathfrak{B} \mid p \in \bar{\mathfrak{B}}_{\mathbb{P}} \right\}. \tag{26}$$

The equivalence of LPV-KR representations can now be introduced in an *almost everywhere* sense:

Definition 14 (Equivalent KR representations): Two kernel representations with polynomials $R, R' \in \mathcal{R}[\xi]^{\bullet \times n_{\mathbb{W}}}$, $\mathbb{P} = \mathbb{R}^{n_{\mathbb{P}}}$ and behaviors $\mathfrak{B}, \mathfrak{B}' \subseteq (\mathbb{R}^{n_{\mathbb{W}}} \times \mathbb{R}^{n_{\mathbb{P}}})^{\mathbb{Z}}$ are called equivalent if $\mathfrak{B} \mid_{\mathfrak{B}_{\mathbb{P}} \cap \mathfrak{B}'_{\mathbb{P}}} = \mathfrak{B}' \mid_{\mathfrak{B}_{\mathbb{P}} \cap \mathfrak{B}'_{\mathbb{P}}}$, i.e. their behaviors are equal for all mutually valid trajectories of p.

Example 11 (Almost everywhere equivalence): By continuing Example 5,

$$\left(\frac{\mathtt{T}_{\mathrm{d}}^2 \mathtt{k}_{\mathrm{s}}}{p} + 1\right) w_1 - \left(\frac{qp}{p} + 1\right) q w_1 + \left(\frac{qp}{p}\right) q^2 w_1 - \frac{\mathtt{T}_{\mathrm{d}}^2}{p} w_2 = 0$$

has the same solutions as (14) except for those trajectories of $p = m$, where $m(k) = 0$ for some $k \in \mathbb{Z}$. Thus, this KR representation and (14) are equivalent in the almost everywhere sense.

To characterize equivalence algebraically, we introduce unimodular transformations just as in the LTI case [15]:

Theorem 5 (Unimodular transformation): Consider $R \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathbb{W}}}$ and $M' \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathrm{r}}}$, $M'' \in \mathcal{R}[\xi]^{n_{\mathbb{W}} \times n_{\mathbb{W}}}$ with M', M'' unimodular. For a given $n_{\mathbb{P}}$, define $R' = M'R$ and $R'' = RM''$. Denote the behaviors corresponding to R, R' and R'' by $\mathfrak{B}, \mathfrak{B}'$ and $\mathfrak{B}''$ with scheduling space $\mathbb{P} \subseteq \mathbb{R}^{n_{\mathbb{P}}}$ and signal space $\mathbb{W} = \mathbb{R}^{n_{\mathbb{W}}}$. Then $\mathfrak{B} \mid_{\mathfrak{B}_{\mathbb{P}} \cap \mathfrak{B}'_{\mathbb{P}}} = \mathfrak{B}' \mid_{\mathfrak{B}_{\mathbb{P}} \cap \mathfrak{B}'_{\mathbb{P}}}$ while $\mathfrak{B} \mid_{\mathfrak{B}_{\mathbb{P}} \cap \mathfrak{B}''_{\mathbb{P}}}$ and $\mathfrak{B}'' \mid_{\mathfrak{B}_{\mathbb{P}} \cap \mathfrak{B}''_{\mathbb{P}}}$ are isomorphic.

The proof of this theorem is given in the Appendix. Furthermore, if $R \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathbb{W}}}$ is not full row rank, i.e. $\mathrm{rank}(R) = n < n_{\mathrm{r}}$, then there exists a unimodular $M \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathrm{r}}}$ such that $M(\xi)R(\xi) = [\ (R'(\xi))^{\top} \quad 0\ ]^{\top}$, where $R' \in \mathcal{R}[\xi]^{n \times n_{\mathbb{W}}}$ is full row rank and the corresponding behaviors are equivalent in terms of Theorem 5.

Definition 15 (Equivalence relation): Introduce the symbol $\overset{n_{\mathbb{P}}}{\sim}$ to denote the equivalence relation on $\bigcup \mathcal{R}[\xi]^{\cdot \times \cdot}$ (all polynomial matrices with finite dimension) for an $n_{\mathbb{P}}$-dimensional scheduling space. $R_1 \in \mathcal{R}[\xi]^{n_1 \times n_{\mathbb{W}}}$ and $R_2 \in \mathcal{R}[\xi]^{n_2 \times n_{\mathbb{W}}}$ with $i = \arg\max_{i \in \{1,2\}}(n_i)$ and $j = \{1,2\} \setminus i$ are called equivalent, i.e. $R_1 \overset{n_{\mathbb{P}}}{\sim} R_2$, if there exists a unimodular matrix function $M \in \mathcal{R}[\xi]^{n_i \times n_i}$ such that

$$M(\xi)R_i(\xi) = \begin{bmatrix} R_j(\xi) \\ 0 \end{bmatrix} \begin{matrix} \updownarrow & n_j \\ \updownarrow & n_i - n_j \end{matrix} . \tag{27}$$

This implies that if $R_1 \overset{n_{\mathbb{P}}}{\sim} R_2$, then the corresponding behaviors with $\mathbb{P} \subseteq \mathbb{R}^{n_{\mathbb{P}}}$ and $\mathbb{W} = \mathbb{R}^{n_{\mathbb{W}}}$ are equal (almost everywhere). Using $\overset{n_{\mathbb{P}}}{\sim}$ we can define equivalence classes as follows:

Definition 16 (Equivalence class): For a given $n_{\mathbb{P}}$, the set $\mathscr{E}^{n_{\mathbb{P}}} \subseteq \bigcup \mathcal{R}[\xi]^{\cdot \times \cdot}$ is called an equivalence class, if it is a maximal subset of $\mathcal{R}[\xi]^{\cdot \times \cdot}$ such that for all $R_1, R_2 \in \mathscr{E}^{n_{\mathbb{P}}}$ it holds that $R_1 \overset{n_{\mathbb{P}}}{\sim} R_2$.

An equivalence class defines the set of all KR representations which have equal behavior. Furthermore it is an obvious consequence, that all R in a given $\mathscr{E}^{n_{\mathbb{P}}}$ have the same Jacobson form. An important subset of an equivalence class contains the so-called minimal representations:

Definition 17 (Minimality): Let $R \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathbb{W}}}$. Then R is called minimal if it has full row rank, i.e. $\operatorname{rank}(R) = n_{\mathrm{r}}$.

Given a minimal $\mathfrak{R}_{\mathrm{K}}(\mathcal{S})$ described by a full row rank $R \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathbb{W}}}$. Assume that $R(\xi) = [R'(\xi) \quad R''(\xi)]$ where $R' \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathrm{r}}}$ has full column rank. Note that such form can always be obtained by the permutation of the signals variables and it is not unique. Consider $n_{\mathrm{deg}} = \deg(r'_n)$ where r'_n results from the Jacobson form (see Theorem 1) of R'. Assume that R' is chosen with respect to R such that n_{deg} is maximal. It follows from Theorem 5, that all KR representations in the equivalence class of $\mathfrak{R}_{\mathrm{K}}(\mathcal{S})$ have the same n_{deg}, hence n_{deg} can be called the degree of these representations. It can be also shown that this degree is equal to the required minimal number of state variables in a SS realization of $\mathfrak{R}_{\mathrm{K}}(\mathcal{S})$, hence n_{deg} can be considered as the order, i.e. *McMillan degree* of $\mathcal{S}$.

Example 12 (LPV equivalence relation and minimality): Let the KR representation $\mathfrak{R}_{\mathrm{K}}(\mathcal{S})$ of an DT-LPV system $\mathcal{S}$ with $\mathbb{P} \subseteq \mathbb{R}$ be given by

$$R(\xi) \diamond p = \begin{bmatrix} qp & -qp \\ p & -p \end{bmatrix} + \begin{bmatrix} 0 & p(qp) \\ p^2 & 0 \end{bmatrix} \xi + \begin{bmatrix} -p(qp^2) & 0 \\ 0 & 0 \end{bmatrix} \xi^2.$$

Then, there exists a unimodular matrix $M \in \mathcal{R}[\xi]^{2 \times 2}$

$$M(\xi) \diamond p = \begin{bmatrix} 0 & 1 \\ 1 & p\xi - \frac{qp}{p} \end{bmatrix} \quad \text{s.t.} \quad (M(\xi)R(\xi)) \diamond p = \begin{bmatrix} p + p^2\xi & -p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R'(\xi) \\ 0 \end{bmatrix}.$$

From Theorem 5 it follows that $R \overset{1}{\sim} R'$. Furthermore, $\operatorname{rank}(R') = 1$ implies that $\operatorname{rank}(R) = 1$, hence R' is minimal while R is not. By computing n_{deg} of R', the McMillan degree of $\mathcal{S}$ is 1.

