5. Solving the SPCP

Given the system

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_1 e(k) + B_2 w(k) + B_3 v(k) \\
y(k) &= C_1 x(k) + D_{11} e(k) + D_{12} w(k) \\
\tilde{z}(k) &= \tilde{C}_2 x(k) + \tilde{D}_{21} e(k) + \tilde{D}_{22} \tilde{w}(k) + \tilde{D}_{23} \tilde{v}(k) \\
\tilde{\phi}(k) &= \tilde{C}_3 x(k) + \tilde{D}_{31} e(k) + \tilde{D}_{32} \tilde{w}(k) + \tilde{D}_{33} \tilde{v}(k) \\
\tilde{\psi}(k) &= \tilde{C}_4 x(k) + \tilde{D}_{41} e(k) + \tilde{D}_{42} \tilde{w}(k) + \tilde{D}_{43} \tilde{v}(k)
\end{aligned}
$$

Minimize performance index

$$
\begin{aligned}
J(v,k) &= \sum_{j=0}^{N-1} \hat{z}^T(k+j|k)\Gamma(j)\hat{z}(k+j|k) \\
&= \tilde{z}^T(k)\bar{\Gamma}\tilde{z}(k)
\end{aligned}
$$

subject to the constraints $\tilde{\phi}(k) = 0$ and $\tilde{\psi}(k) \le \tilde{\Psi}(k)$.

Solving the SPCP

SPCP:

$$
\begin{aligned}
J(v,k) &= \tilde{z}^T(k)\bar{\Gamma}\tilde{z}(k) && (1)\\
\tilde{\phi}(k) &= 0 && (2)\\
\tilde{\psi}(k) &\leq \tilde{\Psi}(k) && (3)
\end{aligned}
$$

Three subproblems:

1. Unconstrained: minimize (1).
2. Equality constrained: minimize (1) s.t. (2).
3. Full SPCP: minimize (1) s.t. (2) & (3).

Unconstrained SPCP

Given the system:

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_1 e(k) + B_2 w(k) + B_3 v(k) \\
y(k) &= C_1 x(k) + D_{11} e(k) + D_{12} w(k) \\
\tilde{z}(k) &= \tilde{C}_2 x(k) + \tilde{D}_{21} e(k) + \tilde{D}_{22} \tilde{w}(k) + \tilde{D}_{23} \tilde{v}(k)
\end{aligned}
$$

Minimize performance index

$$J(v,k) = \tilde{z}^T(k) \bar{\Gamma} \tilde{z}(k)$$

Unconstrained SPCP

Use prediction:

$$\tilde{z}(k) = \tilde{z}_0(k) + \tilde{D}_{23}\,\tilde{v}(k)$$

Then

$$\begin{aligned}
J(v,k) &= \tilde{z}^T(k)\bar{\Gamma}\tilde{z}(k) \\
&= \left(\tilde{z}_0^T + \tilde{v}^T\tilde{D}_{23}^T\right)\bar{\Gamma}\left(\tilde{z}_0 + \tilde{D}_{23}\,\tilde{v}\right) \\
&= \tilde{v}^T\tilde{D}_{23}^T\bar{\Gamma}\tilde{D}_{23}\tilde{v} + \tilde{v}^T\tilde{D}_{23}^T\bar{\Gamma}\tilde{z}_0 + \tilde{z}_0^T\bar{\Gamma}\tilde{D}_{23}\tilde{v} + \tilde{z}_0^T\bar{\Gamma}\tilde{z}_0 \\
&= \tilde{v}^T\tilde{D}_{23}^T\bar{\Gamma}\tilde{D}_{23}\tilde{v} + 2\,\tilde{v}^T\tilde{D}_{23}^T\bar{\Gamma}\tilde{z}_0 + \tilde{z}_0^T\bar{\Gamma}\tilde{z}_0 \\
&= \frac{1}{2}\tilde{v}^T H\tilde{v} + \tilde{v}^T f + c
\end{aligned}$$

where we used $\tilde{z}_0^T\bar{\Gamma}\tilde{D}_{23}\tilde{v} + \tilde{v}^T\tilde{D}_{23}^T\bar{\Gamma}\tilde{z}_0 = 2\,\tilde{v}^T\tilde{D}_{23}^T\bar{\Gamma}\tilde{z}_0$

Unconstrained SPCP

Optimization problem:

$$\min_{\tilde{v}} J(\tilde{v},k) = \min_{\tilde{v}} \frac{1}{2}\tilde{v}^T H \tilde{v} + \tilde{v}^T f + c$$

$$\frac{\partial J}{\partial \tilde{v}} = H\tilde{v}(k) + f(k) = 0$$

Let H be invertible, then:

$$\tilde{v}(k) = -H^{-1} f(k)$$

$$= -\left(2\tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} 2\tilde{D}_{23}^T \bar{\Gamma} \tilde{z}_0(k)$$

Unconstrained SPCP

Controller implementation: Let $E_v = [\ I \quad 0 \quad \ldots \quad 0 \ ]$

and remind that $\tilde{z}_0(k) = \tilde{C}_2 x(k) + \tilde{D}_{21} e(k) + \tilde{D}_{22} \tilde{w}(k)$

The control signal $v(k)$ becomes:

$$
\begin{aligned}
v(k) = &\ E_v \tilde{v}(k) \\
&= -E_v H^{-1} f(k) \\
&= -E_v \left(\tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{z}_0(k) \\
&= -E_v \left(\tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{C}_2 x(k) - E_v \left(\tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{21} e(k) \\
&\quad - E_v \left(\tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{22} \tilde{w}(k)
\end{aligned}
$$

Unconstrained SPCP

Controller implementation: Let $E_v = [\; I \quad 0 \quad \dots \quad 0 \;]$

and remind that $\tilde{z}_0(k) = \tilde{C}_2 x(k) + \tilde{D}_{21} e(k) + \tilde{D}_{22} \tilde{w}(k)$

The control signal $v(k)$ becomes:

$$
\begin{aligned}
v(k) = {} & E_v \tilde{v}(k) \\
= {} & -E_v H^{-1} f(k) \\
= {} & -E_v \left(\tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \right)^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{z}_0(k) \\
= {} & -E_v (\tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23})^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{C}_2 x(k) - E_v (\tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23})^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{21} e(k) \\
& - E_v (\tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23})^{-1} \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{22} \tilde{w}(k) \\
= {} & -F x(k) + D_e e(k) + D_w \tilde{w}(k)
\end{aligned}
$$

Equality constrained optimization

Equality constrained standard problem:

$$\min_{\tilde{v}} J = \min_{\tilde{v}} \tilde{z}^T(k)\bar{\Gamma}\tilde{z}(k)$$

subject to the constraint:

$$\begin{aligned}
\tilde{\phi}(k) = & \underbrace{\tilde{C}_3 x(k) + \tilde{D}_{31} e(k) + \tilde{D}_{32} \tilde{w}(k)} + \tilde{D}_{33} \tilde{v}(k) \\
= & \qquad\qquad \tilde{\phi}_E(k) \qquad\qquad + \tilde{D}_{33} \tilde{v}(k) = 0
\end{aligned}$$

Step 1: Elimination

Step 2: Solve problem analytically

Step 1: Elimination

Singular value decomposition:

$$\tilde{D}_{33} = U \begin{bmatrix} \Sigma & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

and define

$$\tilde{D}_{33}^r = V_1 \Sigma^{-1} U^T \qquad \tilde{D}_{33}^{\perp} = V_2$$

then

$$\tilde{D}_{33}\tilde{D}_{33}^r = U\Sigma V_1^T V_1 \Sigma^{-1} U^T = I$$

$$\tilde{D}_{33}\tilde{D}_{33}^{\perp} = U\Sigma V_1^T V_2 = 0$$

Now $\tilde{v}(k)$ is chosen as

$$\tilde{v}(k) = -\tilde{D}_{33}^r \tilde{\phi}_E(k) + \tilde{D}_{33}^{\perp}\tilde{\mu}(k) \quad = \quad \tilde{v}_E(k) + \tilde{D}_{33}^{\perp}\tilde{\mu}(k)$$

$$\tilde{v}(k) = -\tilde{D}^{r}_{33}\tilde{\phi}_E(k) + \tilde{D}^{\perp}_{33}\tilde{\mu}(k) \quad = \quad \tilde{v}_E(k) + \tilde{D}^{\perp}_{33}\tilde{\mu}(k)$$

