6. Stability

No a priori stability guarantee !!.

Two different cases:

- No constraints/ equality constraints

  - LTI controller

- Inequality constraints

  - Nonlinear controller

Very important to realize:

stability for unconstrained or equality constrained case $\not\Rightarrow$ stability for inequality constrained case

LTI case

Let the process be given by:

$$x(k+1) = Ax(k) + B_1 e(k) + B_2 w(k) + B_3 v(k)$$
$$y(k) = C_1 x(k) + D_{11} e(k) + D_{12} w(k)$$

and the controller by

$$\begin{aligned}
x_c(k+1) & (A - B_3 F - B_1 D_{11}^{-1} C_1 - B_3 D_e D_{11}^{-1} C_1) x_c(k) \\
& + (B_1 D_{11}^{-1} + B_3 D_e D_{11}^{-1}) y(k) \\
& + (B_3 D_w - (B_1 + B_3 D_e) D_{11}^{-1} D_{12} E_w + B_2 E_w) \tilde{w}(k) \\
v(k) & - (F + D_e D_{11}^{-1} C_1) x_c(k) + D_e D_{11}^{-1} y(k) \\
& + (D_w - D_e D_{11}^{-1} D_{12} E_w) \tilde{w}(k)
\end{aligned}$$

Closed loop analysis

Closed loop state space description:

$$\begin{bmatrix} x(k+1) \\ x_c(k+1) \end{bmatrix} = A_{cl} \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix} + B_{1,cl} e(k) + B_{2,cl} \tilde{w}(k)$$

$$\begin{bmatrix} v(k) \\ y(k) \end{bmatrix} = C_{cl} \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix} + D_{1,cl} e(k) + D_{2,cl} \tilde{w}(k)$$

where

$$A_{cl} = \begin{bmatrix} A + KC_1 & -B_3F - KC_1 \\ B_1 D_{11}^{-1} C_1 + KC_1 & A - B_3F - B_1 D_{11}^{-1} C_1 - KC_1 \end{bmatrix}$$

$$K = B_3 D_e D_{11}^{-1}$$

Closed loop analysis

Next we choose a new state:

$$\begin{bmatrix} x(k) \\ x_c(k) - x(k) \end{bmatrix} = T \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}$$

This state transformation with

$$T = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$$

New system matrix:

$$\bar{A}_{cl} = T\,A_{cl}\,T^{-1} = \begin{bmatrix} A - B_3 F & -B_3 F - B_3 D_e D_{11}^{-1} C_1 \\ 0 & A - B_1 D_{11}^{-1} C_1 \end{bmatrix}$$

Closed loop analysis

$$\bar{A}_{cl} = \begin{bmatrix} A - B_3F & -B_3F - B_3D_eD_{11}^{-1}C_1 \\ 0 & A - B_1D_{11}^{-1}C_1 \end{bmatrix}$$

Necessary and sufficient conditions:

1. eigenvalues $(A - B_1D_{11}^{-1}C_1)$ strictly inside the unit circle.
2. eigenvalues $(A - B_3F)$ strictly inside the unit circle.

Closed loop analysis

1. Eigenvalues $(A - B_1 D_{11}^{-1} C_1)$ strictly inside the unit circle.

$$H(q) \equiv \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right]$$

we find

$$H^{-1}(q) \equiv \left[\begin{array}{c|c} A - B_1 D_{11}^{-1} C_1 & B_1 D_{11}^{-1} \\ \hline -D_{11}^{-1} C_1 & D_{11}^{-1} \end{array}\right]$$

and so condition 1 means: H^{-1} must be stable !!!

If not, do spectral factorization find H_{new} with H_{new}^{-1} stable and

$$H_{new}(q)\, H_{new}^T(q^{-1}) = H(q)\, H^T(q^{-1})$$

Closed loop analysis

2. Eigenvalues $(A - B_3 F)$ strictly inside the unit circle.

Three different ways:

1. Careful tuning (chapter 8)
2. End-point constraint (section 6.3)
3. Infinite prediction horizon (section 6.3)

Infinite horizon

Theorem:

LTI system ($z_{ss} = 0$)

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_2 w(k) + B_3 v(k) \\
z(k) &= C_2 x(k) + D_{22} w(k) + D_{23} v(k)
\end{aligned}
$$

Control horizon $N_c \in \mathbb{Z}^+ \cup \{\infty\}$. Performance index is defined as

$$
\min_{\tilde{v}(k)} \sum_{j=0}^{\infty} \Big(\hat{z}(k+j|k) \Big)^T \Gamma(j) \Big(\hat{z}(k+j|k) \Big)
$$

where $\Gamma(i) \geq \Gamma(i+1) \geq 0$.

For $w(k) = w_{ss},\ k \geq 0$, we obtain a stable closed loop.

Define

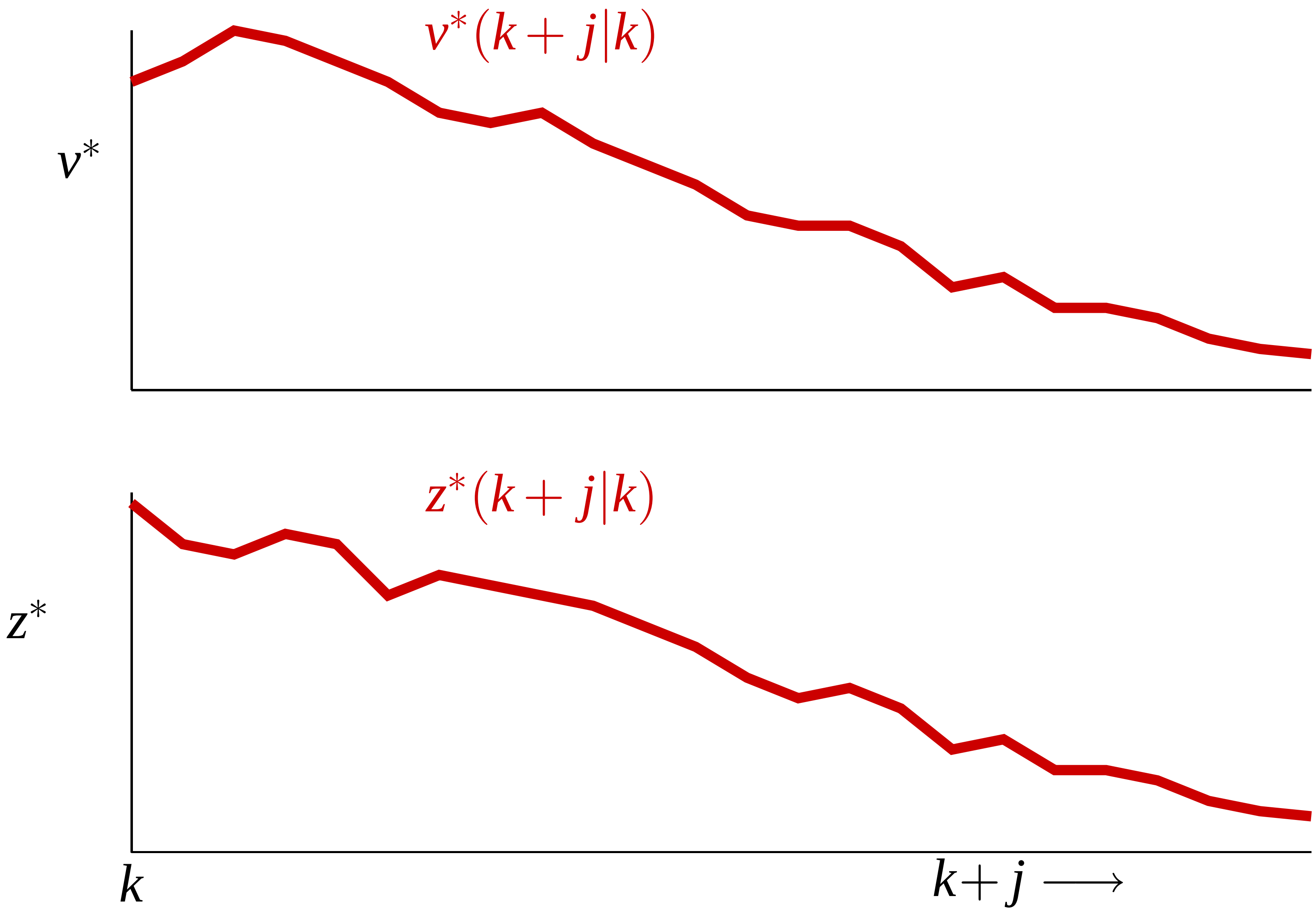
$$V(k) = \min_{\tilde{v}(k)} J(\tilde{v}, k) \geq 0,$$