B. Equivalent IO forms

The introduced equivalence concept generalizes to LPV-IO representations:

Definition 18 (Equivalence relation, LPV-IO): Let $R_{\mathrm{u}}, R'_{\mathrm{u}} \in \mathcal{R}[\xi]^{n_{\mathbb{Y}} \times n_{\mathbb{U}}}$ and $R_{\mathrm{y}}, R'_{\mathrm{y}} \in \mathcal{R}[\xi]^{n_{\mathbb{Y}} \times n_{\mathbb{Y}}}$ with $R_{\mathrm{y}}, R'_{\mathrm{y}}$ full row rank, $\deg(R_{\mathrm{y}}) \geq \deg(R_{\mathrm{u}})$, and $\deg(R'_{\mathrm{y}}) \geq \deg(R'_{\mathrm{u}})$. For a given $n_{\mathbb{P}}$, we call the LPV-IO representations defined via $(R_{\mathrm{y}}, R_{\mathrm{u}})$ and $(R'_{\mathrm{y}}, R'_{\mathrm{u}})$ equivalent, i.e. $(R_{\mathrm{y}}, R_{\mathrm{u}}) \overset{n_{\mathbb{P}}}{\sim} (R'_{\mathrm{y}}, R'_{\mathrm{u}})$, if there exists a unimodular matrix $M \in \mathcal{R}[\xi]^{n_{\mathbb{Y}} \times n_{\mathbb{Y}}}$ such that

$$R'_{\mathrm{y}}(\xi) = M(\xi) R_{\mathrm{y}}(\xi) \quad \text{and} \quad R'_{\mathrm{u}}(\xi) = M(\xi) R_{\mathrm{u}}(\xi). \tag{28}$$

This implies the following minimality concept of LPV-IO representations:

Definition 19 (Minimal LPV-IO representation): An IO representation defined through $R_{\mathrm{y}} \in \mathcal{R}[\xi]^{n_{\mathbb{Y}} \times n_{\mathbb{Y}}}$ and $R_{\mathrm{u}} \in \mathcal{R}[\xi]^{n_{\mathbb{Y}} \times n_{\mathbb{U}}}$ is called minimal for a given scheduling dimension $n_{\mathbb{P}}$, if there are no polynomials $R'_{\mathrm{y}} \in \mathcal{R}[\xi]^{n_{\mathbb{Y}} \times n_{\mathbb{Y}}}$ and $R'_{\mathrm{u}} \in \mathcal{R}[\xi]^{n_{\mathbb{Y}} \times n_{\mathbb{U}}}$ with $\deg(R_{\mathrm{y}}) < \deg(R'_{\mathrm{y}})$ such that

$$(R_{\mathrm{y}}, R_{\mathrm{u}}) \overset{n_{\mathbb{P}}}{\sim} (R'_{\mathrm{y}}, R'_{\mathrm{u}}).$$

Using the IO equivalence relation and minimality, the definition of IO equivalence classes follows naturally.

Example 13 (LPV-IO equivalence and minimality): Let the IO representation $\mathfrak{R}_{\mathrm{IO}}(\mathcal{S})$ of an DT-LPV system $\mathcal{S}$ with $\mathbb{P} \subseteq \mathbb{R}$ be given by

$$R_{\mathrm{y}}(\xi) \diamond p = \begin{bmatrix} p\xi & p^2 \\ p\xi^2 & p(qp)\xi \end{bmatrix}, \; R_{\mathrm{u}}(\xi) \diamond p = \begin{bmatrix} p \\ p(\xi - 1) \end{bmatrix}.$$

Consider the unimodular matrix $M \in \mathcal{R}[\xi]^{2\times 2}$ given by

$$M(\xi) \diamond p = \begin{bmatrix} \frac{1}{p} & 0 \\ \frac{p}{qp}\xi & -1 \end{bmatrix}, \quad \text{then} \quad (M(\xi)R_{\mathrm{y}}(\xi)) \diamond p = \begin{bmatrix} \xi & p \\ 0 & \xi \end{bmatrix}, \quad (M(\xi)R_{\mathrm{u}}(\xi)) \diamond p = \begin{bmatrix} 1 \\ p \end{bmatrix}.$$

This implies that $(R'_{\mathrm{y}}, R'_{\mathrm{u}}) = (MR_{\mathrm{y}}, MR_{\mathrm{u}})$ and $(R_{\mathrm{y}}, R_{\mathrm{u}})$ are equivalent for $n_{\mathbb{P}} = 1$ in terms of Theorem 5. From Definition 19 it follows that $\mathfrak{R}_{\mathrm{IO}}(\mathcal{S})$ is not minimal as $\deg(R_{\mathrm{y}}) = 2$ is larger than $\deg(R'_{\mathrm{y}}) = 1$. On the other hand, it is trivial that $(R'_{\mathrm{y}}, R'_{\mathrm{u}})$ defines a minimal IO representation of $\mathcal{S}$. By computing the Jacobson form of R'_{y}, the McMillan degree of $\mathcal{S}$ is 1.

C. Equivalent state-space forms

We can also generalize the equivalence concept to LPV-SS representations. To do so, we first have to clarify state-transformations in the LPV case.

By definition, the full behavior of LPV-SS representation is represented by a matrix $R_{\mathrm{w}} \in \mathcal{R}^{n_{\mathrm{r}} \times (n_{\mathbb{Y}} + n_{\mathbb{U}})}$ and a first-order polynomial $R_{\mathrm{L}} \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathbb{X}}}$ in the form

$$(R_{\mathrm{w}} \diamond p)\mathrm{col}(u, y) = (R_{\mathrm{L}}(q) \diamond p)x. \tag{29}$$

Similar to the LTI case, left and right side multiplication of R_{w} and R_{L} with unimodular $M_1 \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathrm{r}}}$ and $M_2 \in \mathcal{R}[\xi]^{n_{\mathbb{X}} \times n_{\mathbb{X}}}$ leads to $R'_{\mathrm{w}}(\xi) = M_1(\xi)R_{\mathrm{w}}$, $R'_{\mathrm{L}}(\xi) = M_1(\xi)R_{\mathrm{L}}(\xi)M_2(\xi)$. In terms of Theorem 5, the resulting polynomials R'_{w} and R'_{L} define an equivalent latent variable representation of $\mathcal{S}$, where the new latent variable is given as $x' = (M_2^{\dagger}(q) \diamond p)x$. To guarantee that the resulting latent variable representation qualifies as a SS representation, R'_{L} needs to be monic and $\deg(R'_{\mathrm{L}}) = 1$ with $\deg(R'_{\mathrm{w}}) = 0$ must be satisfied. This implies that the unimodular matrices must have zero order, i.e. $M_1 \in \mathcal{R}^{n_{\mathrm{r}} \times n_{\mathrm{r}}}$ and $M_2 \in \mathcal{R}^{n_{\mathbb{X}} \times n_{\mathbb{X}}}$, and M_1 must have a special structure in order to guarantee that R'_{w} and R'_{L} correspond to an equivalent SS representation. In that case, $x' = (M_2^{\dagger}(q) \diamond p)x$ is called a *state-transformation* and $T = M_2^{\dagger}$ is called the *state transformation matrix* resulting in

$$x' = (T \diamond p)x. \tag{30}$$

A major difference with respect to LTI state-transformations is that, in the LPV case, T is inherently dependent on p and this dependence is dynamic, i.e. $T \in \mathcal{R}^{n_{\mathbb{X}} \times n_{\mathbb{X}}}$. Additionally it can be shown that an invertible $T \in \mathcal{R}^{n_{\mathbb{X}} \times n_{\mathbb{X}}}$ used as a state-transformation is always equivalent with a right and left-side multiplication by unimodular matrix functions yielding a valid SS representation of the LPV system. Based on this, two SS representations are equivalent if and only if their states can be related via an invertible state-transformation (30).