Substitution:

$$\begin{aligned}\tilde{\phi}(k) &= \tilde{\phi}_E(k) + \tilde{D}_{33}\tilde{v}(k) = \tilde{\phi}_E(k) - \tilde{D}_{33}\tilde{D}^{r}_{33}\tilde{\phi}_E(k) + \tilde{D}_{33}\tilde{D}^{\perp}_{33}\tilde{\mu}(k) \\ &= \tilde{\phi}_E(k) - I\,\tilde{\phi}_E(k) + 0\,\tilde{\mu}(k) = 0\end{aligned}$$

Substitution:

$$\begin{aligned}\tilde{z}(k) &= \tilde{z}_0(k) + \tilde{D}_{23}\,\tilde{v}(k) \quad = \quad \tilde{z}_0(k) + \tilde{D}_{23}\,\tilde{v}_E(k) + \tilde{D}_{23}\tilde{D}^{\perp}_{33}\tilde{\mu}(k) \\ &= \tilde{z}_E(k) + \tilde{D}_{23}\tilde{D}^{\perp}_{33}\tilde{\mu}(k)\end{aligned}$$

Then

$$J(v,k) \quad = \quad \tilde{z}^T(k)\bar{\Gamma}\tilde{z}(k) = \frac{1}{2}\tilde{\mu}^T H\tilde{\mu} + \tilde{\mu}^T f + c$$

Step 2: Analytic solution

$$J(v,k) = \frac{1}{2}\tilde{\mu}^T H \tilde{\mu} + \tilde{\mu}^T f + c$$

Let invertible H, then:

$$\tilde{\mu}(k) = -H^{-1} f(k) = -\left(\tilde{D}_{33}^{\perp T} \tilde{D}_{23}^T \bar{\Gamma} \tilde{D}_{23} \tilde{D}_{33}^{\perp}\right)^{-1} \tilde{D}_{33}^{\perp T} \tilde{D}_{23}^T \bar{\Gamma} \left(\tilde{z}_0(k) + \tilde{D}_{23} \tilde{v}_E\right)$$

Use

$$\tilde{v}(k) = -\tilde{D}_{33}^r \tilde{\phi}_E(k) + \tilde{D}_{33}^{\perp} \tilde{\mu}(k)$$
$$v(k) = E_v \tilde{v}(k)$$

then

$$v(k) = -Fx(k) + D_e e(k) + D_w \tilde{w}(k)$$

This is also an LTI-controller !!!

Full SPCP

Given the system:

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_1 e(k) + B_2 w(k) + B_3 v(k) \\
y(k) &= C_1 x(k) + D_{11} e(k) + D_{12} w(k) \\
\tilde{z}(k) &= \tilde{C}_2 x(k) + \tilde{D}_{21} e(k) + \tilde{D}_{22} \tilde{w}(k) + \tilde{D}_{23} \tilde{v}(k) \\
\tilde{\phi}(k) &= \tilde{C}_3 x(k) + \tilde{D}_{31} e(k) + \tilde{D}_{32} \tilde{w}(k) + \tilde{D}_{33} \tilde{v}(k) \\
\tilde{\psi}(k) &= \tilde{C}_4 x(k) + \tilde{D}_{41} e(k) + \tilde{D}_{42} \tilde{w}(k) + \tilde{D}_{43} \tilde{v}(k)
\end{aligned}
$$

Minimize performance index

$$J(v,k) = \tilde{z}^T(k)\bar{\Gamma}\tilde{z}(k)$$

subject to the constraints

$$\tilde{\phi}(k) = 0 \qquad \tilde{\psi}(k) \le \tilde{\Psi}(k)$$

Full SPCP

Equality contraint is satisfied for

$$\tilde{v}(k) = \tilde{v}_E(k) + \tilde{D}_{33}^{\perp}\tilde{\mu}(k)$$

Choose optimization vector $\tilde{\mu}(k)$ as:

$$\tilde{\mu}(k) = \tilde{\mu}_E(k) + \tilde{\mu}_I(k)$$

where $\tilde{\mu}_E(k) = -H^{-1}f(k)$ is equality constrained solution. Now

$$\tilde{\phi}(k) = 0$$

$$\tilde{\psi}(k) = \tilde{\psi}_E(k) + \tilde{D}_{43}\tilde{D}_{33}^{r\perp}\tilde{\mu}_I(k)$$

$$J(v,k) = \frac{1}{2}\tilde{\mu}_I^T H \tilde{\mu}_I$$

Inequality constraint:

$$\tilde{\psi}(k) = \tilde{\psi}_E(k) + \tilde{D}_{43}\tilde{D}_{33}^{r\perp}\tilde{\mu}_I(k) \leq \tilde{\Psi}$$

Now let

$$A_\psi = \tilde{D}_{43}\tilde{D}_{33}^{\perp}$$

$$b_\psi(k) = \tilde{\psi}_E(k) - \tilde{\Psi}(k)$$

This results in the inequality constraint:

$$A_\psi\tilde{\mu}_I(k) + b_\psi(k) \leq 0$$

Full SPCP

Minimization problem:

$$\min_{\tilde{\mu}_I} \frac{1}{2} \tilde{\mu}_I^T(k) H \tilde{\mu}_I(k)$$

subject to

$$A_\psi \tilde{\mu}_I(k) + b_\psi(k) \leq 0$$

This a Quadratic Programming problem.

Solution found in finite number of steps.

The full SPCP results is a nonlinear controller

Infinite horizon SPCP

Steady state $z_{ss} = 0$.

Otherwise: NO SOLUTION !!!

- Steady-state behavior
- Unconstrained: LQ-controller.
- Equality constraints: Structuring of input and get finite degrees of freedom in optimization.
- Equality and inequality constraints: Use finite degrees of freedom in optimization to solve the problem.

Steady state $(v_{ss}, x_{ss}, w_{ss}, z_{ss})$

The quadruple $(v_{ss}, x_{ss}, w_{ss}, z_{ss})$ is a steady state if there holds:

$$\begin{aligned} x_{ss} &= A x_{ss} + B_2 w_{ss} + B_3 v_{ss} \\ z_{ss} &= C_2 x_{ss} + D_{22} w_{ss} + D_{23} v_{ss} \end{aligned}$$

For infinite horizon

$$J = \sum_{j=0}^{\infty} \hat{z}^T(k+j|k)\hat{z}(k+j|k)$$

we need $(v_{ss}, x_{ss}, w_{ss}, z_{ss}) = (v_{ss}, x_{ss}, w_{ss}, 0)$

For infinite horizon:

$$\begin{aligned} x_{ss} &= A x_{ss} + B_2 w_{ss} + B_3 v_{ss} \\ z_{ss} = 0 &= C_2 x_{ss} + D_{22} w_{ss} + D_{23} v_{ss} \end{aligned}$$

Define:

$$M_{ss} = \begin{bmatrix} I - A & -B_3 \\ -C_2 & -D_{23} \end{bmatrix}$$

then:

$$M_{ss} \begin{bmatrix} x_{ss} \\ v_{ss} \end{bmatrix} = \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} w_{ss}$$

Singular value decomposition:

$$M_{ss} = \begin{bmatrix} U_{M1} & U_{M2} \end{bmatrix} \begin{bmatrix} \Sigma_M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{M1}^T \\ V_{M2}^T \end{bmatrix}$$