$$\tilde{v}^*(k) = \arg\min_{\tilde{v}(k)} J(k),$$

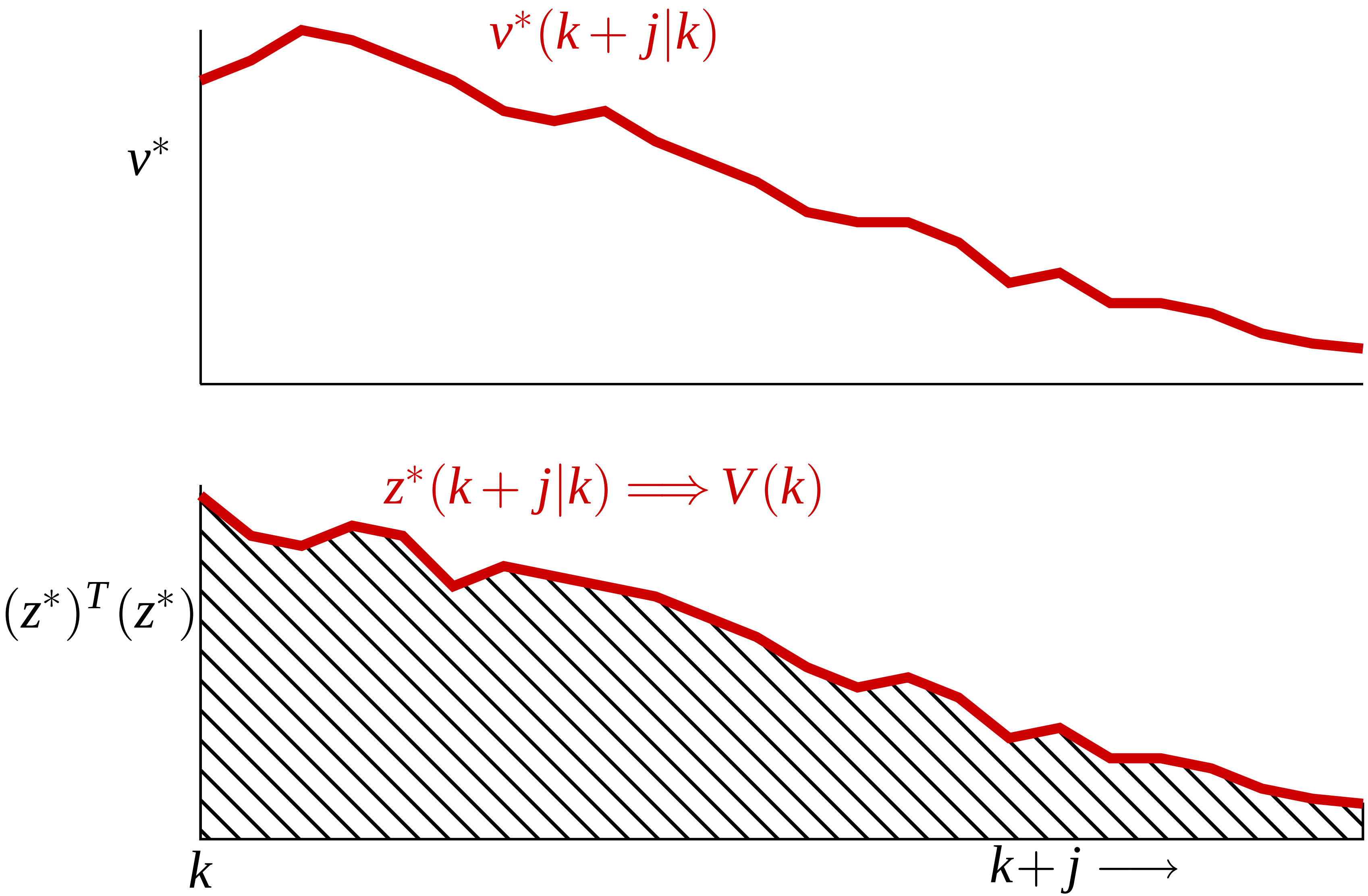
where

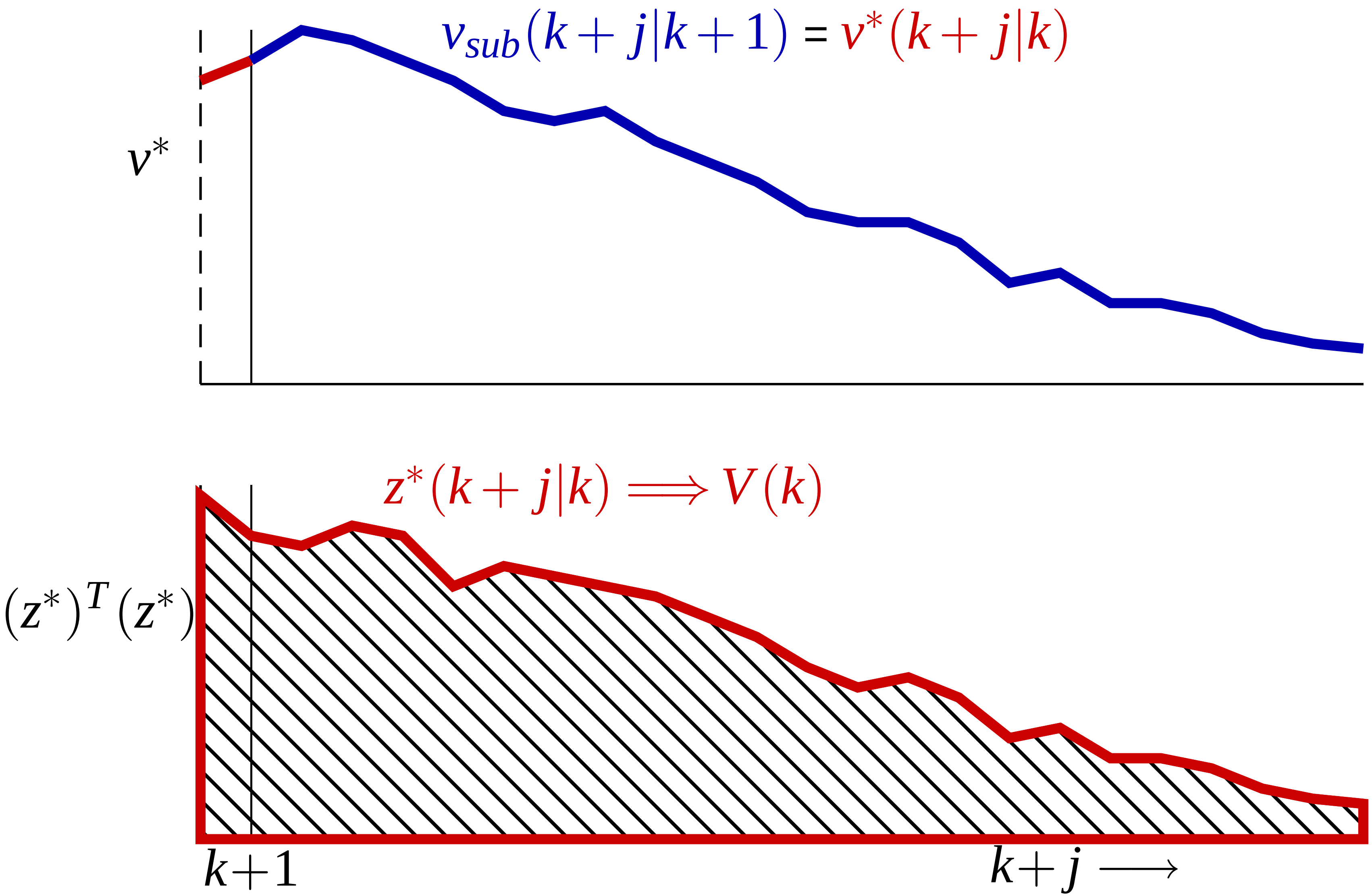
$$\tilde{v}^*(k) = \begin{bmatrix} v^*(k|k) \\ v^*(k+1|k) \\ \vdots \\ v^*(k+N_c-1|k) \end{bmatrix} \text{ for } N_c < \infty,$$