Consider an LPV-SS representation (25a-b). Let $T \in \mathcal{R}^{n_{\mathbb{X}} \times n_{\mathbb{X}}}$ be an invertible matrix function and consider x', given by (30), as a new state variable. Substitution of (30) into (25a) gives

$$q(T^{-1} \diamond p)x' = (A \diamond p)(T^{-1} \diamond p)x' + (B \diamond p)u. \tag{31}$$

Using that $qT^{-1} = \overrightarrow{(T^{-1})}q = \overrightarrow{T}^{-1}q$, (31) yields that the equivalent LPV-SS representation is

$$\left[\begin{array}{c|c} \overrightarrow{T}AT^{-1} & \overrightarrow{T}B \\ \hline CT^{-1} & D \end{array}\right]. \tag{32}$$

Definition 20 (Equivalence relation, LPV-SS): Consider two LPV-SS representations with state-space matrices (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) in $\mathcal{R}^{\cdot \times \cdot}$ where $A_1 \in \mathcal{R}^{n_1 \times n_1}$ and $A_2 \in \mathcal{R}^{n_2 \times n_2}$ and $n_1 \geq n_2$. For a given scheduling dimension $n_{\mathbb{P}}$, these representations are called equivalent,

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \overset{n_{\mathbb{P}}}{\sim} \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right], \tag{33}$$

if there exists an invertible $T \in \mathcal{R}^{n_1 \times n_1}$ such that

$$\overrightarrow{T}A_1T^{-1} = \begin{bmatrix} A_2 & 0 \\ * & * \end{bmatrix}, \quad \overrightarrow{T}B_1 = \begin{bmatrix} B_2 \\ * \end{bmatrix} \begin{matrix} \updownarrow n_2 \\ \updownarrow n_1 - n_2 \end{matrix} \quad C_1T^{-1} = \begin{bmatrix} C_2 & 0 \end{bmatrix}, \quad D_1 = D_2.$$

From the concept of LPV-SS equivalence the concept of minimality directly follows:

Definition 21 (Minimal LPV-SS representation): For a given $n_{\mathbb{P}}$, an SS representation, defined through the matrix functions (A, B, C, D), is called minimal if there exist no (A', B', C', D') with $n'_{\mathbb{X}} < n_{\mathbb{X}}$ such that

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \overset{n_{\mathbb{P}}}{\sim} \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right].$$

Again, using the concept of the SS equivalence relation and minimality, the definition of LPV-SS equivalence classes follows naturally. In addition, the state-dimension $n_{\mathbb{X}}$ of a minimal $\mathfrak{R}_{\mathrm{SS}}(\mathcal{S})$ is equal to the McMillan degree of $\mathcal{S}$.

Example 14 (LPV-SS equivalence and minimality): Consider the LPV-SS representation derived in Example 10. Let $T \in \mathcal{R}^{2\times 2}$ be an invertible state-transformation defined by

$$T \diamond p = \begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{q^{-1}p} \end{bmatrix}, \quad \text{with} \quad T^{-1} \diamond p = \begin{bmatrix} -1 & q^{-1}p \\ 0 & -q^{-1}p \end{bmatrix}, \quad \overrightarrow{T} \diamond p = \begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{p} \end{bmatrix},$$

implying

$$\left[\begin{array}{c|c} \overrightarrow{T} A T^{-1} & \overrightarrow{T} B \\ \hline C T^{-1} & D \end{array}\right] \diamond p = \left[\begin{array}{cc|c} 1 & -\mathrm{T}_{\mathrm{d}}^2 \mathrm{k}_{\mathrm{s}} & \mathrm{T}_{\mathrm{d}}^2 \\ \frac{1}{p} & 1 & 0 \\ \hline 0 & 1 & 0 \end{array}\right].$$

The obtained SS representation is an equivalent minimal SS representation of $\mathcal{S}$ as it is in an equivalence relation with $\mathfrak{R}_{\mathrm{SS}}(\mathcal{S})$ and its state dimension is the same. Note that this realization has only static dependence.

Based on the developed state-transformations and the concepts of state-observability and -reachability matrices, the classical canonical forms can also be defined (see [11], [16]). Furthermore, Definition 20 highlights that applying p-dependent state transformation or system transposition according to the rules of the LTI theory deforms the dynamic relation. This "common practice" leads to inequivalent system representations with arbitrary large difference in terms of manifest behavior (see [11], [16] for illustrative examples).

VI. Equivalence transformations

Next, we introduce equivalence transformations between the SS and IO representation domains. These provide algorithms to obtain an IO (SS) realization of a given LPV-SS (IO) representation, solving the core problem of the existing LPV system theory, motivated in Example 1.

A. State-space to IO

As a consequence of Theorem 3, the following corollary holds:

Corollary 1 (Latent variable elimination): For any latent variable representation (29) with manifest behavior $\mathfrak{B}$ and polynomial matrices $R_{\mathrm{w}} \in \mathcal{R}[\xi]^{n_{\mathrm{r}}\times n_{\mathbb{W}}}$ and $R_{\mathrm{L}} \in \mathcal{R}[\xi]^{n_{\mathrm{r}}\times n_{\mathrm{L}}}$, there exists a unimodular matrix $M \in \mathcal{R}[\xi]^{n_{\mathrm{r}}\times n_{\mathrm{r}}}$ such that

$$M(\xi)R_{\rm w}(\xi)=\begin{bmatrix} R'_{\rm w}(\xi) \\ R''_{\rm w}(\xi) \end{bmatrix}, \quad M(\xi)R_{\rm L}(\xi)=\begin{bmatrix} R'_{\rm L}(\xi) \\ 0 \end{bmatrix}, \tag{34}$$

with $R'_{\rm L}$ of full row rank. The behavior defined by $(R''_{\rm w}(\xi)\diamond p)w = 0$ is equal (almost everywhere) with the manifest behavior of (29).

Due to the latent nature of the variable $w_{\rm L}$, such a transformation is always possible and does not change the manifest behavior, hence it is called an *equivalence transformation*. We can use this result to establish an IO realization of a given SS representation (25a-b) by writing it in the latent form

$$R_{\rm w}(q) = \begin{bmatrix} 0 & B \\ -I & D \end{bmatrix}, \quad R_{\rm L}(q) = \begin{bmatrix} Iq - A \\ -C \end{bmatrix},$$

with $w = \mathrm{col}(u, y)$, $w_{\rm L} = x$, $R_{\rm w} \in \mathcal{R}[\xi]^{(n_{\mathbb{X}}+n_{\mathbb{Y}})\times(n_{\mathbb{X}}+n_{\mathbb{U}})}$, and $R_{\rm L} \in \mathcal{R}[\xi]^{(n_{\mathbb{X}}+n_{\mathbb{Y}})\times n_{\mathbb{X}}}$. According to Corollary 1, there exists a unimodular matrix

$$M(\xi) = \begin{bmatrix} M_{11}(\xi) & M_{12}(\xi) \\ M_{21}(\xi) & M_{22}(\xi) \end{bmatrix} \in \mathcal{R}[\xi]^{(n_{\mathbb{X}}+n_{\mathbb{Y}})\times n_{\mathbb{X}}+n_{\mathbb{Y}}} \tag{35}$$

which in terms of $M(\xi)R_{\rm L}(\xi) = [\ * \quad 0\ ]^\top$ in (34) satisfies $M_{21}(\xi)(I\xi - A) - M_{22}(\xi)C = 0$. This yields that

$$\underbrace{\begin{bmatrix} * & * \\ -M_{21}(\xi) & M_{21}(\xi)B + M_{22}(\xi)D \end{bmatrix}}_{M(\xi)R_{\rm w}(\xi)} = \underbrace{\begin{bmatrix} * \\ 0 \end{bmatrix}}_{M(\xi)R_{\rm L}(\xi)},$$

and $R''_{\rm w}(\xi) = [\ -M_{21}(\xi) \quad M_{21}(\xi)B + M_{22}(\xi)D\ ]$ is in the form of an output side polynomial $R_{\rm y}(\xi) = M_{21}(\xi)$ and an input side polynomial $R_{\rm u}(\xi) = M_{21}(\xi)B + M_{22}(\xi)D$.

