Example exam
Model Predictive Control
(SC4060)

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Exercise 1 — Standard formulation

Consider the IO model

$$x_o(k+1) = A_o x_o(k) + K_o e_o(k) + L_o d_o(k) + B_o u(k) \quad (1)$$
$$y(k) = C_o x_o(k) + e_o(k) \quad (2)$$

where $A_o \in \mathbb{R}^{n_a \times n_a}$, $K_o \in \mathbb{R}^{n_a \times 1}$, $L_o \in \mathbb{R}^{n_a \times 1}$, $B_o \in \mathbb{R}^{n_a \times 1}$ and $C_o \in \mathbb{R}^{1 \times n_a}$. The state is $x_o(k)$, $d_o(k)$ is the known disturbance, $u(k)$ is the input signal and $e_o(k)$ is zero-mean white noise. Note that this is a SISO model.

A performance index $J(k)$ is given by:

$$J(u,k) = \sum_{j=N_m}^{N} \Big(\hat{\varepsilon}(k+j|k)\Big)^2 + \lambda^2 \sum_{j=1}^{N} \Big(u(k+j-1|k)\Big)^2 \quad (3)$$

where ˆ denotes prediction, and $\varepsilon(k)$ contribute to the performance index only if $y(k) - r(k) < \delta_{\text{min}}$ or $y(k) - r(k) > \delta_{\text{max}}$:

$$\varepsilon(k) = \begin{cases} 0 & \text{for} \quad \delta_{\text{min}} \le y(k) - r(k) \le \delta_{\text{max}} \\ y(k) - r(k) - \delta_{\text{max}} & \text{for} \quad y(k) - r(k) > \delta_{\text{max}} \\ y(k) - r(k) - \delta_{\text{min}} & \text{for} \quad y(k) - r(k) < \delta_{\text{min}} \end{cases}$$

so

$$|\varepsilon(k)| = \min_{\delta_{\text{min}} \le \delta(k) \le \delta_{\text{max}}} |y(k) - r(k) - \delta(k)|$$

(1.1) Describe the problem of finding the control-law that minimizes performance index (3) as a standard predictive control problem, and give the corresponding system matrices and the selection matrices $\Gamma(j)$.

Exercise 2 — Solution of the SPCP

Consider the IIO ($v = \Delta u$) system (5.1) - (5.3) with a steady-state

$$(v_{ss}, x_{ss}, w_{ss}, z_{ss}) = (v_{ss}, x_{ss}, w_{ss}, 0)$$

at time k with state $x(k)$, external signal $\tilde{w}(k)$ and ZMWN signal $e(k)$.
Define

$$J_i^*(k) = \min_{\tilde{v}} J_i(k)$$

where

- J_1 is the performance index for the SPCP with prediction horizon $N = 40$, control horizon $N_c = 20$, and finite minimum cost horizon $N_m = 2$.
- J_2 is the performance index for the SPCP with infinite prediction horizon ($N = \infty$), control horizon $N_c = 20$, and finite minimum cost horizon $N_m = 1$.
- J_3 is the performance index for the SPCP with infinite prediction horizon ($N = \infty$), finite control horizon $N_c = 20$, and finite minimum cost horizon $N_m = 2$ subject to inequality constraints $\psi(k) \leq \Psi(k)$.
- J_4 is the performance index for the unconstrained SPCP with prediction horizon $N = 40$, control horizon $N_c = 20$, and finite minimum cost horizon $N_m = 4$.
- J_5 is the performance index for the SPCP with infinite prediction horizon ($N = \infty$), control horizon $N_c = 20$, and finite minimum cost horizon $N_m = 1$, subject to inequality constraints $\psi(k) \leq \Psi(k)$.

(2.1)
Describe how the performance index $J^*(k)$ of the SPCP changes when

- the prediction horizon is increased (decreased),
- the minimum cost horizon is increased (decreased),
- an inequality constraint is introduced (removed).

(2.2) Find a ranking in magnitude $J_i^* \leq J_j^*$ for all combinations $i = 1, \ldots, 5$, $j = 1, \ldots, 5$.
(For example, four fictive performance indices J_7^*, J_8^*, J_9^* and J_{10}^* give a ranking: $J_7^* \leq J_8^* \leq J_9^*$ and $J_7^* \leq J_{10}^* \leq J_9^*$ if J_7^* is the smallest, J_9^* is the largest, but the order in J_8^* and J_{10}^* is indefinite). Comment on your choices.

Exercise 3 **Model predictive control of a bicycle**

(3.1) If you ride a bicycle and you want to make a turn to the left, you first have to steer a little bit to the right to make your bicycle tilt to the left. This means that, if we use MPC (with a sufficiently large prediction horizon), the optimal control sequence will be to initially steer to the right and later steer to the left.
If we use this optimal control sequence in receding horizon, only the first control action will be implemented. As a consequence we will always steer the right, and we will make a turn to the right instead of a turn to the left.

Is the above reasoning correct and may we conclude that MPC does not work for controlling a bicycle?

Exercise 4 **Stability of infinite horizon MPC**

Given the following theorem:

Theorem

Consider a LTI system given in the state space description

$$\begin{aligned} x(k+1) &= Ax(k) + B_2\, w(k) + B_3\, v(k) \\ z(k) &= C_2\, x(k) + D_{22}\, w(k) + D_{23}\, v(k) \end{aligned}$$

with $z_{ss} = 0$ in steady state. A performance index is defined as

$$\min_{\tilde{v}(k)} J(\tilde{v}, k) = \min_{\tilde{v}(k)} \sum_{j=0}^{N-1} \hat{z}(k+j|k)\Big)^T \Big(\hat{z}(k+j|k) \tag{4}$$

where an additional equality constraint is given by:

$$x(k+N) = x_{ss} \tag{5}$$

Finally, let $w(k) = w_{ss}$ for $k \geq 0$. Then, the predictive control law, minimizing this performance index subject to the equality constraint results in a stable closed loop.

(4.1)

Proof this theorem.