Written exam
Model Predictive Control
(SC4060)

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Exercise 1 **Standard formulation** **(25%)**

Consider the IO model

$$\begin{aligned} x_o(k+1) &= A_o x_o(k) + K_o e_o(k) + L_o d_o(k) + B_o u(k) \\ y(k) &= C_o x_o(k) + e_o(k) \end{aligned}$$

where $A_o \in \mathbb{R}^{n_a \times n_a}$, $K_o \in \mathbb{R}^{n_a \times 1}$, $L_o \in \mathbb{R}^{n_a \times 1}$, $B_o \in \mathbb{R}^{n_a \times 1}$ and $C_o \in \mathbb{R}^{1 \times n_a}$. The state is $x_o(k)$, $d_o(k)$ is the known disturbance, $u(k)$ is the input signal and $e_o(k)$ is zero-mean white noise. Note that this is a SISO model.

A performance index $J(k)$ is given by:

$$\begin{aligned} J(u,k) = \sum_{j=1}^{N} \Big(& \hat{x}_o^T(k+j|k) Q \hat{x}_o(k+j|k) + u^T(k+j-1|k) R u(k+j-1|k) \\ & + \hat{y}^T(k+j|k) S \hat{y}(k+j|k) \Big) \end{aligned} \tag{1}$$

where ˆ denotes prediction, and Q, R, S are positive definite matrices.
Consider the following constraint:

$$u(k+j) = u(k+j-1) \ , \text{ for } \ j \geq N_c \tag{2}$$

(1.1) Describe the problem of finding the control-law that minimizes performance index (1) subject to (2) as a standard predictive control problem, and give the corresponding system matrices and the selection matrices $\Gamma(j)$.

Exercise 2 **Solution of the SPCP** **(25%)**

Consider the system

$$\begin{aligned} x(k+1) &= A\,x(k) + B_1\,e(k) + B_2\,w(k) + B_3\,v(k) \\ y(k) &= C_1\,x(k) + D_{11}\,e(k) + D_{12}\,w(k) \\ z(k) &= C_2\,x(k) + D_{21}\,e(k) + D_{22}\,w(k) + D_{23}\,v(k) \end{aligned}$$

with $v = \Delta u$ (so we have an IIO system) and a steady-state

$$(v_{ss}, x_{ss}, w_{ss}, z_{ss}) = (v_{ss}, x_{ss}, w_{ss}, 0)$$

at time k with state $x(k)$, external signal $\tilde{w}(k)$ and ZMWN signal $e(k)$.
Define

$$J_i^*(k) = \min_{\tilde{v}} J_i(k)$$

where

- J_1 is the performance index for the SPCP with prediction horizon $N = 30$ and control horizon $N_c = 15$.
- J_2 is the performance index for the SPCP with infinite prediction horizon ($N = \infty$) and finite control horizon $N_c = 15$.
- J_3 is the performance index for the unconstrained SPCP with prediction and control horizon $N = N_c = 30$.
- J_4 is the performance index for the SPCP with infinite prediction horizon ($N = \infty$) and infinite control horizon($N_c = \infty$) .
- J_5 is the performance index for the SPCP with infinite prediction horizon ($N = \infty$) and finite control horizon $N_c = 15$, subject to inequality constraints $\psi(k) \le \Psi(k)$.
- J_6 is the performance index for the SPCP with finite prediction and control horizon $N = N_c = 30$ subject to inequality constraints $\psi(k) \le \Psi(k)$.

(2.1) Describe how the performance index of the SPCP changes when

- the prediction horizon is increased (decreased),
- the control horizon is increased (decreased), (but $N_c < N$),
- an inequality constraint is introduced (removed).

(2.2) Find a ranking in magnitude $J_i^* \le J_j^*$ for all combinations $i = 1, \ldots, 6$, $j = 1, \ldots, 6$.
(For example, four fictive performance indices J_7^*, J_8^*, J_9^* and J_{10}^* give a ranking: $J_7^* \le J_8^* \le J_9^*$ and $J_7^* \le J_{10}^* \le J_9^*$ if J_7^* is the smallest, J_9^* is the largest, but the order in J_8^* and J_{10}^* is indefinite). Comment on your choices.

Exercise 3 — **Five items in model predictive control** — **(25%)**

(3.1) In all model predictive controllers five important items are part of the design procedure. Give all five items.

(3.2) Consider a game of chess. Explain that a chess player can be seen as a predictive controller. Identify the five basic items.

(3.3) Consider driving a car. Explain that a car driver can be seen as a predictive controller. Identify the five basic items.

Exercise 4 — **Stability of infinite horizon MPC** — **(25%)**

Given the following theorem:

Theorem

Consider a LTI system given in the state space description

$$\begin{aligned} x(k+1) &= Ax(k) + B_2\,w(k) + B_3\,v(k), \\ z(k) &= C_2\,x(k) + D_{22}\,w(k) + D_{23}\,v(k), \end{aligned}$$

with $z_{ss} = 0$ in steady state. A performance index is defined as

$$\min_{\tilde{v}(k)} J(\tilde{v}, k) = \min_{\tilde{v}(k)} \sum_{j=0}^{\infty} \hat{z}^T(k+j|k)\,\hat{z}(k+j|k)$$

with a control horizon $N_c < \infty$. Finally assume $w(k+j) = w_{ss}$ for all $j \geq 0$. The predictive control law minimizing this performance index results in a stabilizing controller.

(4.1)

Proof this theorem.