Corollary 2 (IO equivalence transformation): Let $\mathfrak{R}_{\rm SS}(\mathcal{S})$ be a state-space representation with manifest behavior $\mathfrak{B}$ and system matrices (A, B, C, D) where $A \in \mathcal{R}^{n_{\mathbb{X}}\times n_{\mathbb{X}}}$. Then there exists a monic polynomial $\bar{R}_{\rm y} \in \mathcal{R}[\xi]^{n_{\mathbb{Y}}\times n_{\mathbb{Y}}}$ with $\deg(\bar{R}_{\rm y}) = n_{\mathbb{X}}$ and a $\bar{R}_{\rm u} \in \mathcal{R}[\xi]^{n_{\mathbb{Y}}\times n_{\mathbb{X}}}$ with $\deg(\bar{R}_{\rm u}) \leq n_{\mathbb{X}} - 1$ such that

$$\bar{R}_{\rm y}(\xi)C = \bar{R}_{\rm u}(\xi)(I\xi - A). \tag{36}$$

Let $R_{\rm c} = \mathrm{diag}(r_1, \ldots, r_{n_{\mathbb{Y}}})$, $r_i \in \mathcal{R}[\xi]$, be the greatest common divisor of $\bar{R}_{\rm y}$ and $\bar{R}_{\rm u}B$ such that there exist $R_{\rm y}, R_{\rm u} \in \mathcal{R}[\xi]$ satisfying

$$R_{\rm c}(\xi)R_{\rm y}(\xi) = \bar{R}_{\rm y}(\xi), \quad R_{\rm c}(\xi)R_{\rm u}(\xi) = \bar{R}_{\rm u}(\xi)B + \bar{R}_{\rm y}(\xi)D. \tag{37}$$

Then the IO representation, given by $(R_{\rm y}(q) \diamond p)y = (R_{\rm u}(q) \diamond p)u$, defines a behavior equal to the manifest behavior of (25a-b), thus it is an IO representation of $\mathcal{S}$.

The algorithm defined by (36) and (37) is structurally similar to the LTI case (see [23], [24]), but it is more complicated as it involves multiplication with the time operators on the coefficients. Thus, this transformation can result in an increased complexity (like dynamic dependence) of the coefficient functions in the equivalent IO representation.

Example 15 (IO equivalence transformation): Consider the LPV-SS representation derived in Example 14. Let r be the identity function so $r \diamond p = p$. In terms of (36), we are looking for a $\bar{R}_\mathrm{u} \in \mathcal{R}[\xi]^{1\times 2}$ with $\deg(\bar{R}_\mathrm{u}) = 1$ and a monic polynomial $\bar{R}_\mathrm{y} \in \mathcal{R}[\xi]$ with $\deg(\bar{R}_\mathrm{y}) = 2$. Parameterize these polynomials as

$$\bar{R}_\mathrm{y}(\xi) = \xi^2 + a_1\xi + a_0, \quad \bar{R}_\mathrm{u}(\xi) = \begin{bmatrix} b_{11}\xi + b_{12} & b_{21}\xi + b_{22} \end{bmatrix}.$$

Then in terms of (36):

$$(\xi^2 + a_1\xi + a_0)\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} b_{11}\xi + b_{12} & b_{21}\xi + b_{22} \end{bmatrix} \underbrace{\begin{bmatrix} \xi - 1 & \mathtt{T}_\mathrm{d}^2\mathtt{k}_\mathrm{s} \\ -\frac{1}{r} & \xi - 1 \end{bmatrix}}_{I\xi - A}.$$

Solving this equation system it follows that

$$a_1 = -\frac{r}{\overrightarrow{r}} - 1, \quad a_0 = \frac{\mathtt{T}_\mathrm{d}^2\mathtt{k}_\mathrm{s} + r}{\overrightarrow{r}}, \quad b_{11} = 0, \; b_{12} = \frac{1}{\overrightarrow{r}}, \quad b_{21} = 1, \quad b_{22} = -\frac{r}{\overrightarrow{r}}.$$

The resulting polynomials $\bar{R}_\mathrm{u}$ and $\bar{R}_\mathrm{y}$ are left coprime, hence

$$R_\mathrm{y}(\xi) = \bar{R}_\mathrm{y}(\xi) = \xi^2 + a_1\xi + a_0, \quad R_\mathrm{u}(\xi) = \bar{R}_\mathrm{u}(\xi)B + \bar{R}_\mathrm{y}(\xi)D = \frac{\mathtt{T}_\mathrm{d}^2}{\overrightarrow{r}}. \tag{38}$$

After left-multiplying these polynomials with $\overrightarrow{r}$, the IO representation in the form of (19) with $n_\mathrm{a} = 2$ and $n_\mathrm{b} = 0$ has the coefficients

$$a_2 \diamond p = qp, \quad a_1 \diamond p = -qp - p, \quad a_0 \diamond p = \mathtt{T}_\mathrm{d}^2\mathtt{k}_\mathrm{s} + p, \quad b_0 \diamond p = \mathtt{T}_\mathrm{d}^2.$$

In terms of $w = \mathrm{col}(y, u)$, the resulting LPV-IO representation is equal to (14) which shows its equivalence with the LPV-SS representation in Example 14.

B. IO to state-space

Finding an equivalent SS representation of a given IO representation is accomplished by constructing a state mapping. This construction can be seen as the counterpart of the latent variable elimination. The aim is to introduce a latent variable into (18) such that it satisfies the state property, i.e. it defines a SS representation (Theorem 4). Similar to the LTI case (see [23], [24]), the central idea of such a state construction is the *cut-and-shift-map* $\varrho_- : \mathcal{R}[\xi]^{\cdot\times\cdot} \to \mathcal{R}[\xi]^{\cdot\times\cdot}$ that acts on polynomial matrices as:

$$\varrho_-(\underbrace{r_0 + r_1\xi + \ldots + r_n\xi^n}_{R(\xi)}) = \overleftarrow{r_1} + \ldots + \overleftarrow{r_n}\xi^{n-1}.$$

This operator can be seen as an intuitive way to introduce state variables for a kernel representation associated with R, as $w_{\mathrm{L}} = \varrho_{-}(R(q) \diamond p)w$ implies that $(R(q) \diamond p)w = (r_0 \diamond p)w + qw_{\mathrm{L}}$. Repeated use of ϱ_{-} and stacking the resulting polynomial matrices gives

$$\underbrace{\begin{bmatrix} \varrho_{-}(R) \\ \varrho_{-}^{2}(R) \\ \vdots \\ \varrho_{-}^{n-2}(R) \\ \varrho_{-}^{n-1}(R) \end{bmatrix}}_{\Sigma_{-}(R)}(\xi) = \begin{bmatrix} r_1^{[1]} + \ldots + r_{n-1}^{[1]}\xi^{n-2} + r_n^{[1]}\xi^{n-1} \\ r_2^{[2]} + \ldots + r_{n-1}^{[2]}\xi^{n-3} + r_n^{[2]}\xi^{n-2} \\ \vdots \\ r_{n-1}^{[n-1]} + r_n^{[n-1]}\xi \\ r_n^{[n]} \end{bmatrix}.$$

where $r_i^{[j]}$ denotes the backward shift operation $\overleftarrow{\bullet}$ applied on r_i for j-times. In case $R \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathbb{W}}}$ with $n_{\mathrm{r}} = 1$, the rows of Σ_{-} are independent, thus it can be shown that $X = \Sigma_{-}(R)$ defines a minimal state-map in the form of

$$x = (X(q) \diamond p)w. \tag{39}$$

In other cases (MIMO case), independent rows of $\Sigma_{-}(R)$ are selected to define a minimal X, but this selection is generally not unique. Later it is shown that a given state-map implies a unique SS representation. Before that, we characterize all possible minimal state maps that lead to an equivalent SS representation.