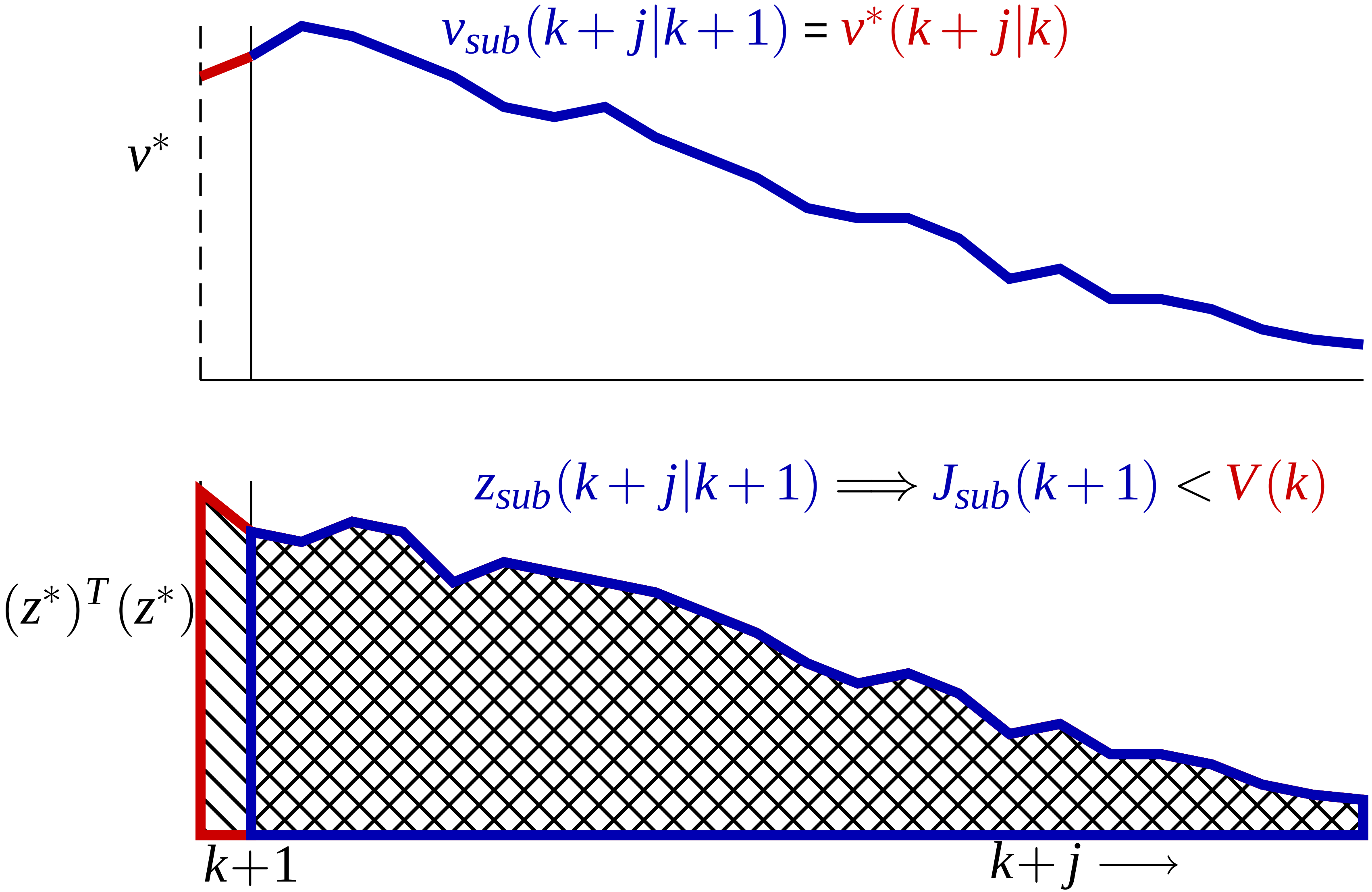
Let $\hat{z}^*(k+j|k)$ be the (optimal) performance signal for $\tilde{v}^*(k)$.



$v^*(k+j|k)$
v^*
$z^*(k+j|k)$
z^*
k
$k+j \longrightarrow$


$v^*(k+j|k)$
v^*
$z^*(k+j|k) \Longrightarrow V(k)$
$(z^*)^T(z^*)$
k
$k+j \longrightarrow$


$v_{sub}(k+j|k+1) = v^*(k+j|k)$
v^*
$z^*(k+j|k) \Longrightarrow V(k)$
$(z^*)^T(z^*)$
$k+1$
$k+j \longrightarrow$


$v_{sub}(k+j|k+1) = v^*(k+j|k)$
v^*
$z_{sub}(k+j|k+1) \Longrightarrow J_{sub}(k+1) < V(k)$
$(z^*)^T(z^*)$
$k+1$
$k+j \longrightarrow$

Infinite horizon

$$V(k+1) = \min_{\tilde{v}(k+1)} J(\tilde{v}, k+1) \leq J_{sub}(\tilde{v}, k+1) < V(k)$$

So

$$V(k) \geq 0$$

$$V(k+1) - V(k) < 0$$

Therefore $V(k)$ is a Lyapunov function and the closed loop system is stable.

End-point constraint

Equality constraint $\Longrightarrow$ LTI controller.

Based on monotonicity of performance index.

Idea: Force state to steady state at $k = N$.

Steady state:

$$\begin{aligned} x_{ss} &= A x_{ss} + B_2 w_{ss} + B_3 v_{ss} \\ z_{ss} &= C_2 x_{ss} + D_{22} w_{ss} + D_{23} v_{ss} \end{aligned}$$

$(v_{ss}, x_{ss}, w_{ss}, z_{ss}) = (v_{ss}, x_{ss}, w_{ss}, 0)$

is a steady state of the system.

End-point constraint

Theorem: LTI system ($z_{ss} = 0$)

$$\begin{aligned} x(k+1) &= Ax(k) + B_2\,w(k) + B_3\,v(k) \\ z(k) &= C_2\,x(k) + D_{22}\,w(k) + D_{23}\,v(k) \end{aligned}$$

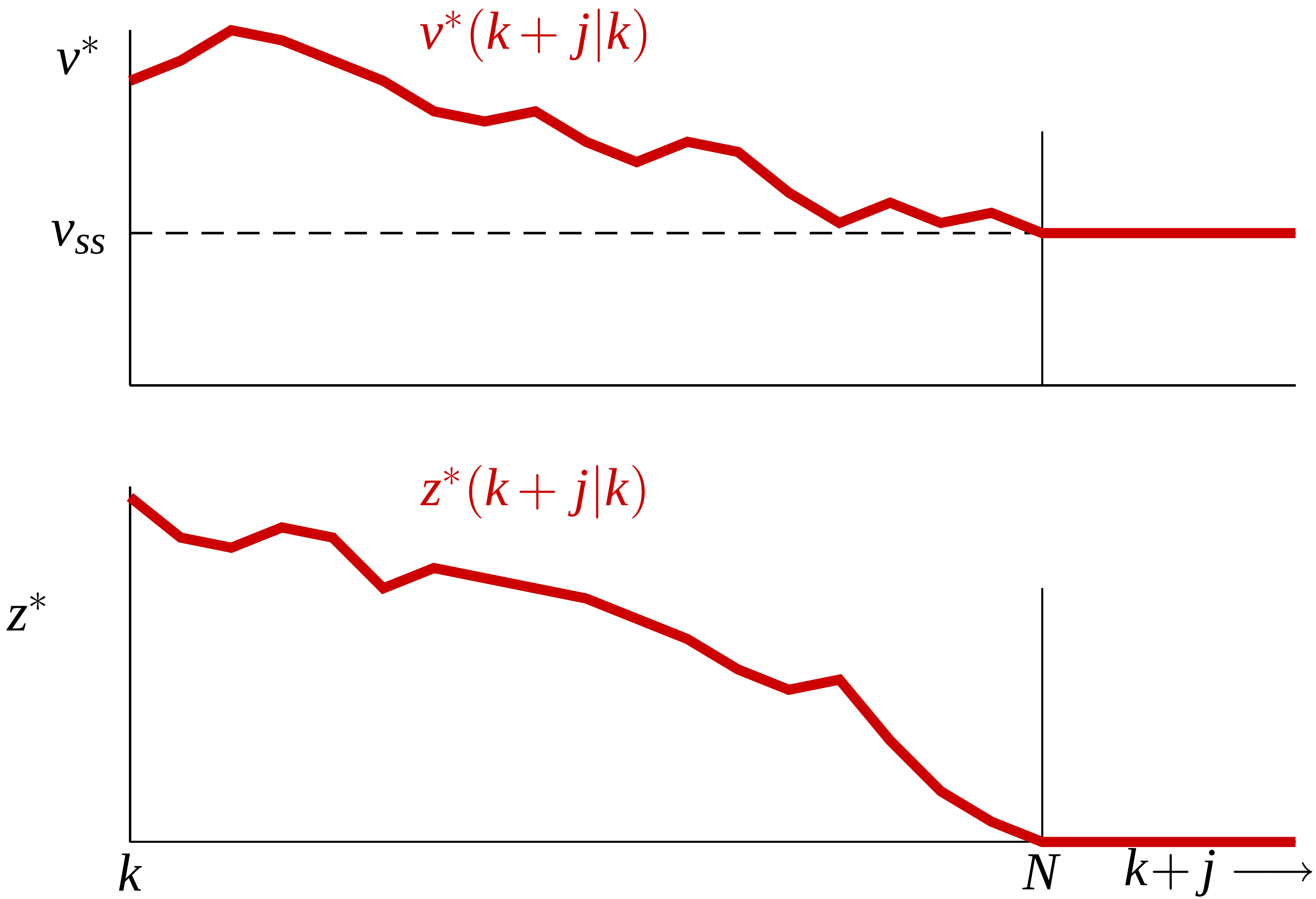
Control horizon $N_c \in \mathbb{Z}^+$. Performance index is defined as