Necessary and sufficient condition:

$$U_{M2}^T \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} = 0$$

The solution is given by:

$$\begin{bmatrix} x_{ss} \\ v_{ss} \end{bmatrix} = V_{M1} \Sigma_M^{-1} U_{M1}^T \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} w_{ss} + V_{M2}\, \alpha_M$$

Example

Consider IO-system:

$$\begin{aligned} x_o(k+1) &= 0.5x_o(k) + 2u(k) \\ y(k) &= 2x_o(k) \end{aligned}$$

GPC-criterion:

$$J(k) = \sum (y(k+1) - r(k+1))^2 + (\Delta u(k))^2$$

IIO-system:

$$x(k) = \begin{bmatrix} y(k-1) \\ \Delta x_o(k) \end{bmatrix} \;,\; v(k) = \Delta u(k) \;,\; w(k) = r(k+1)$$

where

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 1 & 2 \\ 0 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} v(k) \\
y(k) &= \begin{bmatrix} 1 & 2 \end{bmatrix} x(k) \\
y(k+1) &= \begin{bmatrix} 1 & 3 \end{bmatrix} x(k) + 6v(k) \\
z(k) &= \begin{bmatrix} y(k+1) - r(k+1) \\ v(k) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} w(k) + \begin{bmatrix} 6 \\ 1 \end{bmatrix} v(k)
\end{aligned}
$$

We compute:

$$M_{ss} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0.5 & -2 \\ -1 & -3 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

and

$$U_2^T = \frac{1}{9} \begin{bmatrix} 1 & 4 & 0 & 8 \end{bmatrix}$$

Note that

$$U_2^T \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} = 0$$

Consider the criterion:

$$J(k) = \sum (y(k+1) - r(k+1))^2 + (u(k))^2$$

Now we find for $x(k) = x_o(k)$, $v(k) = u(k)$, $w(k) = r(k+1)$:

$$\begin{aligned} y(k+1) &= x(k) + 4v(k) \\ z(k) &= \begin{bmatrix} y(k+1) - r(k+1) \\ v(k) \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(k) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} w(k) + \begin{bmatrix} 4 \\ 1 \end{bmatrix} v(k) \end{aligned}$$

We compute:

$$M_{ss} = \begin{bmatrix} 0.5 & 2 \\ 1 & 4 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

and

$$U_2^T = \sqrt{\frac{1}{69}} \begin{bmatrix} -2 & 1 & 8 \end{bmatrix}$$

Note that

$$U_2^T \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} = -\sqrt{\frac{1}{69}} \neq 0$$

Steady-state:

Let $w_{ss} = D_{ssw}\tilde{w}(k)$ and define

$$\begin{bmatrix} D_{ssx} \\ D_{ssv} \end{bmatrix} = V_{M1}\Sigma_M^{-1}U_{M1}^T \begin{bmatrix} B_2 \\ D_{22} \end{bmatrix} D_{ssw}$$

$$D_{ssy} = C_1 D_{ssx} + D_{12} D_{ssw} + D_{13} D_{ssv}$$

Then:

$$\begin{bmatrix} x_{ss} \\ v_{ss} \\ w_{ss} \\ y_{ss} \end{bmatrix} = \begin{bmatrix} D_{ssx} \\ D_{ssv} \\ D_{ssw} \\ D_{ssy} \end{bmatrix} \tilde{w}(k)$$

Unconstrained infinite horizon SPCP

Let

$$w(k+j)=0 \quad \text{for all} \quad j \geq 0$$

$$z_{ss}=0$$

$$\Gamma(j)=I \quad \text{for all} \quad j \geq 0$$

Performance index:

$$J(v,k)=\sum_{j=0}^{\infty} \hat{z}^T(k+j|k)\hat{z}(k+j|k)$$

$$=\sum_{j=0}^{\infty} \Big(\hat{x}^T(k+j|k)C_2^T C_2 \hat{x}(k+j|k) + 2\hat{x}^T(k+j|k)C_2^T D_{23} v(k+j)$$

$$+ v^T(k+j)D_{23}^T D_{23} v(k+j)\Big)$$

Now define

$$\bar{v}(k) = v(k) + (D_{23}^T D_{23})^{-1} D_{23}^T C_2 \hat{x}(k|k)$$

$$\bar{A} = A - B_3 (D_{23}^T D_{23})^{-1} D_{23}^T C_2$$

$$\bar{Q} = C_2^T \left(I - D_{23} (D_{23}^T D_{23})^{-1} D_{23}^T\right) C_2$$

$$\bar{R} = D_{23}^T D_{23}$$

Then the performance index becomes

$$\begin{aligned} J(v,k) &= \sum_{j=0}^{\infty} \hat{x}^T(k+j|k) C_2^T \left(I - D_{23} (D_{23}^T D_{23})^{-1} D_{23}^T\right) C_2 \hat{x}(k+j|k) \\ &\quad + \bar{v}^T(k+j) D_{23}^T D_{23} \bar{v}(k+j) \\ &= \sum_{j=0}^{\infty} \hat{x}^T(k+j|k) \bar{Q} \hat{x}(k+j|k) + \bar{v}^T(k+j) \bar{R} \bar{v}(k+j) \end{aligned}$$

This is a standard LQG problem.

The optimal control signal $\bar{v}$ is given by

$$\bar{v}(k) = -(B_3^T P B_3 + \bar{R})^{-1} B_3^T P \bar{A} \hat{x}_\Delta(k|k)$$

where P is the smallest positive semi-definite solution of the discrete time Riccati equation

$$P = \bar{A}^T P \bar{A} - \bar{A}^T P B_3 (B_3^T P B_3 + \bar{R})^{-1} B_3^T P \bar{A} + \bar{Q}$$

which exists due to stabilizability of (A, B_3) and invertibility of $\bar{R}$.

After substitution we obtain a linear controller:

$$v(k) = -F\,x(k) + D_w\,w(k) + D_e\,e(k|k)$$

Infinite horizon SPCP with control horizon

Consider a system

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_1 e(k) + B_2 w(k) + B_3 v(k) \\
y(k) &= C_1 x(k) + D_{11} e(k) + D_{12} w(k) \\
z(k) &= C_2 x(k) + D_{21} e(k) + D_{22} w(k) + D_{23} v(k)
\end{aligned}
$$

Minimize $\quad J(v,k) = \sum_{j=0}^{\infty} \hat{z}^T(k+j|k)\Gamma(j)\hat{z}(k+j|k)$

subject to $v(k+j|k) = 0$, for $j \geq N_c$ Control horizon constraint

$$
\begin{aligned}
\psi(k+j) &= C_4 x(k+j) + D_{41} e(k+j) + D_{42} w(k+j) + D_{43} v(k+j) \\
&\leq \Psi \ , \quad \text{for } j \geq 0
\end{aligned}
$$

Inequality constraints

For simplicity: Consider $e(k) = 0$, $w(k) = 0$, and A is stable.

The performance index can be split in two parts:

$$J(v,k) = \sum_{j=0}^{\infty} \hat{z}^T(k+j|k)\Gamma(j)\hat{z}(k+j|k) = J_1(v,k) + J_2(v,k)$$

where

$$\begin{aligned} J_1(v,k) &= \sum_{j=0}^{N_c-1} \hat{z}^T(k+j|k)\Gamma(j)\hat{z}(k+j|k) \\ J_2(v,k) &= \sum_{j=N_c}^{\infty} \hat{z}^T(k+j|k)\Gamma_{ss}\hat{z}(k+j|k) \end{aligned}$$

We will consider the derivation of criterion J_2 before we derive criterion J_1.