Denote the left-side multiplication of $R(\xi)$ by ξ as ϱ_{+} and introduce $\mathrm{module}_{\mathcal{R}[\xi]}(R)$ as the left module in $\mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathbb{W}}}$ spanned by the rows of $R \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathbb{W}}}$, i.e. $\mathrm{module}_{\mathcal{R}[\xi]}(R) = \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}([\ R^{\top} \ \ \varrho_{+}(R)^{\top} \ \ \ldots\ ]^{\top})$. This module represents the set of equivalence classes on $\mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(\Sigma_{-}(R))$. Let $X \in \mathcal{R}[\xi]^{\cdot \times n_{\mathbb{W}}}$ be a polynomial matrix with independent rows (full row-rank) and such that

$$\mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(X) \oplus \mathrm{module}_{\mathcal{R}[\xi]}(R) = \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(\Sigma_{-}(R)) + \mathrm{module}_{\mathcal{R}[\xi]}(R), \tag{40}$$

where $\oplus$ denotes direct sum. Then, similar to the LTI case (see [23], [24]), it is possible to show that X is a minimal state-map of the LPV system $\mathcal{S}$ and it defines a state variable by (39) [16]. This way, it is possible to obtain all minimal, equivalent SS realizations of $\mathcal{S}$ which have a kernel representation associated with R.

The next step is to characterize these SS representations with respect to an IO partition. For a given kernel representation associated with the polynomial $R \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathbb{W}}}$, a valid input-output partition (u, y) of the representation is characterized by choosing a selector matrix $S_\mathrm{u} \in \mathbb{R}^{\cdot \times n_\mathbb{W}}$ giving $u = S_\mathrm{u} w$ and a complementary matrix $S_\mathrm{y} \in \mathbb{R}^{\cdot \times n_\mathbb{W}}$ giving $y = S_\mathrm{y} w$.

Assume that a full row rank $X \in \mathcal{R}[\xi]^{\cdot \times n_\mathbb{W}}$ is given which satisfies (40). Then X and S_u jointly lead to

$$\mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(\varrho_+(X)) \subseteq \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(X) \oplus \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(S_\mathrm{u}) \oplus \mathrm{module}_{\mathcal{R}[\xi]}(R). \tag{41}$$

On the other hand, S_y gives

$$\mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(S_\mathrm{y}) \subseteq \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(X) \oplus \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(S_\mathrm{u}) \oplus \mathrm{module}_{\mathcal{R}[\xi]}(R). \tag{42}$$

These inclusions imply that there exist unique matrix functions $\{A, B, C, D\}$ in $\mathcal{R}^{\cdot \times \cdot}$ and polynomial matrix functions $X_\mathrm{u}, X_\mathrm{y} \in \mathcal{R}[\xi]^{\cdot \times \cdot}$ with appropriate dimensions such that

$$\xi X(\xi) \;=\; AX(\xi) + BS_\mathrm{u} + X_\mathrm{u}(\xi)R(\xi), \tag{43a}$$

$$S_\mathrm{y} \;=\; CX(\xi) + DS_\mathrm{u} + X_\mathrm{y}(\xi)R(\xi). \tag{43b}$$

Then the resulting matrix function $\{A, B, C, D\}$ define a minimal state-representation of the LPV system $\mathcal{S}$. This algorithm provides an SS realization of both LPV-IO and LPV-KR representations. Specific choices of X leads to specific canonical forms. Note that a similar algorithm can be deduced for a realization in a image type of representation, i.e. latent variable representation (29) where $R_\mathrm{w}(q) = I$.

Example 16 (SS equivalence transformation): Consider the LPV-IO representation derived in Example 15:

$$R_\mathrm{y}(\xi) = \xi^2 - \left(1 + \tfrac{r}{\overrightarrow{r}}\right)\xi + \tfrac{\mathrm{T}_\mathrm{d}^2 \mathrm{k}_\mathrm{s} + r}{\overrightarrow{r}}, \quad R_\mathrm{u}(\xi) = \tfrac{\mathrm{T}_\mathrm{d}^2}{\overrightarrow{r}}.$$

Denote $R(\xi) = [\; R_\mathrm{y}(\xi) \quad -R_\mathrm{u}(\xi) \;]$, and generate the state-map

$$X(\xi) = \Sigma_-(R(\xi)) = \begin{bmatrix} \xi - (1 + \frac{\overleftarrow{r}}{r}) & 0 \\ 1 & 0 \end{bmatrix}.$$

Now with $S_\mathrm{y} = [\; 1 \quad 0 \;]$ and $S_\mathrm{u} = [\; 0 \quad 1 \;]$, equations (43a-b) read as

$$\underbrace{\begin{bmatrix} \xi^2 - \left(1 + \frac{r}{\overrightarrow{r}}\right)\xi & 0 \\ \xi & 0 \end{bmatrix}}_{\xi X(\xi)} = \underbrace{\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} \xi - \left(1 + \frac{\overleftarrow{r}}{r}\right) & 0 \\ 1 & 0 \end{bmatrix}}_{X(\xi)} + \underbrace{\begin{bmatrix} 0 & \beta_1 \\ 0 & \beta_2 \end{bmatrix}}_{BS_\mathrm{u}} + \begin{bmatrix} X_{\mathrm{u}1}(\xi) \\ X_{\mathrm{u}2}(\xi) \end{bmatrix} R(\xi),$$

$$\underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{S_\mathrm{y}} = \underbrace{\begin{bmatrix} c_1 & c_2 \end{bmatrix}}_{C} \cdot \underbrace{\begin{bmatrix} \xi - (1 + \frac{\overleftarrow{r}}{r}) & 0 \\ 1 & 0 \end{bmatrix}}_{X(\xi)} + \underbrace{\begin{bmatrix} 0 & d_1 \end{bmatrix}}_{DS_\mathrm{u}} + X_\mathrm{y}(\xi)R(\xi).$$

By solving these equations, it follows that

$$\alpha_{11}=0 \qquad \alpha_{12}=-\frac{\mathtt{T}_{\mathrm{d}}^2\mathtt{k}_{\mathrm{s}}+r}{\overrightarrow{r}} \qquad \alpha_{21}=1 \qquad \alpha_{22}=1+\frac{\overleftarrow{r}}{r} \qquad \beta_1=\frac{\mathtt{T}_{\mathrm{d}}^2}{\overrightarrow{r}} \qquad \beta_2=0$$

$$c_1=0 \qquad c_2=1 \qquad d_1=0 \qquad X_{\mathtt{u}1}(\xi)=1 \qquad X_{\mathtt{u}2}(\xi)=0 \qquad X_{\mathtt{y}}(\xi)=0$$

Then, the obtained LPV-SS representation is

$$\mathfrak{R}_{\mathrm{SS}}(\mathcal{S})=\left[\begin{array}{cc|c} 0 & -\frac{\mathtt{T}_{\mathrm{d}}^2\mathtt{k}_{\mathrm{s}}+p}{qp} & \frac{\mathtt{T}_{\mathrm{d}}^2}{qp} \\ 1 & 1+\frac{q^{-1}p}{p} & 0 \\ \hline 0 & 1 & 0 \end{array}\right], \text{ which through } T\diamond p=\begin{bmatrix} p & q^{-1}p \\ 0 & 1 \end{bmatrix},$$

is in equivalence relation with the LPV-SS representation of Example 14 follows. The latter proves that the IO representation given by R_{y} and R_{u} has the same manifest behavior as $\mathfrak{R}_{\mathrm{SS}}(\mathcal{S})$.

VII. Conclusion

In this paper, we have extended the behavioral approach to LPV systems in order to lay the foundations of a LPV system theory which provides a clear understanding of this system class and the relations of its representations. We have defined LPV systems as the collection of signal and scheduling trajectories and it has been shown that representations of these systems need dynamic dependence on the scheduling variable. By the use of such system descriptions, it has been proven that equivalence relations and transformations between these descriptions can be developed, giving a common ground where model structures of LPV system identification and concepts of LPV control can be compared, analyzed, and further developed.