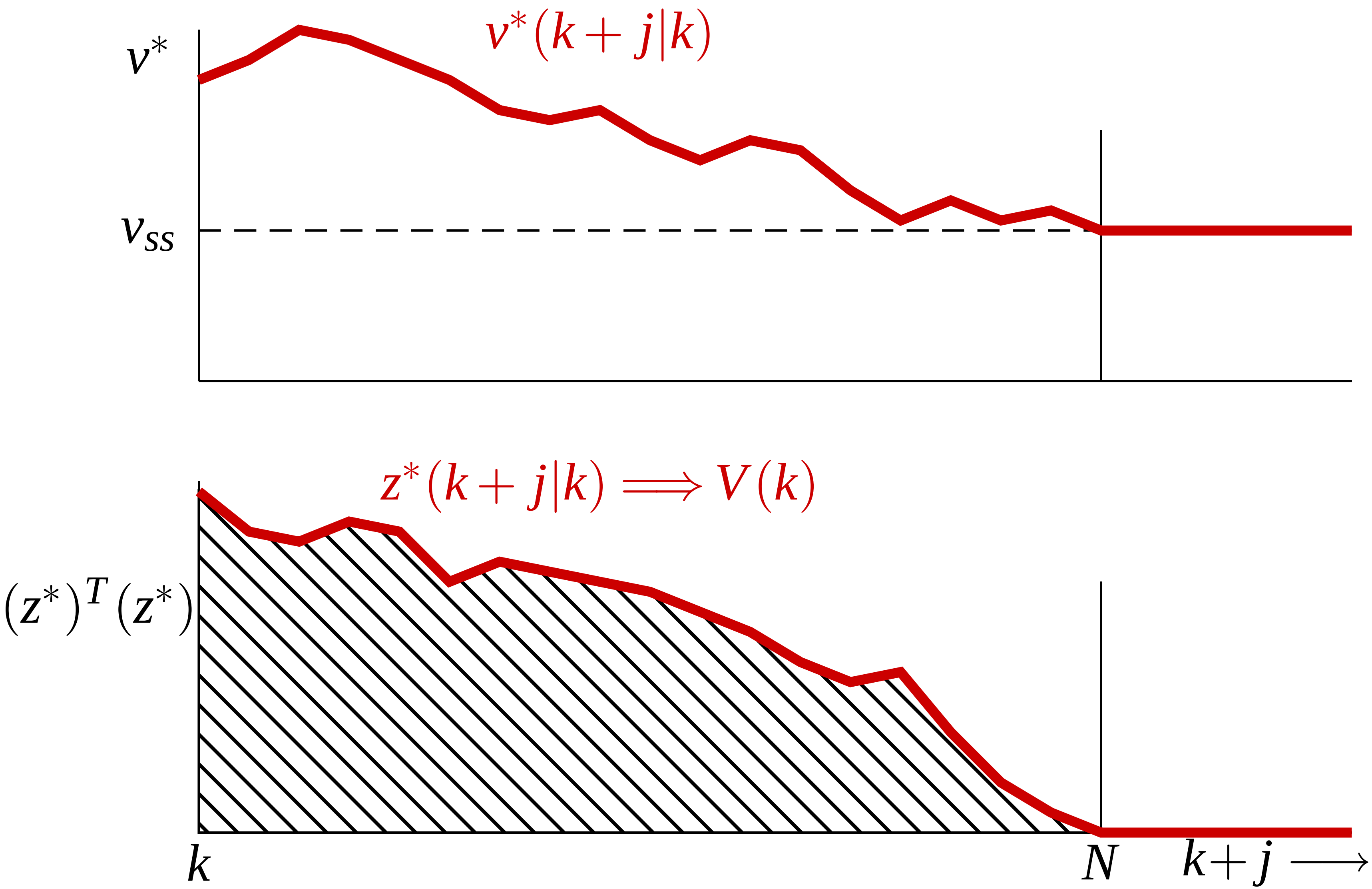
$$\min_{\tilde{v}(k)} \sum_{j=0}^{N-1} \Big(\hat{z}(k+j|k)\Big)^T \Gamma(j) \Big(\hat{z}(k+j|k)\Big)$$

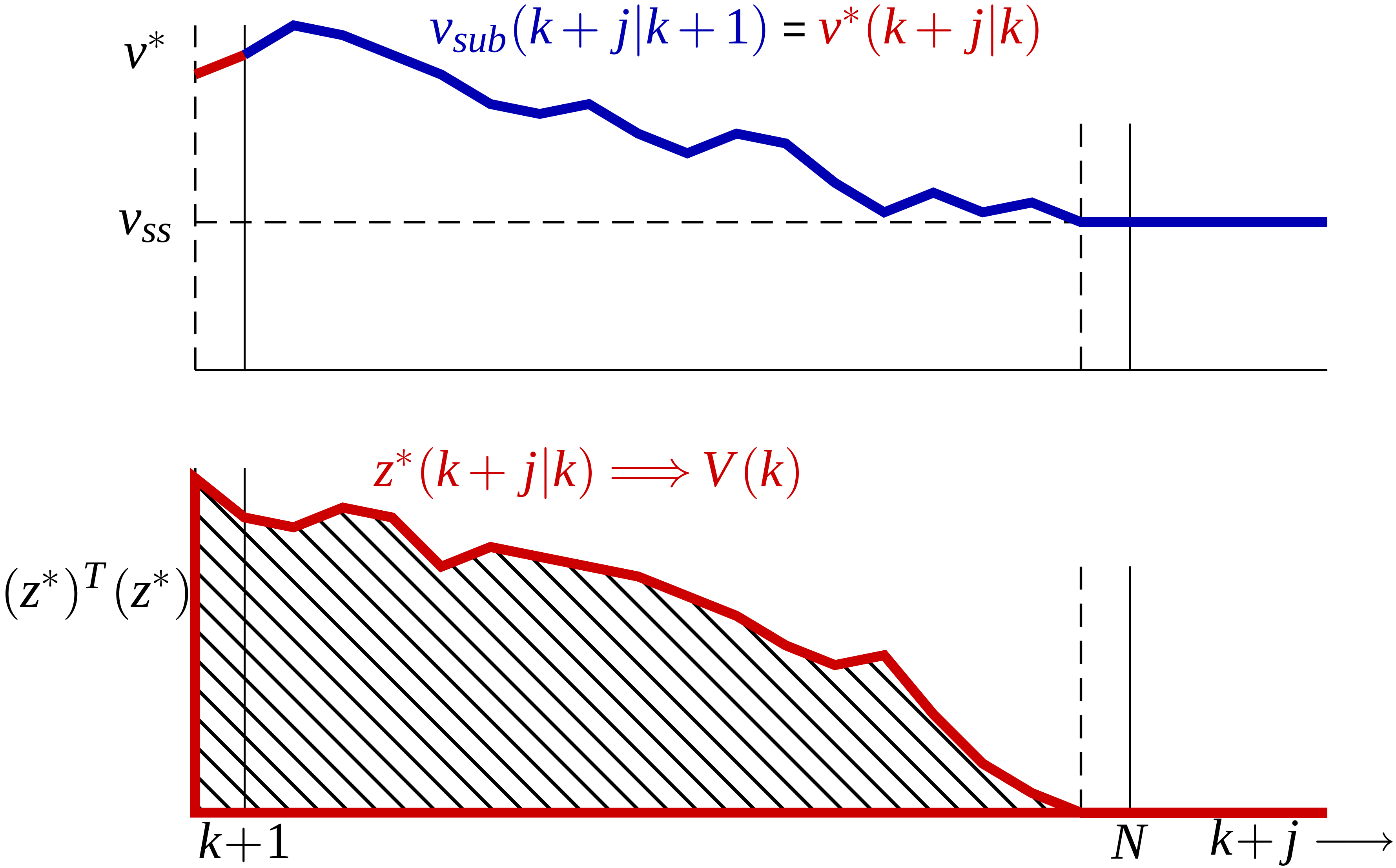
where $\Gamma(i) > \Gamma(i+1) \geq 0$. Additional end-point constraint:

$$x(k+N) = x_{ss}$$

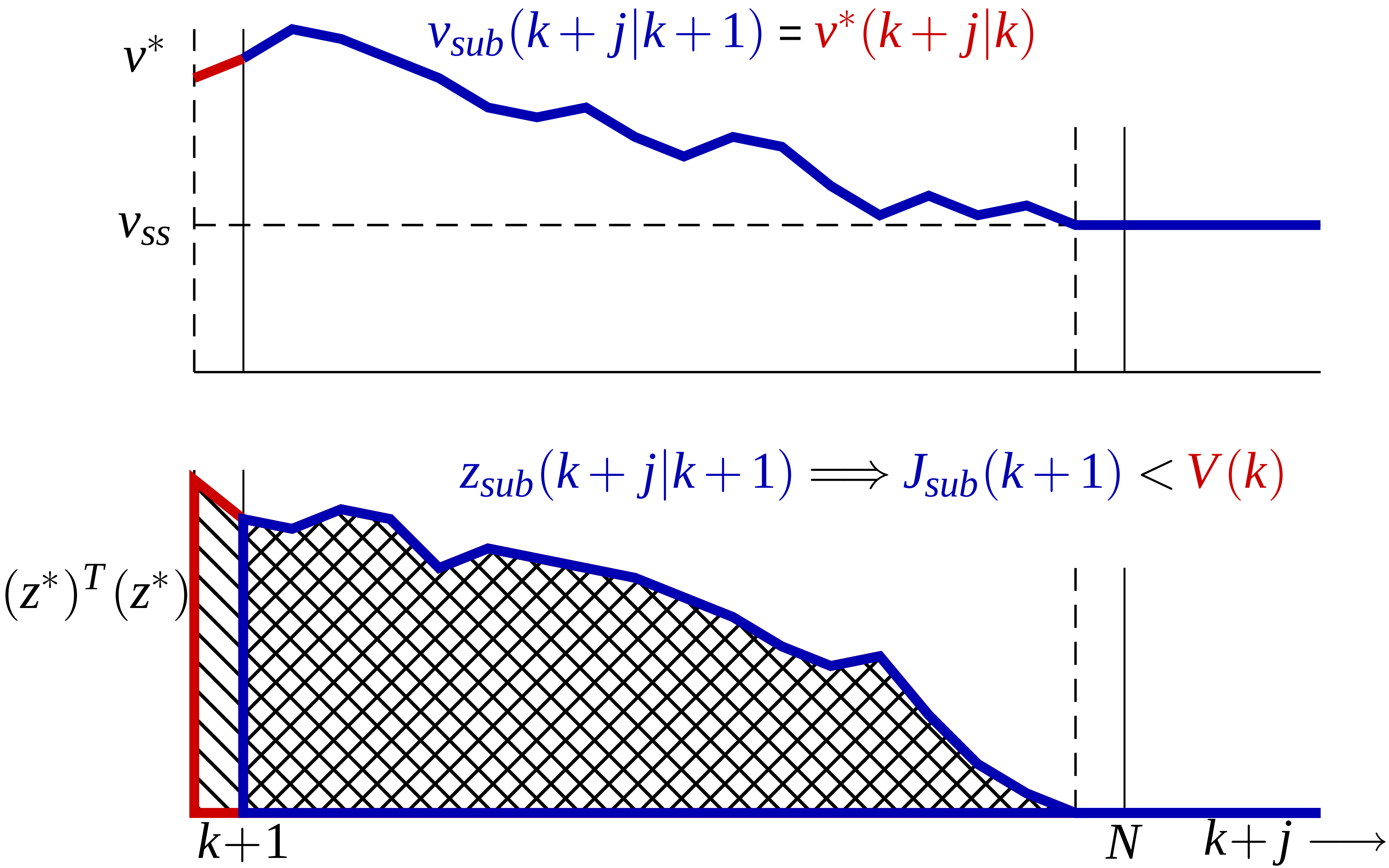
For $w(k) = w_{ss},\ k \geq 0$, we obtain a stable closed loop


$v^*(k+j|k)$
v^*
v_{ss}
$z^*(k+j|k)$
z^*
k
N
$k+j \longrightarrow$


v^*
$v^*(k+j|k)$
v_{ss}
$z^*(k+j|k) \Longrightarrow V(k)$
$(z^*)^T(z^*)$
k
N
$k+j \longrightarrow$



$v_{sub}(k+j|k+1) = v^*(k+j|k)$
v^*
v_{ss}
$z^*(k+j|k) \Longrightarrow V(k)$
$(z^*)^T(z^*)$
$k+1$
N
$k+j \longrightarrow$


$v_{sub}(k+j|k+1) = v^*(k+j|k)$
v^*
v_{ss}
$z_{sub}(k+j|k+1) \Longrightarrow J_{sub}(k+1) < V(k)$
$(z^*)^T(z^*)$
$k+1$
N
$k+j \longrightarrow$

End-point constraint

$$V(k+1) = \min_{\tilde{v}(k+1)} J(\tilde{v}, k+1) \leq J_{sub}(\tilde{v}, k+1) < V(k)$$

So

$$V(k) \geq 0$$

$$V(k+1) - V(k) < 0$$

Therefore $V(k)$ is a Lyapunov function and the closed loop system is stable.

End-point constraint

$$W\,x(k+N|k) - W\,x_{ss} = 0$$

W has full column-rank. Equivalent to the end-point constraint.

$$\begin{bmatrix} \hat{y}(k+N|k) \\ \hat{y}(k+N+1) \\ \vdots \\ \hat{y}(k+N+n-1|k) \end{bmatrix} - \begin{bmatrix} y_{ss} \\ \vdots \\ y_{ss} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_1 A \\ \vdots \\ C_1 A^{n-1} \end{bmatrix} x(k+N|k) - \begin{bmatrix} C_1 \\ C_1 A \\ \vdots \\ C_1 A^{n-1} \end{bmatrix} x_{ss} = 0$$

Clarke & Scattolini [17], Mosca & Zhang [61]

Drawbacks:

End-point constraint:

- Tuning rules change
- Constrained case: Infeasibility possible

Infinite horizon:

- Tuning rules change
- For $N_c = \infty$; No constraint handling
- For $N_c < \infty$; Constrained case: Infeasibility possible

Inequality constrained case

- Stability for nonlinear system
- Feasibility of optimization problem

Solutions:

- LMI-approach (chapter 8).
- Add feasibility guarantee.

Shift-invariant constraint:

Any $\tilde{v}(k)$ that makes

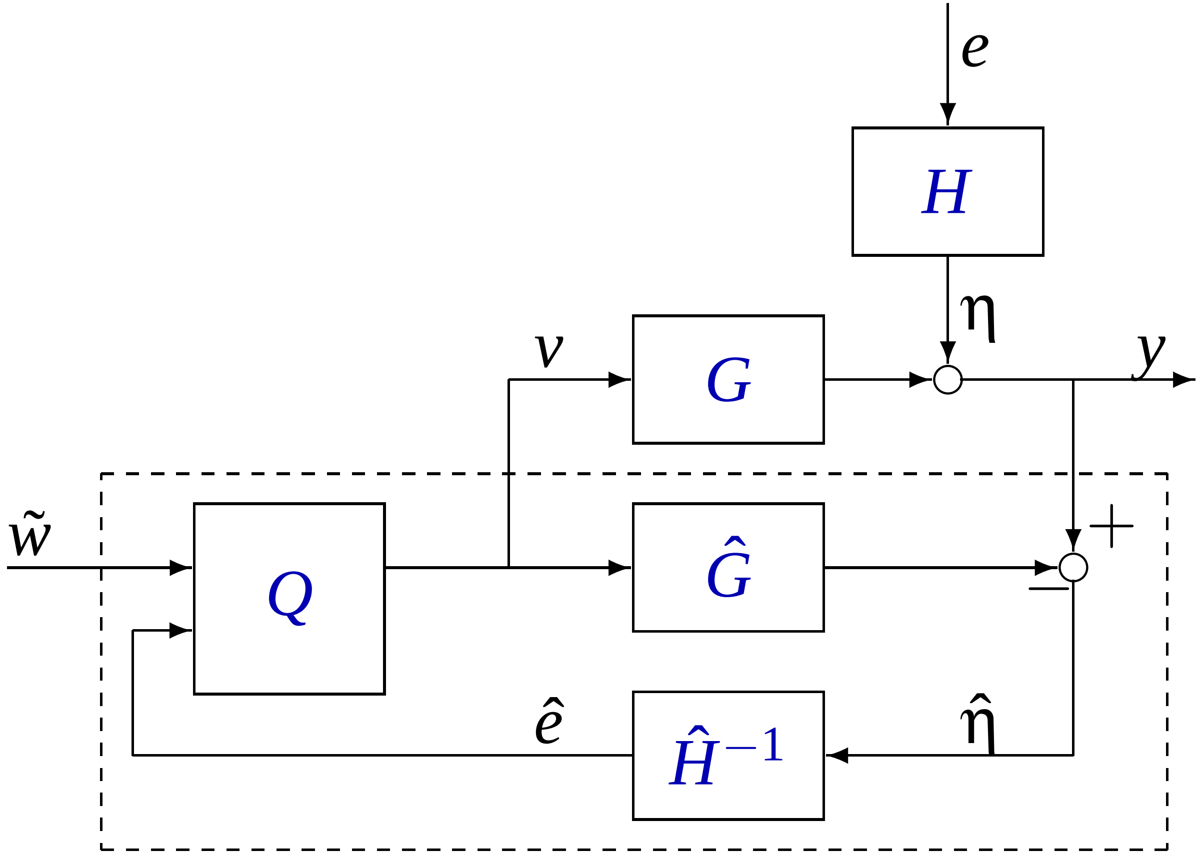
$$\tilde{\psi}(k) \le \tilde{\Psi}(k)$$

we find that for $\tilde{v}_{sub}(k+1)$:

$$\tilde{\psi}(k+1) \le \tilde{\Psi}(k+1)$$

IMC scheme and MPC

Let G, $\hat{G}$, H, $\hat{H}$, $\hat{H}^{-1}$ be stable and $F = 0$.