Derivation of J_2:

For $j \geq N_c$ this system is autonomous. This means that $\hat{z}(k+j|k)$ can be computed for all $j \geq N_c$ if initial state $\hat{x}(k+N_c)$ is known. This state $\hat{x}(k+N_c|k)$ can be found by successive substitution:

$$\hat{x}(k+N_c|k) = A^{N_c} x(k|k) + \begin{bmatrix} A^{N_c-1}B_3 & \cdots & B_3 \end{bmatrix} \tilde{v}(k)$$
$$= \tilde{C}_{N_c,3}\, x(k|k) + \tilde{D}_{N_c,33}\, \tilde{v}(k)$$

The prediction $\hat{z}(k+j|k)$ for $j \geq N_c$ is obtained by successive substitution, resulting in:

$$\hat{z}(k+j|k) = C_2 A^{j-N_c} \hat{x}(k+N_c|k)$$

then

$$J_2(v,k) = \sum_{j=N_c}^{\infty} \hat{z}^T(k+j|k)\Gamma_{ss}\hat{z}(k+j|k)$$

$$= \sum_{j=N_c}^{\infty} \hat{x}^T(k+N_c|k)(A^T)^{j-N_c} C_2^T \Gamma_{ss} C_2 A^{j-N_c} \hat{x}(k+N_c|k)$$

$$= \hat{x}^T(k+N_c|k)\bar{M}\hat{x}(k+N_c|k)$$

where

$$\bar{M} = \sum_{j=N_c}^{\infty} (A^T)^{j-N_c} C_2^T \Gamma_{ss} C_2 A^{j-N_c}$$

The matrix $\bar{M}$ is solution of the discrete time Lyapunov equation

$$A^T \bar{M} A - \bar{M} + C_2^T \Gamma_{ss} C_2 = 0$$

With

$$\hat{x}(k+N_c|k) = \tilde{C}_{N_c,3}\,x(k|k) + \tilde{D}_{N_c,33}\,\tilde{v}(k)$$

we derive

$$\begin{aligned} J_2(v,k) &= \hat{x}^T(k+N_c|k)\,\bar{M}\,\hat{x}(k+N_c|k) \\ &= \frac{1}{2}\tilde{v}^T(k)H_2\tilde{v}(k) + \tilde{v}^T(k)\,f_2(k) + c_2(k) \end{aligned}$$

where

$$\begin{aligned} H_2 &= 2\tilde{D}^T_{N_c,33}\,\bar{M}\,\tilde{D}_{N_c,33} \\ f_2(k) &= 2\tilde{D}^T_{N_c,33}\,\bar{M}\,\tilde{C}_{N_c,3}\,x(k|k) \\ c_2(k) &= x^T(k|k)\,\tilde{C}^T_{N_c,3}\,\bar{M}\,\tilde{C}_{N_c,3}\,x(k|k) \end{aligned}$$

Derivation of J_1:

Define $\tilde{z}(k)$ for horizon N_c:

$$\tilde{z}(k) = \left[\begin{array}{cccc} \hat{z}^T(k|k) & \hat{z}^T(k+1|k) & \ldots & \hat{z}^T(k+N_c-1|k) \end{array} \right]$$

Using the results of chapter 3 we derive:

$$\tilde{z}(k) = \tilde{C}_{N_c,2}\, x(k|k) + \tilde{D}_{N_c,23}\, \tilde{v}(k)$$

where $\tilde{C}_{N_c,2}$ and $\tilde{D}_{N_c,23}$ are given by

$$\tilde{C}_{N_c,2} = \begin{bmatrix} C_2 \\ C_2 A \\ \vdots \\ C_2 A^{N_c-1} \end{bmatrix}, \quad \tilde{D}_{N_c,23} = \begin{bmatrix} D_{23} & 0 & \cdots & 0 \\ C_2 B_3 & D_{23} & \cdots & 0 \\ \vdots & & \ddots & 0 \\ C_2 A^{N_c-2} B_3 & & \cdots & D_{23} \end{bmatrix}$$

Now J_1 becomes

$$\begin{aligned} J_1(v,k) &= \sum_{j=0}^{N_c-1} \hat{z}^T(k+j|k)\Gamma(j)\hat{z}(k+j|k) \\ &= \tilde{z}^T(k)\bar{\Gamma}_{N_c}\tilde{z}(k) \\ &= \frac{1}{2}\tilde{v}^T(k)H_1\tilde{v}(k) + \tilde{v}^T(k)f_1(k) + c_1(k) \end{aligned}$$

where

$$\begin{aligned} H_1 &= 2\tilde{D}^T_{N_c,23}\,\bar{\Gamma}_{N_c}\,\tilde{D}_{N_c,23} \\ f_1(k) &= 2\tilde{D}^T_{N_c,23}\,\bar{\Gamma}_{N_c}\,\tilde{C}_{N_c,2}\,x(k|k) \\ c_1(k) &= x^T(k|k)\,\tilde{C}^T_{N_c,2}\,\bar{\Gamma}_{N_c}\,\tilde{C}_{N_c,2}\,x(k|k) \end{aligned}$$

Minimization of $J_1 + J_2$:

Combining the results we obtain the problem of minimizing

$$
\begin{aligned}
J_1 + J_2 &= \frac{1}{2}\tilde{v}^T(k)(H_1+H_2)\tilde{v}(k) + \tilde{v}^T(k)(f_1(k)+f_2(k)) + c_1(k) + c_2(k) \\
&= \frac{1}{2}\tilde{v}^T(k)\, H\, \tilde{v}(k) + \tilde{v}^T(k)\, f(k) + c(k)
\end{aligned}
$$

In the absence of inequality constraints, the optimal $v(k)$ is found for

$$
\begin{aligned}
v(k) &= -E_v\,(H_1 + H_2)^{-1}(f_1(k) + f_2(k)) \\
&= -E_v H^{-1} f(k) \\
&= -E_v \left(\tilde{D}^T_{N_c,23}\bar{\Gamma}_{N_c}\tilde{D}_{N_c,23} + \tilde{D}^T_{N_c,33}\bar{M}\tilde{D}_{N_c,33}\right)^{-1} \\
&\quad \times \left(\tilde{D}^T_{N_c,23}\bar{\Gamma}_{N_c}\tilde{C}_{N_c,2} + \tilde{D}^T_{N_c,33}\bar{M}\tilde{C}_{N_c,3}\right) x(k|k) \\
&= -F\, x(k|k)
\end{aligned}
$$

If $e(k) \neq 0$ and $w(k) \neq 0$ we obtain

$$v(k) = -F\,x(k|k) + D_e\,e(k) + D_w\,\tilde{w}(k)$$

where

$$F = E_v\,\Lambda\left(\tilde{D}^T_{N_c,23}\bar{\Gamma}_{N_c}\,\tilde{C}_{N_c,2} + \tilde{D}^T_{N_c,33}\,\bar{M}\,\tilde{C}_{N_c,3}\right)$$

$$D_e = E_v\,\Lambda\left(\tilde{D}^T_{N_c,23}\bar{\Gamma}_{N_c}\,\tilde{D}_{N_c,21} + \tilde{D}^T_{N_c,33}\,\bar{M}\,\tilde{D}_{N_c,31}\right)$$

$$D_w = E_v\,\Lambda\left(\tilde{D}^T_{N_c,23}\bar{\Gamma}_{N_c}\,\tilde{D}_{N_c,22} + \tilde{D}^T_{N_c,33}\,\bar{M}\,\tilde{D}_{N_c,32}\right)$$

$$\Lambda = \left(\tilde{D}^T_{N_c,23}\bar{\Gamma}_{N_c}\tilde{D}_{N_c,23} + \tilde{D}^T_{N_c,33}\,\bar{M}\,\tilde{D}_{N_c,33}\right)^{-1}$$

Unstable poles:

The prediction for $j \geq N_c$ resulted in:

$$\hat{z}(k+j|k) = C_2 A^{j-N_c} \hat{x}(k+N_c|k)$$

If A has unstable poles:

$$A = \begin{bmatrix} T_s & T_u \end{bmatrix} \begin{bmatrix} \bar{A}_s & 0 \\ 0 & \bar{A}_u \end{bmatrix} \begin{bmatrix} \bar{T}_s \\ \bar{T}_u \end{bmatrix}$$

where $\bar{A}_s$ is stable and $\bar{A}_u$ is unstable.

$$x_{unstable}(k+N_c|k) = T_u x(k+N_c|k) = 0$$

This is an equality constraint.