VIII. Appendix

A. *Proof of the injective cogenerator property*

The concept of the proof is based on [20]. Let $\mathbb{R}_\infty=\mathbb{R}\cup\{-\infty,\infty\}$ and denote by $\mathcal{Q}_n$ all maps from $(\mathbb{Z},\mathbb{R}^n)$ to $\mathbb{R}_\infty$ that are bounded except for a discrete set of points on $\mathbb{R}^n$, i.e. for each $w\in\mathcal{Q}_n$ there exists a discrete set $\mathbb{E}(w)\subset\mathbb{R}^n$ such that $w\in\mathbb{R}^{\mathbb{Z}\times(\mathbb{R}^n\setminus\mathbb{E}(w))}$. The set $\mathcal{Q}_n$ is a real vector space for each $n\in\mathbb{N}$. Denote $\bar{\mathcal{Q}}_n\subset\mathcal{Q}_n$ all $w\in\mathcal{Q}_n$ for which there exist a $k\in\mathbb{Z}$ and $\mathtt{x}_1,\dots\mathtt{x}_n\in\mathbb{R}$ such that $w(k,\mathtt{x}_1,\dots,\mathtt{x}_n)\neq w(k,\mathtt{x}_1,\dots,\mathtt{x}_{n-1},0)$. Denote $\mathcal{Q}=\bigcup_{n\in\mathbb{N}}\bar{\mathcal{Q}}_n$. $\mathcal{Q}$ is an (additive) Abelian group.

Consider a $R\in\mathcal{R}[\xi]^{n_{\mathrm{r}}\times n_{\mathbb{W}}}$ with $\mathbb{P}=\mathbb{R}^{n_{\mathbb{P}}}$. For a $w\in\mathcal{Q}$, $R\odot w=0$ means that any $(w,p)\in(\mathbb{R}_\infty^{n_{\mathbb{W}}}\times\mathbb{R}^{n_{\mathbb{P}}})^{\mathbb{Z}}$ satisfying

$$w(k)=w(k,[\begin{array}{cccc} p(k) & p(k+1) & p(k-1) & \dots \end{array}]), \tag{44}$$

for all $k \in \mathbb{Z}$, also satisfies $(R(q) \diamond p)(k)w(k) = 0$ for all $k \in \mathbb{Z} \backslash \mathbb{J}(w, p)$, where $\mathbb{J}(w, p) = \{k \in \mathbb{Z} \mid [\; p(k) \;\; p(k+1) \;\; p(k-1) \;\; \ldots \;] \in \mathbb{E}(w)\}$. As $\mathbb{E}(w)$ is discrete, this means that there exists also a (bounded solution) $(w, p) \in (\mathbb{R}^{n_\mathbb{W}} \times \mathbb{R}^{n_\mathbb{P}})^{\mathbb{Z}}$ satisfying (44) such that $(R(q) \diamond p)(k)w(k) = 0$ holds for all $k \in \mathbb{Z}$. The set $\mathfrak{B}_*$ given as $\mathfrak{B}_* = \{w \in \mathcal{Q}^{n_\mathbb{W}} \mid R \odot w = 0\}$, is called the complete solution space of the linear system of PV difference equations (KR-representation) $(R(q) \diamond p)w = 0$. Note that the behavior $\mathfrak{B}$ of R defined by (16), contain the set of trajectories (w, p) that satisfy $w \in \mathfrak{B}_*$ and bounded, while $\mathfrak{B}_*$ describes the relationship of the trajectories containing the descriptions of possible solutions which fall out from $\mathfrak{B}$ due to the singularity of the coefficients in R.

Let $M_1 \in \mathcal{R}[\xi]^{n_\mathrm{r} \times n_\mathrm{r}}$ and $M_2 \in \mathcal{R}[\xi]^{n_\mathbb{W} \times n_\mathbb{W}}$ be unimodular matrices such that (15) is the Jacobson form of R with $Q = \mathrm{diag}(r_1, \ldots, r_n) \in \mathcal{R}[\xi]^{n \times n}$. It can be shown (see [19], [16]), that $(R(q) \diamond p)w = 0$ has the same solutions as

$$(M_1(q)R(q) \diamond p)w = (Q(q)M_2^\dagger(q) \diamond p)w = 0, \tag{45}$$

so there is an isomorphism of solution spaces

$$\mathfrak{B}_* \cong \tilde{\mathfrak{B}}_* := \{\tilde{w} \in \mathcal{Q}^{n_\mathbb{W}} \mid [\; Q \quad 0 \;] \odot \tilde{w} = 0\}, \tag{46a}$$

$$w \to \tilde{w} := M_2^\dagger(q)w, \tag{46b}$$

where $r_i \odot \tilde{w}_i = 0$ for $i \in \{1, \ldots, n\}$. Introduce $\mathcal{M}_R = \mathrm{module}_{\mathcal{R}[\xi]}(R)$ as the left module in $\mathcal{R}[\xi]^{n_\mathrm{r} \times n_\mathbb{W}}$ spanned by the rows of $R \in \mathcal{R}[\xi]^{n_\mathrm{r} \times n_\mathbb{W}}$. Then

$$\mathfrak{B}_* \cong \hom_{\mathcal{R}[\xi]}(\mathcal{M}_R, \mathcal{Q}^{n_\mathbb{W}}), \tag{47}$$

which corresponds to the so-called Malgrange isomorphism. Explicitly, (47) assigns to each $w \in \mathfrak{B}_*$ the linear map $\phi_w : \mathcal{M}_R \mapsto \mathcal{Q}$ defined by $\phi_w([r]) := r(q)w$ where $[r]$ denotes the residue class of $r \in \mathcal{R}[\xi]^{1 \times n_\mathbb{W}}$ in $\mathcal{M}_R$, and the well definedness of ϕ_w follows from

$$[r_1] = [r_2] \;\; \to \;\; r_1 - r_2 \in \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(R) \;\; \to \;\; r_1(q)w = r_2(q)w,$$

for all $w \in \mathfrak{B}_*$ which also implies that $\mathcal{Q}^{n_\mathbb{W}} \cong \hom_{\mathcal{R}[\xi]}(\mathcal{R}[\xi]^{1 \times n_\mathbb{W}}, \mathcal{Q}^{n_\mathbb{W}})$. Conversely, for a linear map $\phi : \mathcal{M}_R \mapsto \mathcal{Q}$ one defines $w_i := \phi([e_i])$, where e_i is the i-th natural basis vector of $\mathcal{R}[\xi]^{1 \times n_\mathbb{W}}$. Then we have

$$\phi([r]) = \phi\left([\textstyle\sum_{i=1}^{n_\mathbb{W}} r_i e_i]\right) = \textstyle\sum_{i=1}^{n_\mathbb{W}} r_i(q)\phi([e_i]) = \textstyle\sum_{i=1}^{n_\mathbb{W}} r_i(q)w_i = r(q)w.$$

Due to (44), the above equation implies an isomorphism of left modules:

$$\text{module}_{\mathcal{R}[\xi]}(R) \cong \text{module}_{\mathcal{R}[\xi]}([Q \quad 0]), \tag{48a}$$

$$[r] \to [rM_2]. \tag{48b}$$

Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ be left modules in $\mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathbb{W}}}$ and let $\phi_{12} : \mathcal{M}_1 \mapsto \mathcal{M}_2$ and $\phi_{23} : \mathcal{M}_2 \mapsto \mathcal{M}_3$ be linear maps, i.e. left module homomorphisms. Then

$$\mathcal{M}_1 \overset{\phi_{12}}{\to} \mathcal{M}_2 \overset{\phi_{23}}{\to} \mathcal{M}_3 \tag{49}$$

is exact if $\text{im}(\phi_{12}) = \ker(\phi_{23})$. The same notion can be used if $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ are Abelian groups and ϕ_{12}, ϕ_{23} are group homomorphisms. Then $\mathcal{Q}$ is called an injective cogenerator if the sequence

$$\mathcal{M}_1 \mapsto \mathcal{M}_2 \mapsto \mathcal{M}_3, \tag{50}$$

is exact iff the sequence

$$\text{hom}_{\mathcal{R}[\xi]}(\mathcal{M}_1, \mathcal{Q}^{n_{\mathbb{W}}}) \leftarrow \text{hom}_{\mathcal{R}[\xi]}(\mathcal{M}_2, \mathcal{Q}^{n_{\mathbb{W}}}) \leftarrow \text{hom}_{\mathcal{R}[\xi]}(\mathcal{M}_3, \mathcal{Q}^{n_{\mathbb{W}}})$$

of Abelian groups is exact.