$$v(k) = \kappa_1(q)\tilde{w}(k) + \kappa_2(q)y(k)$$

Relation κ and Q:

$$\kappa_1(q) = \left(I + Q_2(q)\hat{H}^{-1}(q)\hat{G}(q)\right)^{-1} Q_1(q)$$

$$\kappa_2(q) = \left(I + Q_2(q)\hat{H}^{-1}(q)\hat{G}(q)\right)^{-1} Q_2(q)\hat{H}^{-1}(q)$$

$$= Q_2(q)\left(\hat{H}(q) + \hat{G}(q)Q_2(q)\right)^{-1}$$

with the inverse relation:

$$Q_1(q) = \left(I - \kappa_2(q)\hat{G}(q)\right)^{-1} \kappa_1(q)$$

$$Q_2(q) = \left(I - \kappa_2(q)\hat{G}(q)\right)^{-1} \kappa_2(q)\hat{H}(q)$$

Closed loop behavior:

$$\begin{aligned} v(k) &= M^{-1}(q)Q_1(q)\tilde{w}(k) + M^{-1}(q)Q_2(q)\hat{H}^{-1}(q)H(q)e(k) \\ y(k) &= G(q)M^{-1}(q)\,Q_1(q)\tilde{w}(k) + \Big(G(q)M^{-1}(q)\,Q_2(q)\hat{H}^{-1}(q) + I\Big)H(q)e(k) \\ M(q) &= \Big(I - Q_2(q)\hat{H}^{-1}(q)(G(q) - \hat{G}(q))\Big) \end{aligned}$$

For $G = \hat{G}$ and $H = \hat{H}$:

$$\begin{aligned} v(k) &= Q_1(q)\tilde{w}(k) + Q_2(q)e(k) \\ y(k) &= G(q)Q_1(q)\tilde{w}(k) + \Big(G(q)\,Q_2(q) + H(q)\Big)e(k) \\ M(q) &= I \end{aligned}$$

Closed loop is stable iff Q is stable

For SPCP problem with

$$G(q) \equiv \left[\begin{array}{c|c} A & B_3 \\ \hline C_1 & 0 \end{array}\right] \quad , \quad H(q) \equiv \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right]$$

with $G(q)$, $H(q)$ and $H^{-1}(q)$ are stable, and solution matrices

$$F \ , \ D_e \ , \ D_w$$

we find:

$$Q_1(q) \equiv \left[\begin{array}{c|c} A - B_3 F & B_3 D_w \\ \hline -F & D_w \end{array}\right] \ , \ Q_2(q) \equiv \left[\begin{array}{c|c} A - B_3 F & (B_1 + B_3 D_e) \\ \hline -F & D_e \end{array}\right]$$

so the SPCP controller is stable if

$$|\, \lambda_i (A - B_3 F) \,| < 1$$

Robust predictive control

- concept of robustness
- uncertainty descriptions
- robustness by tuning
- robustness in LMI-MPC (see chapter 7)

Concepts of Robustness

$$\boxed{\text{Small change of process}}$$
$$\Downarrow$$
$$\boxed{\text{Small change of behaviour}}$$

G_t **in uncertainty set** $\mathcal{G}$

Robust stability:

Stability for all systems in $\mathcal{G}$

Robust Performance:

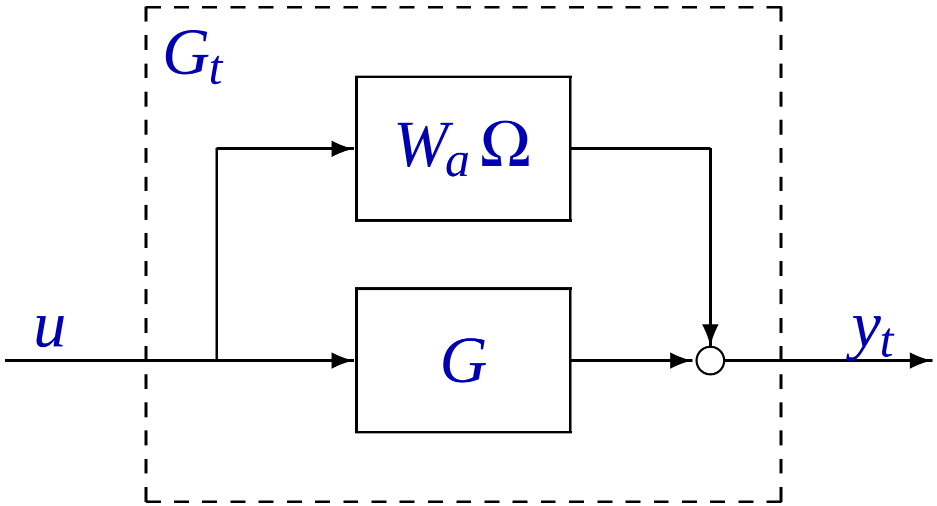
Performance level for all systems in $\mathcal{G}$

Unstructured model uncertainty

Uncertainty described by transferfunction.

Example:

$$G_t = G + W_a \Omega \qquad \text{where} \qquad \| \Omega \|_\infty \leq 1$$



Unstructured model uncertainty

General plant description:

$$
\begin{aligned}
x(k+1) &= Ax(k)+B_1e(k)+B_2w(k)+B_3v(k)+B_4\delta(k)\\
y(k) &= C_1x(k)+D_{11}e(k)+D_{12}w(k)\\
z(k) &= C_2x(k)+D_{21}e(k)+D_{22}w(k)+D_{23}v(k)+D_{24}\delta(k)\\
\psi(k) &= C_4x(k)+D_{41}e(k)+D_{42}w(k)+D_{43}v(k)+D_{44}\delta(k)\\
\varepsilon(k) &= C_5x(k)+D_{53}v(k)+D_{54}\,\delta(k)\\
\delta(k) &= \Omega(q)\varepsilon(k)
\end{aligned}
$$

where $\Omega(q)$ is unknown, but bounded:

$$
\|\Omega\|_\infty = \max_{\omega}\bar{\sigma}\left(\Omega(e^{j\omega})\right) \leq 1
$$

Structured model uncertainty

Uncertainty described by parameterset.

Example:

$$A = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{aligned} a_{1,min} &\le a_1 \le a_{1,max} \\ a_{2,min} &\le a_2 \le a_{2,max} \end{aligned}$$

Structured model uncertainty

$$
\begin{aligned}
x(k+1) &= Ax(k) + B_1 e(k) + B_2 w(k) + B_3 v(k) \\
y(k) &= C_1 x(k) + D_{11} e(k) + D_{12} w(k) \\
z(k) &= C_2 x(k) + D_{21} e(k) + D_{22} w(k) + D_{23} v(k) \\
\psi(k) &= C_4 x(k) + D_{41} e(k) + D_{42} w(k) + D_{43} v(k)
\end{aligned}
$$

where

$$
P = \begin{bmatrix} A & B_1 & B_2 & B_3 \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_4 & D_{41} & D_{42} & D_{43} \end{bmatrix}
$$

belongs to a set $\mathcal{P}$, given by $\mathcal{P} = \text{Co}\,\{P_1, P_2, \ldots, P_L\}$