Inequality constraints:

For constraints there exists n_ψ such that

If	$\psi(k+j\|k) \leq \Psi(k+j\|k)$ for $j = 0,\ldots,N_c+n_\psi$
then	$\psi(k+j\|k) \leq \Psi(k+j\|k)$ for $j > N_c+n_\psi$

The infinite horizon predictive control problem turns out to be a Quadratic Programming (QP) problem.

Quadratic programming (QP)

Various algorithms to solve the quadratic programming problem:

1. **The modified simplex method:**

   Most efficient for small and medium-sized problems.

   The algorithm will find the optimum in a finite number of steps.

2. **The interior point method:**

   For large-sized quadratic programming problems.

   Disadvantage: the optimum can only be approximated. Bounds for approximation can be derived (Nesterov & Nemirovsky, 1994).

3. **Other convex optimization methods:**

   For example: cutting plane algorithm (Boyd & Barratt, 1991 / Demyanov & Vasilev, 1985) or the ellipsoid algorithm (Boyd & Barratt, 1991 / Grötschel, Lovász & Schrijver, 1988)

Implementation of LTI SPCP 1

$$v(k) = -F\,x(k) + D_e\,e(k) + D_w\,\tilde{w}(k)$$

State $x(k)$ and true noise signal $e(k)$ are unknown.

Introduce

- controller state $x_c(k)$

- noise estimate $e_c(k)$.

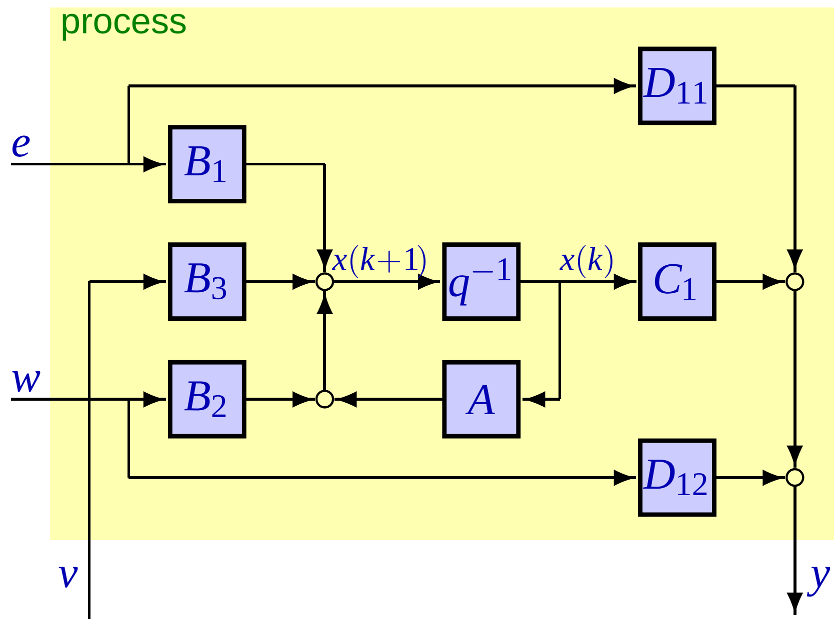
Substitution gives controller equations:

$$\begin{aligned}
x_c(k+1) &= Ax_c(k) + B_1 e_c(k) + B_2 w(k) + B_3 v(k) \\
e_c(k) &= D_{11}^{-1}\Big(y(k) - C_1 x_c(k) - D_{12} w(k)\Big) \\
v(k) &= -F x_c(k) + D_e e_c(k) + D_w \tilde{w}(k)
\end{aligned}$$