For injectivity, one needs to prove according to Corollary 3.17 of [25]: For every $0 \neq R \in \mathcal{R}[\xi]$ and every $w_{\mathrm{u}} \in \mathcal{Q}$, there exists a $w_{\mathrm{y}} \in \mathcal{Q}$ such that $R \odot w_{\mathrm{y}} = w_{\mathrm{u}}$. Let $R(\xi) = \sum_{i=0}^{n_\xi} r_i \xi^i$ be given with $r_{n_\xi} \neq 0$. If $n_\xi = 0$, there is nothing to prove. Since $\mathcal{R}$ is a field, assume that $r_{n_\xi} = 1$. Then $R \odot w_{\mathrm{y}} = w_{\mathrm{u}}$ can be rewritten as a first-order system

$$(r_{\mathrm{x},1} q + r_{\mathrm{x},0}) \odot w_{\mathrm{x}} = r_{\mathrm{u}} \odot w_{\mathrm{u}}, \tag{51}$$

where $w_{\mathrm{x}} = [\; w_{\mathrm{y}} \quad \ldots \quad q^{-(n_\xi - 1)} w_{\mathrm{y}} \;]^\top$, $r_{\mathrm{x},1} = I$, $r_{\mathrm{u}} = [\; 0 \quad \ldots \quad 0 \quad 1 \;]^\top \in \mathbb{R}^{n_\xi}$ and

$$r_{\mathrm{x},0} = \begin{bmatrix} 0 & -I \\ r_0 & r_* \end{bmatrix} \in \mathcal{R}^{n_\xi \times n_\xi}, \quad \text{with} \quad r_* = [\; r_1 \quad \ldots \quad r_n \;]. \tag{52}$$

Let $\mathbb{E}(R)$ denote the discrete set of singularities of the meromorphic coefficients r_i in R. Let $\mathbb{E}(w_{\mathrm{y}}) := \mathbb{E}(w_{\mathrm{u}}) \cup \mathbb{E}(R)$ which is still discrete. Hence, $\mathbb{R}^{\cdot} \setminus \mathbb{E}(w_{\mathrm{y}})$ is a countable union of open intervals $I_i \in \mathbb{R}^{\cdot}$ and on each I_i it holds that R_0 and w_{u} are bounded. Therefore there exists a bounded solution $w_{\mathrm{x}} : (\mathbb{Z} \times I_i) \mapsto \mathbb{R}^{n_\xi}$ to (51) on each I_i. By concatenating them, one gets a solution $w_{\mathrm{x}} \in \mathcal{Q}^{n_\xi}$ and thus $w_{\mathrm{y}} \in \mathcal{Q}$.

For the cogenerator property, it has to be shown that if for some $R \in \mathcal{R}[\xi]$, $R \odot w_{\mathrm{y}} = 0$ has only the zero solution, then this implies that $R \in \mathcal{R}$ and $R \neq 0$. Assume the contrary and let $\deg(R) = n_\xi \geq 1$. Then one can rewrite $R \odot w_{\mathrm{y}} = 0$ as $q w_{\mathrm{x}} = -r_{\mathrm{x},0} \odot w_{\mathrm{x}}$ like in the previous part. Let $\mathbb{E}(w_{\mathrm{y}}) = \mathbb{E}(R)$, then on each of the intervals I_i, the solution set of this homogenous equation is an n_ξ-dimensional subspace of $(\mathbb{R}^{n_\xi})^{\mathbb{R} \times I_i}$, in particular there exist non-zero solutions. By concatenating them, we get a non-zero solution $w_{\mathrm{x}} \in \mathcal{Q}^n$. If $w_{\mathrm{y}} = w_{\mathrm{x},1}$ was identically zero, then $w_{\mathrm{x}} = [\; w_{\mathrm{y}} \quad \ldots \quad q^{-(n_\xi - 1)} w_{\mathrm{y}} \;]^\top$ would be identically zero which leads to a contradiction. ■

B. Proof of Theorem 2

Consider $\mathfrak{R}_{\mathrm{K}}(\mathcal{S})$ with $R \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathbb{W}}}$, $\mathbb{P} = \mathbb{R}^{n_{\mathbb{P}}}$, and behavior $\mathfrak{B}$ in terms of (16). Without loss of generality, let $R \neq 0$ as the behavior $\mathfrak{B} = (\mathbb{R}^{n_{\mathbb{W}}} \times \mathbb{R}^{n_{\mathbb{P}}})^{\mathbb{Z}}$ can be represented by the empty matrix which is full rank by convention. Let $M_1 \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathrm{r}}}$ and $M_2 \in \mathcal{R}[\xi]^{n_{\mathbb{W}} \times n_{\mathbb{W}}}$ be unimodular matrices such that (15) is the Jacobson form of R in terms of Theorem 1 with $Q = \mathrm{diag}(r_1, \ldots, r_n) \in \mathcal{R}[\xi]^{n \times n}$. Partition $M_2^{\dagger} = [\ W_1 \quad W_2\ ]^{\top}$ according to the partition of the Jacobson form. Since M_1 is unimodular, the solution space of $(R(q) \diamond p)w = 0$ is equal to the solution space of $(M_1(q)R(q) \diamond p)w = 0$ (see the previous proof). Thus $R' := QW_1$ also represents $\mathfrak{B}$ in an almost everywhere sense, i.e. for all trajectories of $p \in \mathfrak{B}_{\bar{\mathbb{P}}}$ for which the coefficients of R' are bounded, and $\mathrm{rank}(R') = n$. ■

C. Proof of Theorem 3

Based on the proof of the injective cogenerator property, consider

$$\mathfrak{B}_* = \{w \in \mathcal{Q}^{n_{\mathbb{W}}} \mid \exists w_{\mathrm{L}} \mathcal{Q}^{n_{\mathrm{L}}} : R_{\mathrm{w}} \odot w = R_{\mathrm{L}} \odot w_{\mathrm{L}}\}, \tag{53}$$

where $R_{\mathrm{w}} \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathbb{W}}}$ and $R_{\mathrm{L}} \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathrm{L}}}$ defines an LPV latent variable representation in the form of (20) with $\mathbb{P} = \mathbb{R}^{n_{\mathbb{P}}}$. Then showing that $\mathfrak{B}_*$ has a kernel representation is equivalent with showing that the manifest behavior of (20) has a kernel representation in an almost everywhere sense. Define the left kernel of R_{L} as

$$\ker_{\mathcal{R}[\xi]}(R_{\mathrm{L}}) = \{r \in \mathcal{R}[\xi]^{1 \times n_{\mathrm{r}}} \mid rR_{\mathrm{L}} = 0\}, \tag{54}$$

which is a left submodule of $\mathcal{R}[\xi]^{1 \times n_{\mathrm{r}}}$. Thus, it is finitely generated, i.e. there exists a $Q \in \mathcal{R}[\xi]^{n \times n_{\mathrm{r}}}$ such that $\mathrm{im}_{\mathcal{R}[\xi]}(Q) = \ker_{\mathcal{R}[\xi]}(R_{\mathrm{L}})$. Then we have an exact sequence

$$\mathcal{R}[\xi]^{1 \times n} \overset{\cdot Q}{\to} \mathcal{R}[\xi]^{1 \times n_{\mathrm{r}}} \overset{\cdot R_{\mathrm{L}}}{\to} \mathcal{R}[\xi]^{1 \times n_{\mathrm{L}}} \tag{55}$$

and therefore the sequence $\mathcal{Q}^{n} \overset{Q(q)}{\leftarrow} \mathcal{Q}^{n_{\mathrm{r}}} \overset{R_{\mathrm{L}}(q)}{\leftarrow} \mathcal{Q}^{n_{\mathrm{L}}}$ is also exact. This signifies that $R_{\mathrm{w}}(q)w \in \mathrm{im}_{\mathcal{Q}}(R_{\mathrm{L}}) := \{R_{\mathrm{L}}(q)w_{\mathrm{L}} \mid w_{\mathrm{L}} \in \mathcal{Q}^{n_{\mathrm{L}}}\}$ iff $R_{\mathrm{w}}(q)w \in \ker_{\mathcal{Q}}(Q)$, i.e. $\mathfrak{B}_* = \{w \in \mathcal{Q}^{n_{\mathbb{W}}} \mid QR_{\mathrm{w}} \odot w = 0\}$. ■