Convex closure

$$H \in \mathrm{Co}\{H_1, H_2, \ldots, H_L\}$$

means $\exists \lambda_i$ such that

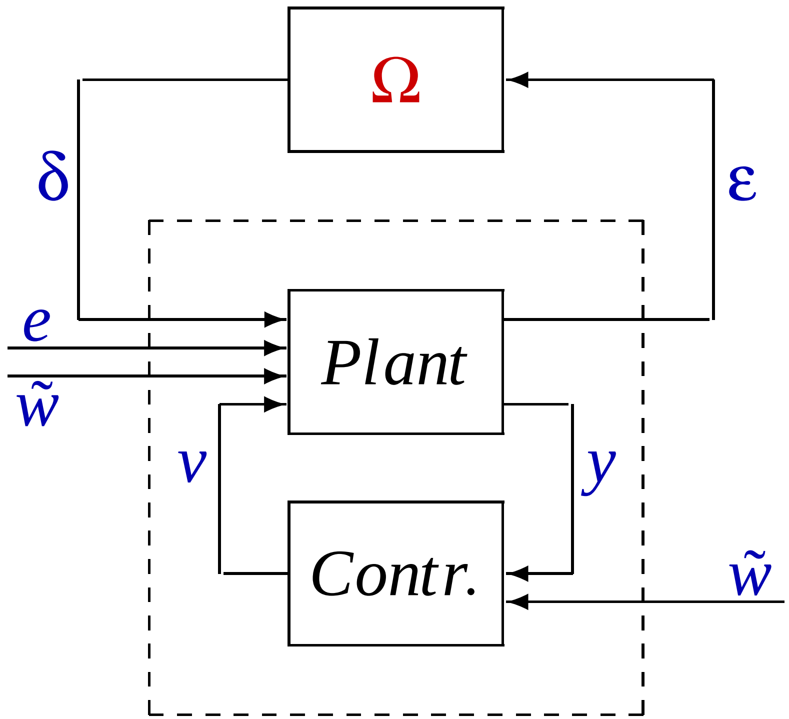
$$H = \lambda_1 H_1 + \lambda_2 H_2 + \ldots + \lambda_L H_L$$

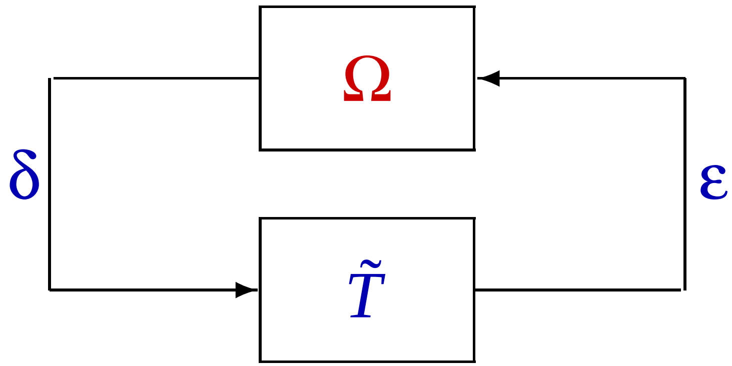
where

$$0 \leq \lambda_i \leq 1 \quad , \quad \sum_{i=1}^{L} \lambda_i = 1$$

A polytope with vertices in H_1, H_2, $\ldots$, H_L.

Robustness by tuning





Robustness by tuning

Closed loop:

$$(I - \tilde{T}\Omega)^{-1}$$

Stability condition:

$$|\tilde{T}(\theta,\eta,z)\Omega(z)| \leq 1 \quad \text{for all } |z| = 1$$

so:

$$\|\tilde{T}(\theta,\eta)\Omega\|_\infty \leq 1$$

Necessary and sufficient condition:

$$\|\tilde{T}(\theta,\eta)\|_\infty \leq 1$$

Tuning of θ and η, where $\eta = (N_m, N, N_c)$ and $\theta = (\lambda, p_0, p_1, \ldots, p_{n_p})$ for GPC and $\theta = (R^{1/2}, Q^{1/2})$ for LQPC.

Additive model uncertainty

$$G_t(q) \;=\; G(q) + W_a(q)\,\Omega(q)$$

or

$$\Omega(q) \;=\; W_a^{-1}(q)\Big(G_t(q) - G(q)\Big)$$

with $\|\Omega\|_\infty \leq 1$.

For SISO systems $W_a(e^{j\omega})$ gives na upper bound on the magnitude of uncertainty

$$|G_t(e^{j\omega}) - G(e^{j\omega})| \leq |W_a(e^{j\omega})| \quad \forall\, \omega \in \mathrm{R}$$

Stable G_t, G, H_t, H, H^{-1}.

MPC controller stabilizes model G, H, so Q_1 Q_2 are stable.

Closed loop in IMC scheme:

$$v(k) = M^{-1}(q)Q_1(q)\tilde{w}(k)+M^{-1}(q)Q_2(q)H^{-1}(q)H_t(q)e(k)$$
$$y(k) = G_t(q)M^{-1}(q)Q_1(q)\tilde{w}(k)+\Big(G_t(q)M^{-1}(q)\,Q_2(q)H^{-1}(q)+I\Big)H_t(q)e(k)$$

where Loop gain $M(q)$ in IMC scheme is given by

$$\begin{aligned} M(q) &= I-Q_2(q)H^{-1}(q)(G_t(q)-G(q)) \\ &= I-Q_2(q)H^{-1}(q)W_a(q)\,\Omega(q) \end{aligned}$$

$$M(q) = I - Q_2(q)H^{-1}(q)W_a(q)\,\Omega(q)$$

stability condition for all $\|\Omega\|_\infty \leq 1$ is given by:

$$\|Q_2 H^{-1} W_a\|_\infty < 1$$

Note that $Q_2(\theta, \eta)$, and so robust stability can be obtained by tuning θ and η.

8. Tuning

Reasons for popularity: easy tuning !!

(Basic parameters: N, N_c and λ)

Purpose of tuning the parameters:

- acquire good signal tracking
- sufficient disturbance rejection
- robustness against model mismatch

Rules of thumb for initial parameter setting

Fine-tuning is done on-line

Tuning of GPC parameters

GPC performance index:

$$J(u,k) = \sum_{j=N_m}^{N} |\hat{y}_p(k+j|k) - r(k+j|k)|^2 + \lambda^2 \sum_{j=1}^{N_c} |\Delta u(k+j-1)|^2$$

$\Delta u(k+j) = 0$ for $j \geq N_c$

N_m	$=$	minimum-cost horizon	λ	$=$	weighting factor
N	$=$	prediction horizon	$P(q)$	$=$	tracking filter
N_c	$=$	control horizon			

Rules-of-thumb

- Clarke & Mothadi (1987)
- Soeterboek (1991)

Tuning of LQPC parameters

LQPC performance index:

$$J(u,k) = \sum_{j=N_m}^{N} \hat{x}^T(k+j|k) Q \hat{x}(k+j|k) + \sum_{j=1}^{N} u^T(k+j-1) R u(k+j-1)$$

$$u(k+j) = u(k+Nc-1) \qquad \text{for } \; j \geq N_c$$

N_m	$=$	minimum-cost horizon	Q	$=$	state weighting matrix
N	$=$	prediction horizon	R	$=$	control weighting matrix
N_c	$=$	control horizon			

Tuning rules

- Lee & Yu (1994)

Summation parameters

Summation parameters: N_m, N and N_c

Parameter N_m:

$N_m = 1$ usually

$N_m > 1$ for dead-time or inverse response

Parameter N:

$N >$ length step response

$[N_m, N]$ contains crucial dynamics of

- response $v(k), e(k), w(k) \longrightarrow z(k)$
- response $u(k), e_o(k), d_o(k) \longrightarrow x(k)$ (LQPC)
- response $\Delta u(k), e_i(k), d_i(k) \longrightarrow y(k)$ (GPC)

Summation parameters

Parameter N_c:

$N_c \leq N$ is control horizon. Control signal is forced to steady-state value

$$u(k+j) = u(k+N_c-1) \text{ for } j \geq N_c \quad \text{or} \quad \Delta u(k+j) = 0 \text{ for } j \geq N_c$$

Small N_c: smoothing, stabilizing effect

Large N_c: faster disturbance rejection

Small N_c: reduction computational effort

Summation parameters

Consider a process where

$$
\begin{aligned}
d &= \text{the dead time of the process} \\
n &= \text{the number of poles of the process} \\
&= \text{the dimension of the matrix } A_o \\
&= \text{the order of polynomial } a_o(q) \\
t_s &= \text{the 5\% settling time of the DT process } G_o(q) \\
\omega_s &= \text{the sampling-frequency} \\
\omega_b &= \text{the bandwidth of the CT process } G_{c,o}(s)
\end{aligned}
$$

then the following initial setting are recommended (Soeterboek [81]):

$$
\begin{aligned}
N_m &= 1+d \\
N_c &= n \\
N &= \alpha_N \, t_s \quad , \quad \alpha_N \in [1.5, 2] \quad \text{(well-damped process)} \\
N &= \text{int}(\beta_N \, \omega_s / \omega_b) \quad , \quad \beta_N \in [4, 25] \quad \text{(badly-damped or unstable process)}
\end{aligned}
$$

GPC signal weighting parameters

parameter λ:

λ as small as possible

$\lambda = 0$ for minimum phase process

$\lambda > 0$ for non-minimum phase process

The filter $P(q)$:

$$P(q) = 1 + p_1 q^{-1} + p_2 q^{-2} + \cdots + p_{n_p} q^{-n_p}$$

Roots $P(q)$ = desired closed loop poles

$y(k) \approx P^{-1}(q) r(k)$

Thus: low-pass filtering

LQPC signal weighting matrices

SISO case:

Q is a $n \times n$ matrix , R is a scalar

GPC-like PI (for $r(k) = 0$):

$$\sum_{j=N_m}^{N} \hat{y}^T \hat{y} + \lambda^2 \sum_{j=1}^{N_c} u^T u$$

Then choose: $Q = C_1^T C_1$ and $R = \lambda^2 I$

Tuning λ: same as for GPC.