Implementation of LTI SPCP 2

$$x(k+1) = Ax(k) + B_1 e(k) + B_2 w(k) + B_3 v(k)$$

$$y(k) = C_1 x(k) + D_{11} e(k) + D_{12} w(k)$$

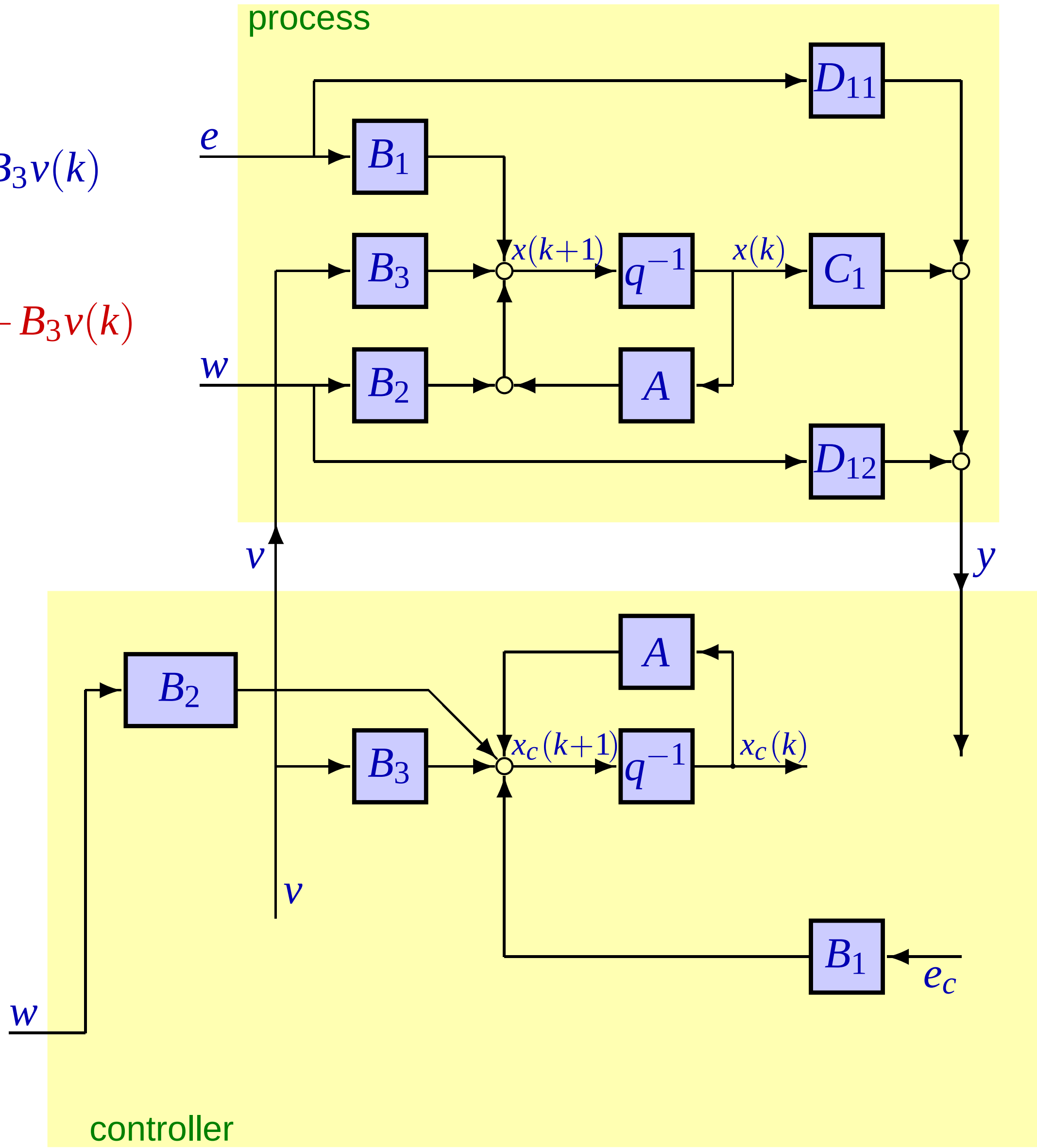


Implementation of LTI SPCP 3

$$x(k+1) = Ax(k) + B_1 e(k) + B_2 w(k) + B_3 v(k)$$

$$y(k) = C_1 x(k) + D_{11} e(k) + D_{12} w(k)$$

$$x_c(k+1) = Ax_c(k) + B_1 e_c(k) + B_2 w(k) + B_3 v(k)$$



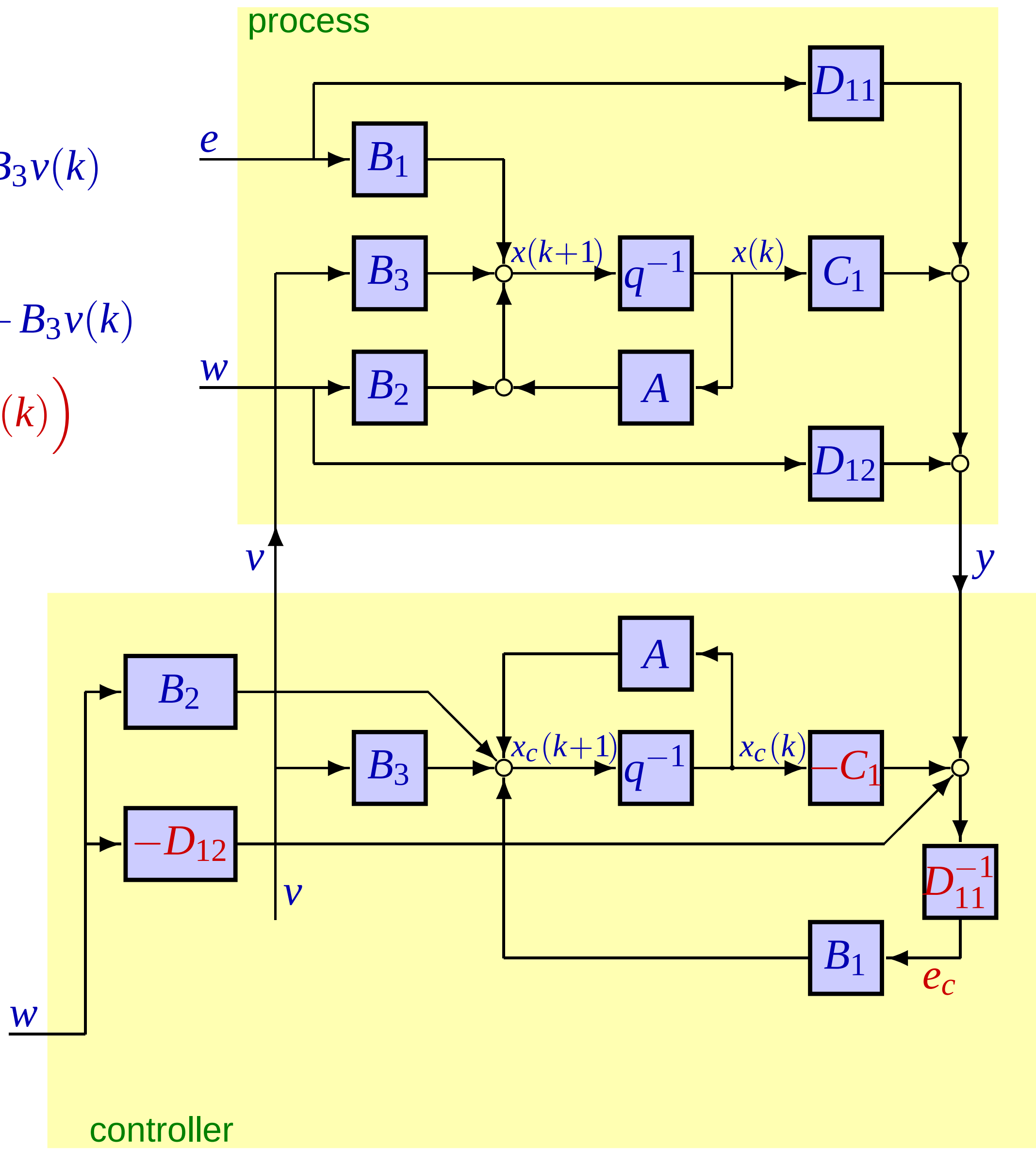
Implementation of LTI SPCP 4

$$x(k+1) = Ax(k) + B_1 e(k) + B_2 w(k) + B_3 v(k)$$

$$y(k) = C_1 x(k) + D_{11} e(k) + D_{12} w(k)$$

$$x_c(k+1) = Ax_c(k) + B_1 e_c(k) + B_2 w(k) + B_3 v(k)$$

$$e_c(k) = D_{11}^{-1}\Big(y(k) - C_1 x_c(k) - D_{12} w(k)\Big)$$



Implementation of LTI SPCP 5

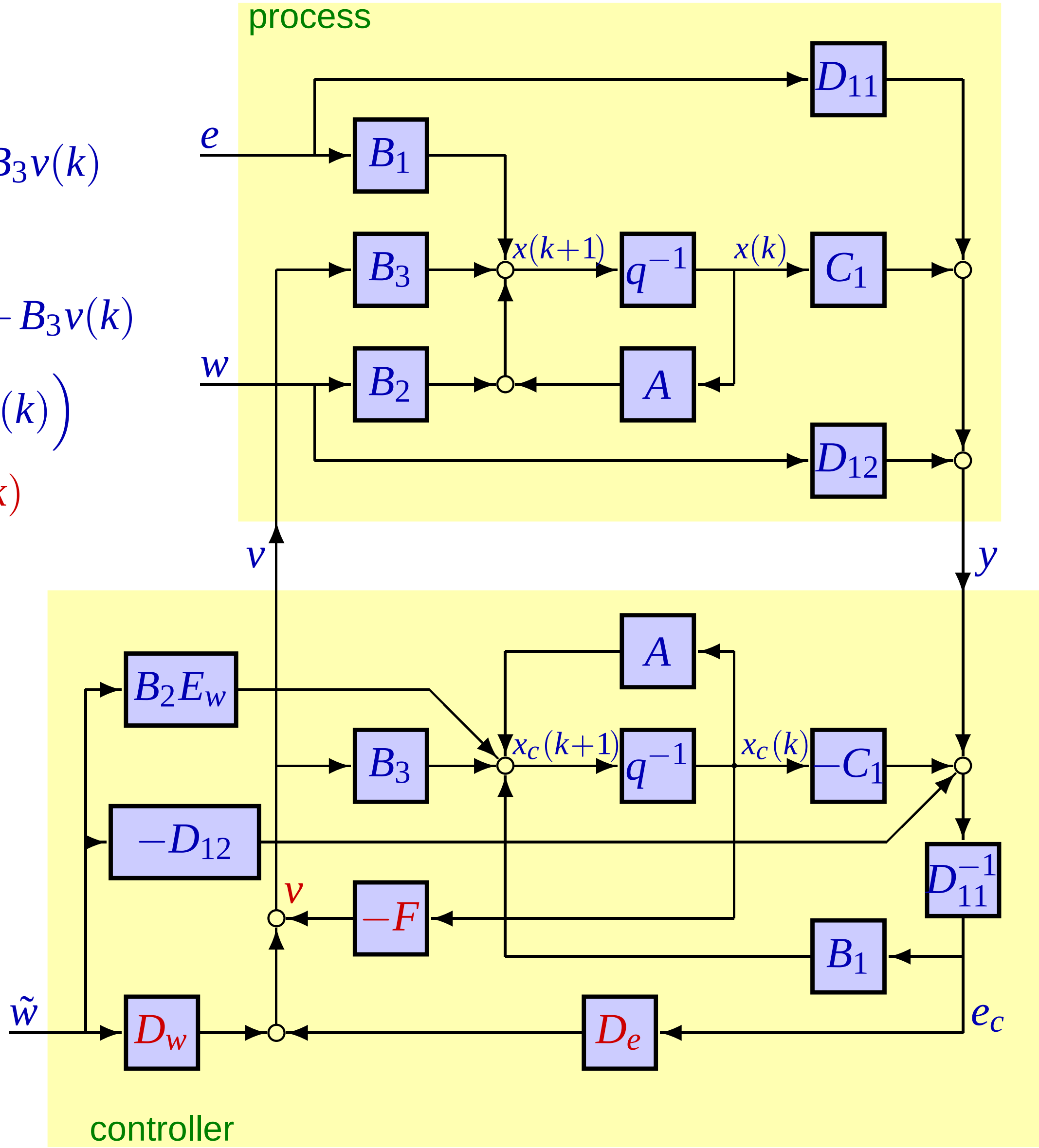
$$x(k+1) = Ax(k) + B_1 e(k) + B_2 w(k) + B_3 v(k)$$