D. Proof of Theorem 4

The concept of the proof is based on [23]. To simplify the discussion, we prove only the so-called *Markovian case* as the state case follows trivially from this concept due to the linearity and time-invariance of LPV systems. We call the discrete-time LPV system $\mathcal{S} = (\mathbb{Z}, \mathbb{P}, \mathbb{W}, \mathfrak{B})$ Markovian, if for all $p \in \mathfrak{B}_{\mathbb{P}}$

$$(w_1, w_2 \in \mathfrak{B}_p) \wedge (w_1(0) = w_2(0)) \rightarrow (w_1 \underset{0}{\wedge} w_2) \in \mathfrak{B}_p.$$

In the following, we prove that $\mathcal{S}$ is Markovian, iff there exist matrices $r_0, r_1 \in \mathcal{R}^{n_\mathrm{r} \times n_\mathbb{W}}$ such that $\mathfrak{B}$ has the kernel representation: $r_0 w + r_1 \xi w = 0$. where $\xi = q$. The "if" part is trivial. To show the "only if" case, assume that a KR representation of $\mathcal{S}$ is given with $R \in \mathcal{R}[\xi]^{n_\mathrm{r} \times n_\mathbb{W}}$ for which the solutions of (13) satisfy the above given connectability condition. Without loss of generality it can be assumed that R is full row rank. Also, there exists a unimodular $M \in \mathcal{R}[\xi]^{\cdot \times n_\mathrm{r}}$ such that $R' = MR$ is in a row reduced form, meaning that the matrix formed by the coefficient functions of the highest powers in ξ of the rows $R'(\xi)$ has full row rank. Due to the fact that M is a left-side unimodular transformation, the behaviors of R and R' are equivalent.

We show now that $\deg(R') = 1$. Assume the contrary and write R' in the IO form:

$$(R_1(q) \diamond p) w_1 = (R_2(q) \diamond p) w_2, \tag{56}$$

where $\mathrm{col}(w_1, w_2) = w$ corresponds to an IO partition and $\deg(R_1) \geq \deg(R_2)$. The assumption that $\deg(R') > 1$ implies that $\deg(R_1) > 1$. Similarly, the assumption of $(R'(q) \ \diamond \ p)w = 0$ is Markovian implies that $(R_1(q) \ \diamond \ p)w_1 = 0$ is Markovian.

Now let w_1', w_1'' be the solutions of $(R_1(q) \diamond p)w_1 = 0$ for a $p \in \mathfrak{B}_\mathbb{P}$ with $w_1'(0) = w_1''(0)$. Since (w_1, w_2) is an IO partition of $\mathcal{S}$, thus $\mathrm{col}(w_1', 0)$ and $\mathrm{col}(w_1'', 0)$ are also solutions of $(R'(q) \diamond p)w = 0$ and in order to obtain contradiction it suffices to prove contradiction for autonomous systems. Let $n_\xi = \deg(R_1)$ and by assumption $n_\xi > 1$. Introduce auxiliary variables $\breve{w}_{ij}$ defined as

$$\breve{w}_{ij} := q^i w_j, \quad (i, j) \in \mathbb{I}_0^{n_\xi} \times \mathbb{I}_1^{n_\mathbb{W}}, \tag{57}$$

where $w = [w_1 \ \ldots \ w_{n_\mathbb{W}}]^\top$. Collect these variables in a column vector

$$\breve{w} = \begin{bmatrix} \breve{w}_{01} & \breve{w}_{02} & \ldots & \breve{w}_{0n_\mathbb{W}} & \breve{w}_{11} & \ldots & \breve{w}_{n_\xi n_\mathbb{W}} \end{bmatrix}^\top. \tag{58}$$

Now consider the system with latent variable $\breve{w}$ as

$$q\breve{w} = (r \diamond p)\breve{w}, \tag{59a}$$

$$w_j = \breve{w}_{0j}, \quad \forall j \in \mathbb{I}_1^{n_\mathbb{W}}. \tag{59b}$$

where the coefficient $r \in \mathcal{R}^{(n_\xi n_\mathbb{W}) \times (n_\xi n_\mathbb{W})}$ is determined from the coefficients of $R_1(\xi)$ and the definition (57). The manifest behavior of (59a) is equivalent with the manifest behavior of $R_1(\xi)$, which can be checked by elimination of the latent variables of (59a-b). However, the manifest behavior can not be Markovian as (59a-b) has exactly one solution $(w, \breve{w})$ for each initial condition $\breve{w}(0)$ and scheduling trajectory $p \in \mathfrak{B}_{\mathbb{P}}$. This contradicts Markovianity, since two solutions $(w, \breve{w})$ and $(w', \breve{w}')$ with $\breve{w}_{0j}(0) = \breve{w}'_{0j}(0)$, $\forall j \in \mathbb{I}_1^{n_{\mathbb{W}}}$ cannot be connected unless also $\breve{w}_{ij}(0) = \breve{w}'_{ij}(0)$, $\forall (i, j) \in \mathbb{I}_1^{n_\xi - 1} \times \mathbb{I}_1^{n_{\mathbb{W}}}$. ■

E. Proof of Theorem 5

First consider the left side transformation. Let $R \in \mathcal{R}[\xi]^{n_{\mathrm{r}} \times n_{\mathbb{W}}}$ and $R' \in \mathcal{R}[\xi]^{n \times n_{\mathrm{r}}}$ and $\mathbb{P} = \mathbb{R}^{n_{\mathbb{P}}}$. Based on the proof of the injective cogenerator property, consider $\mathfrak{B}_*$ and $\mathfrak{B}'_*$ as the complete behaviors of R and R'. Then the inclusion $\mathfrak{B}'_* \subseteq \mathfrak{B}_*$ can be expressed as an exact sequence

$$0 \rightarrow \mathfrak{B}'_* \rightarrow \mathfrak{B}_*, \tag{60}$$

which is equivalent to the exact sequence

$$0 \leftarrow \mathrm{module}_{\mathcal{R}[\xi]}(R') \leftarrow \mathrm{module}_{\mathcal{R}[\xi]}(R). \tag{61}$$

Equivalently, we have $\mathrm{span}^{\mathrm{row}}_{\mathcal{R}}(R') \supseteq \mathrm{span}^{\mathrm{row}}_{\mathcal{R}}(R)$ or $R' = QR$ for some $Q \in \mathcal{R}[\xi]^{n \times n_{\mathrm{r}}}$. If $\mathfrak{B}_* = \mathfrak{B}'_*$, then $R' = Q_1 R$ and $R = Q_2 R'$, which shows that R and R' has the same rank. If additionally, R and R' are full rank, than this implies that $Q_1 = Q_2^\dagger$, ergo Q_1 and Q_2 are unimodular. As the complete behaviors are equal therefore this implies that the behaviors of R and R' for each commonly valid trajectories of p are equal.

Consider the right side transformation. Based on the proof of the injective cogenerator property, there is a homomorphism between the the complete behaviors of R and $R' = RQ_1$ and also between $R = R'Q_2$ and R'. This implies that if $Q_1 = Q_2^\dagger$, ergo Q_1 and Q_2 are unimodular, then there exists a isomorphism between the behaviors. ■

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