$$y(k) = C_1 x(k) + D_{11} e(k) + D_{12} w(k)$$

$$x_c(k+1) = Ax_c(k) + B_1 e_c(k) + B_2 w(k) + B_3 v(k)$$

$$e_c(k) = D_{11}^{-1}\Big(y(k) - C_1 x_c(k) - D_{12} w(k)\Big)$$

$$v(k) = -F x_c(k) + D_e e_c(k) + D_w \tilde{w}(k)$$



Implementation of full SPCP

Minimization problem:

$$\begin{aligned} J(v,k) &= \tilde{z}^T(k)\bar{\Gamma}\tilde{z}(k) \\ &= \frac{1}{2}\tilde{\mu}_I^T H \tilde{\mu}_I + c_I \end{aligned}$$

subject to

$$A_\psi \tilde{\mu}_I(k) + b_\psi(k) \leq 0$$

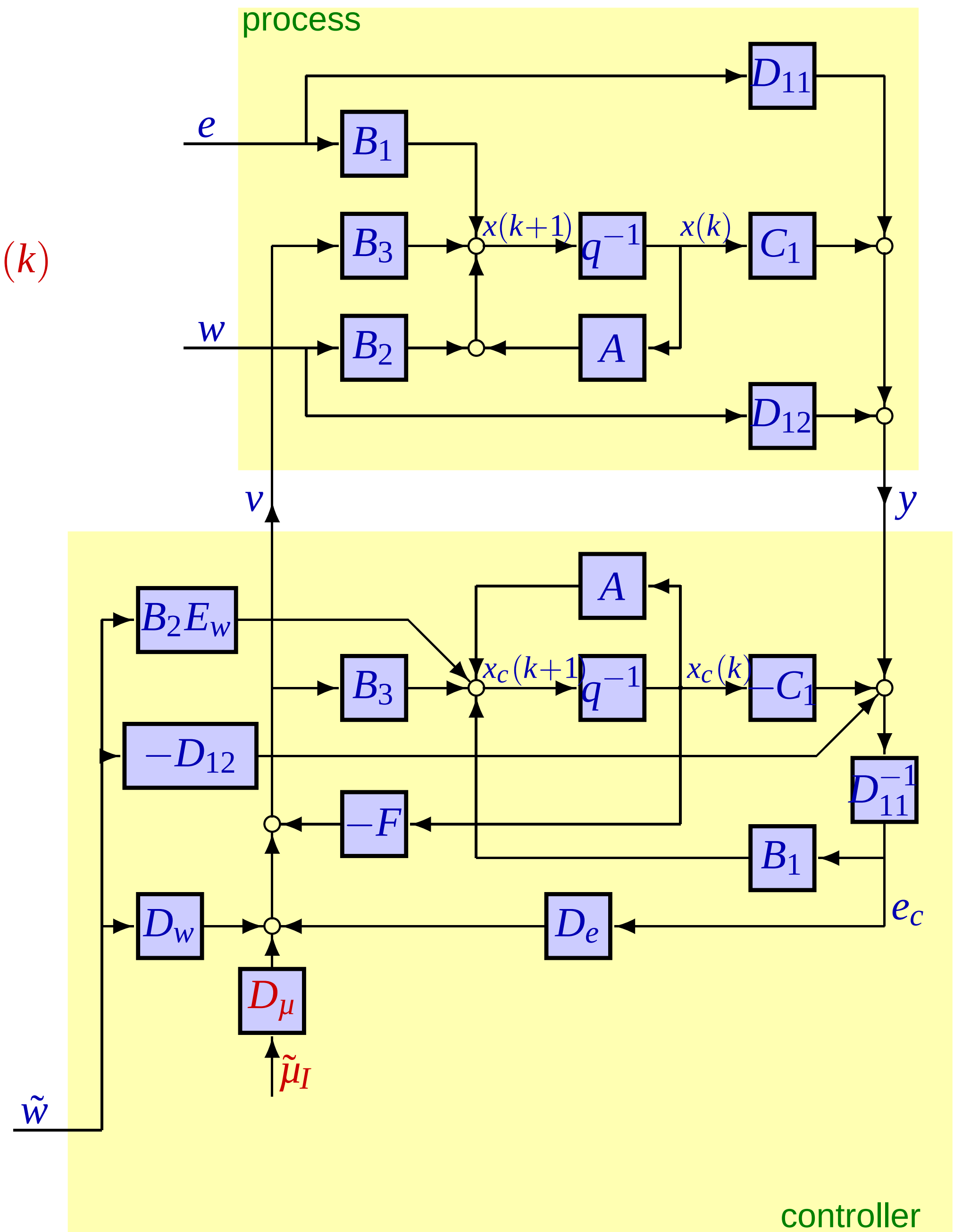
The full SPCP results is a nonlinear controller

$$v(k) = -Fx_c(k) + D_e e_c(k) + D_w \tilde{w}(k) + D_\mu \tilde{\mu}_I(k)$$

where

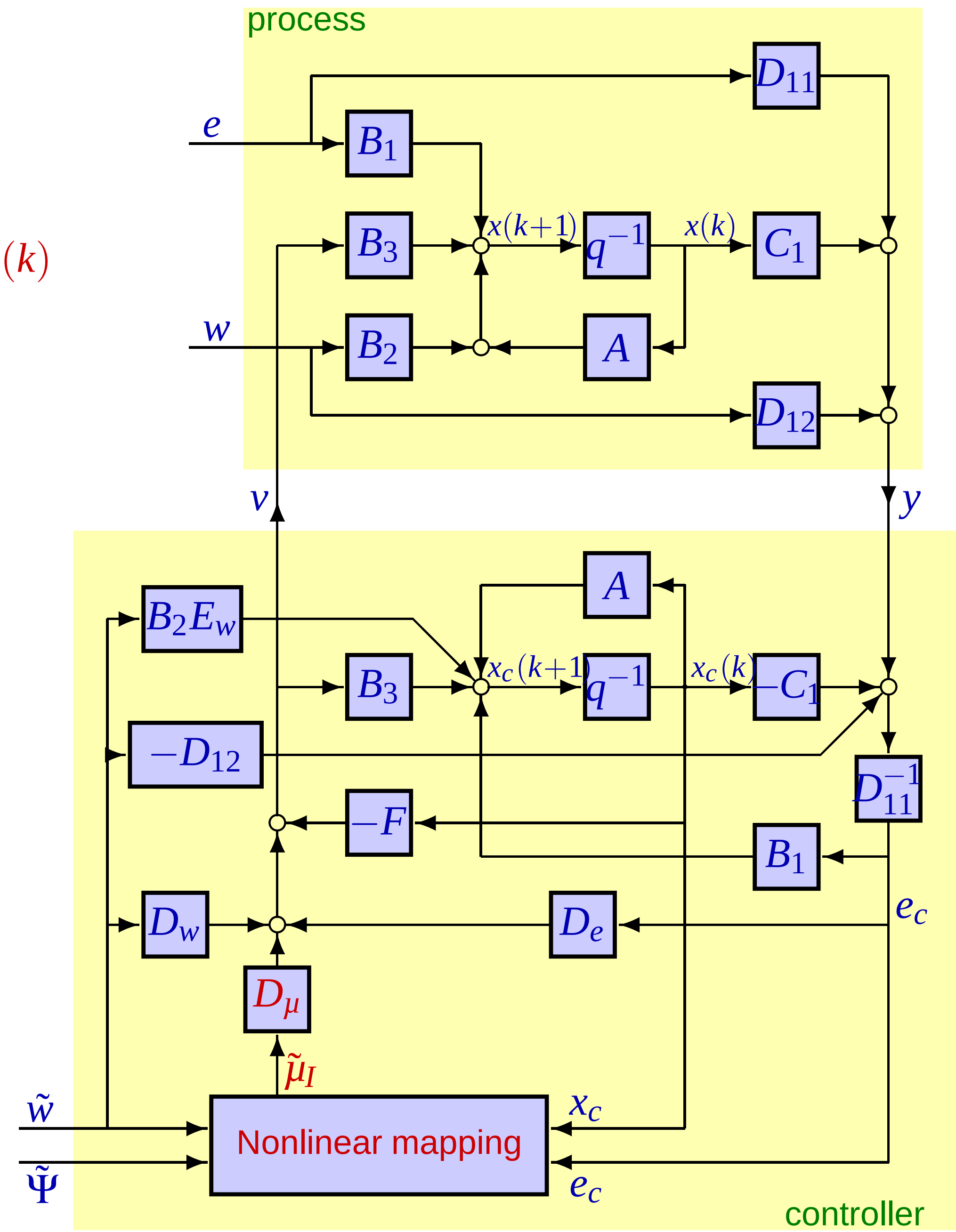
$$\tilde{\mu}_I(k) = h(x_c(k), e_c(k), \tilde{w}(k), \tilde{\Psi}(k))$$

$$v(k) = -Fx_c(k) + D_e e_c(k) + D_w \tilde{w}(k) + D_\mu \tilde{\mu}_I(k)$$



$$v(k) = -F x_c(k) + D_e e_c(k) + D_w \tilde{w}(k) + D_\mu \tilde{\mu}_I(k)$$

$$\tilde{\mu}_I(k) = h(x_c(k), e_c(k), \tilde{w}(k), \tilde{\Psi}(k))$$



Analytic solution to full SPCP

Analytic solution for the mapping

$$\tilde{\mu}_I(k) = h(x_c(k), e_c(k), \tilde{w}(k), \tilde{\Psi}(k))$$

The function h is a continuous and piecewise affine function of $x_c(k)$, $e_c(k)$, $\tilde{w}(k)$, and $\tilde{\Psi}(k)$.

Partitions the state space into polyhedral sets and compute the coefficients of the affine function for every set. The result is a search tree.

Feasibility

$$\min_{v} J \quad \text{subject to} \quad \tilde{\psi} \leq \tilde{\Psi}$$

if inequality constraints too stringent

$$\Downarrow$$

no feasible solution to QP problem

Three algorithms to handle infeasibility:

- soft-constraint approach
- minimal time approach
- constraint prioritization

Soft-constraint algorithm

Based on penalty-function description.

New problem becomes:

$$\min_{v,\alpha} J + c\,\alpha^2$$

subject to

$$\tilde{\psi} \leq \tilde{\Psi} + \alpha$$

$$\alpha \geq 0$$

$$c \gg 0$$

Minimal time algorithm

Based on minimum duration of infeasibility.

Compute minimal time j_{mt}:

$$\min_{v, j_{mt}} j_{mt} \quad \text{subject to}$$

$$\psi(k+j) \leq \Psi(k+j) \ \text{ for } \ j > j_{mt}$$

and optimize new problem:

$$\min_{v} J \quad \text{subject to}$$

$$\psi(k+j) \leq \Psi(k+j) \ \text{ for } \ j > j_{mt}$$

Constraint prioritization

New algorithm:

1. order constraints from lowest to highest priority.
2. solve optimization problem.

   in case of feasible solution: goto step 4

   in case of infeasibility: goto step 3
3. drop constraint with lowest priority, goto step 2
4. implement solution

Solve a sequence of quadratic programming problems

Algorithm minimizes the violations in priority order.

Example: elevator

Triple integrator chain

$$\dddot{y}(t) = u(t) + \varepsilon(t)$$

$$\varepsilon(t) = \dddot{e}(t) + 0.5\ddot{e}(t) + \dot{e}(t) + 0.1e(t)$$

where $e(t)$ is a continuous time ZMWN signal.

There are constraints on y, $\dot{y}$, $\ddot{y}$ and $\dddot{y}$.

Define the state

$$x(t) = \begin{bmatrix} \ddot{y}(t) \\ \dot{y}(t) \\ y(t) \end{bmatrix}^T$$

Continuous-time state space description:

$$\begin{aligned} \dot{x}_1(t) &= u(t) + 0.1e(t) \\ \dot{x}_2(t) &= x_1(t) + e(t) \\ \dot{x}_3(t) &= x_2(t) + 0.5e(t) \\ y(t) &= x_3(t) + e(t) \end{aligned}$$

Example: elevator

Matrix form:

$$
\begin{aligned}
\dot{x}(t) &= A_c x(t) + K_c e(t) + B_c u(t) \\
y(t) &= C_c x(t) + e(t)
\end{aligned}
$$

where

$$
A_c = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

$$
K_c = \begin{bmatrix} 0.1 \\ 1 \\ 0.5 \end{bmatrix} \quad C_c = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}
$$

Example: elevator

Zero-order hold transformation:

$$\begin{aligned} x_o(k+1) &= A_o x_o(k) + K_o e(k) + B_o u(k) \\ y(k) &= C_o x_o(k) + e(k) \end{aligned}$$

where

$$A_o = \begin{bmatrix} 1 & 0 & 0 \\ T & 1 & 0 \\ T^2/2 & T & 1 \end{bmatrix} \quad B_o = \begin{bmatrix} T \\ T^2/2 \\ T^3/6 \end{bmatrix}$$

$$K_o = \begin{bmatrix} 0.1T \\ T + T^2/20 \\ T/2 + T^2/6 + T^3/60 \end{bmatrix} \quad C_o = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Example: elevator

Constraints:

$$
\begin{aligned}
|\dddot{y}(t)| &\leq 0.4 && \text{(jerk)} \\
|\ddot{y}(t)| &\leq 0.3 && \text{(acceleration)} \\
|\dot{y}(t)| &\leq 0.4 && \text{(speed)} \\
y(t) &\leq 1.01 && \text{(overshoot)}
\end{aligned}
$$

translated in:

$$
\begin{aligned}
u(k+j-1) &\leq 0.4 && \text{(positive jerk)} \\
-u(k+j-1) &\leq 0.4 && \text{(negative jerk)} \\
\hat{x}_1(k+j) &\leq 0.3 && \text{(positive acceleration)} \\
-\hat{x}_1(k+j) &\leq 0.3 && \text{(negative acceleration)} \\
\hat{x}_2(k+j) &\leq 0.4 && \text{(positive speed)} \\
-\hat{x}_2(k+j) &\leq 0.4 && \text{(negative speed)} \\
\hat{x}_3(k+j) &\leq 1.01 && \text{(overshoot)}
\end{aligned}
$$

for all $j = 1, \ldots, N$.

Example: elevator

Sampling-time $T = 0.1$

Prediction & control horizon $N = N_c = 30$.

Minimum cost-horizon $N_m = 1$.

Reference signal $r(k) = 1$ for all $k > 0$

Weighting parameters $P(q) = 1$, $\lambda = 0.1$.

The GPC performance index (for an IO model):

$$J = \sum_{j=1}^{30} \left| \hat{y}(k+j|k) - r(k+j|k) \right|^2 +$$

$$+ \lambda^2 u^2(k+j-1)$$

subject to the above linear constraints.

Example: elevator