Adding term Q_1 to the weighting matrix

$$Q = C_1^T C_1 + Q_1$$

introduces additional weighting $\hat{x}^T(k+j|k) Q_1 \hat{x}(k+j|k)$.

LQPC signal weighting matrices

MIMO case:

Scaling the input and output signals !!!

If one requires precision $|\,y_i\,| \leq d_i$ for $i = 1, \ldots, m$

introduce scaling matrix:

$$S = \text{diag}(d_1, d_2, \ldots, d_m)$$

Weighting matrix on state:

$$Q = C_1^T S^{-2} C_1$$

First term of performance index:

$$\hat{x}^T Q \hat{x} = \hat{x}^T C_1^T S^{-2} C_1 \hat{x} = \sum_{i=1}^{m} \Big(|\hat{y}_i| / d_i \Big)^2$$

We can do the same for the input.

Weighting matrix on input:

$$R = \lambda^2 \, \mathrm{diag}(r_1^2, r_2^2, \ldots, r_p^2)$$

Second term of performance index:

$$u^T R u = \sum_{i=1}^{m} \Big(|u_i| / r_i \Big)^2$$

Example: tuning GPC

IO-ystem:

$$a_o(q)\,y(k) = b_o(q)\,u(k) + f_o(q)\,d_o(k) + c_o(q)\,e_o(k)$$

where

$$\begin{aligned} a_o(q) &= (1-0.9q^{-1}) \qquad & b_o(q) &= 0.03\,q^{-1} \\ c_o(q) &= (1-0.7q^{-1}) \qquad & f_o(q) &= 0 \end{aligned}$$

Transform IO model to IIO model:

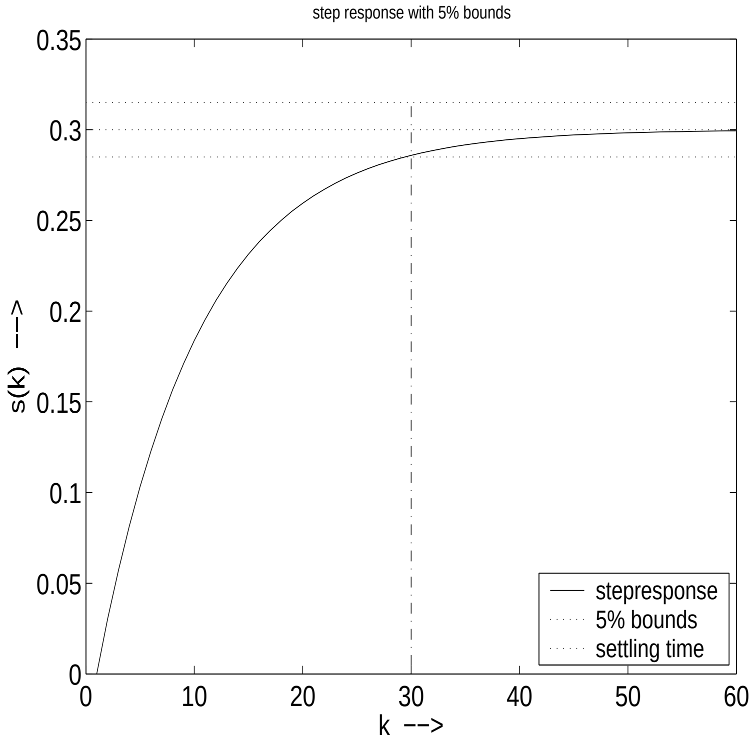
$$a_i(q)\,y(k) = b_i(q)\,\Delta u(k) + f_i(q)\,d_i(k) + c_i(q)\,e_i(k)$$

where

$$\begin{aligned} a_i(q) &= (1-q^{-1})a_o(q) \\ &= (1-q^{-1})(1-0.9q^{-1}) \\ b_i(q) &= b_o(q) = 0.03\,q^{-1} \end{aligned}$$

$$
\begin{aligned}
c_i(q) &= c_o(q) &= (1 - 0.7q^{-1}) \\
f_i(q) &= f_o(q) &= 0
\end{aligned}
$$

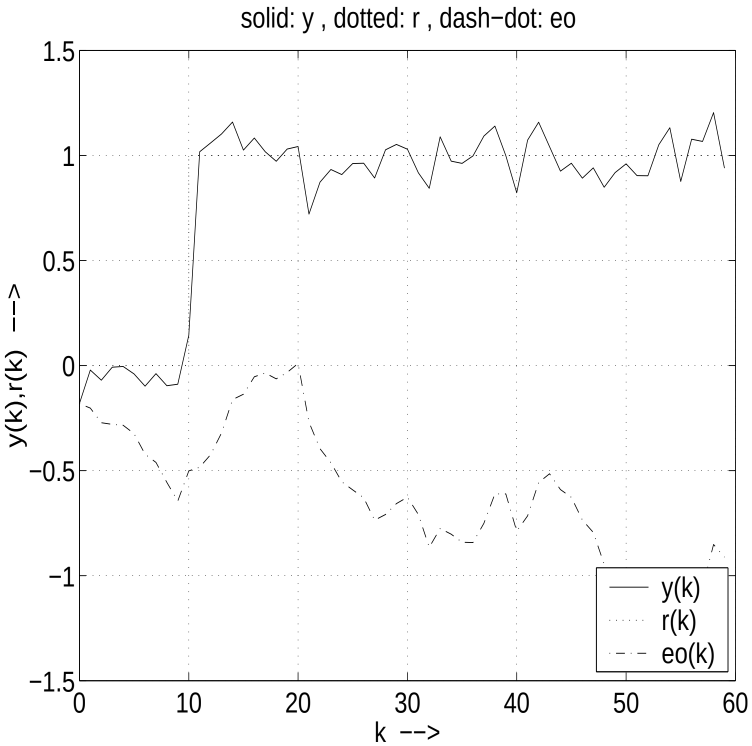
Step response of the IO-model:



So settling-time $t_s = 30$ and dead time $d = 0$.

Initial setting:

$$
\begin{aligned}
N_m &= 1 + d = 1 \\
N_c &= n = 2 \\
N &= \text{int}(\alpha_N t_s) = \text{int}(1.5\,30) = 45
\end{aligned}
$$



Example: tuning LQPC

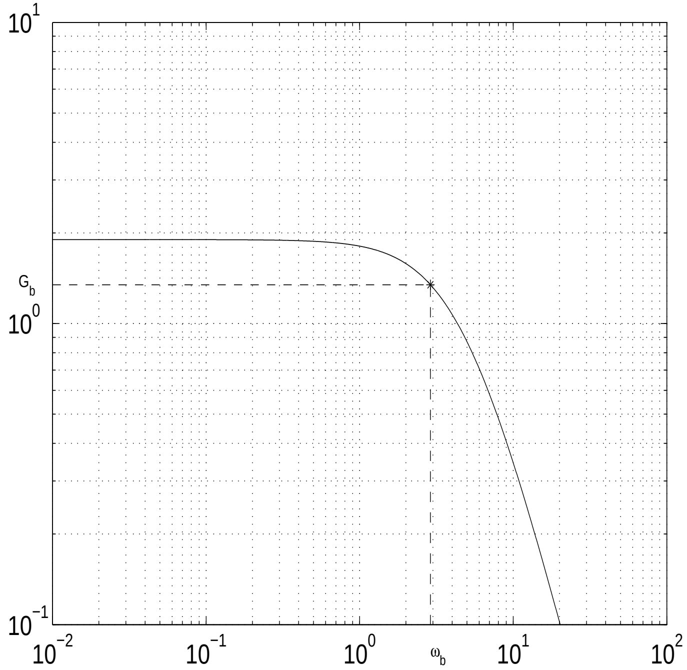
IO system:

$$A_o = \begin{bmatrix} 2.3000 & 1.0000 \\ -1.2000 & 0 \end{bmatrix} \quad C_o = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$B_o = \begin{bmatrix} 0.1 \\ 0.09 \end{bmatrix} \quad K_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad L_o = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

System is unstable ($\lambda_1 = 0.8$ and $\lambda_2 = 1.5$)

Bode-diagram of the C-T model ($\omega_s = 100$ rad/s):

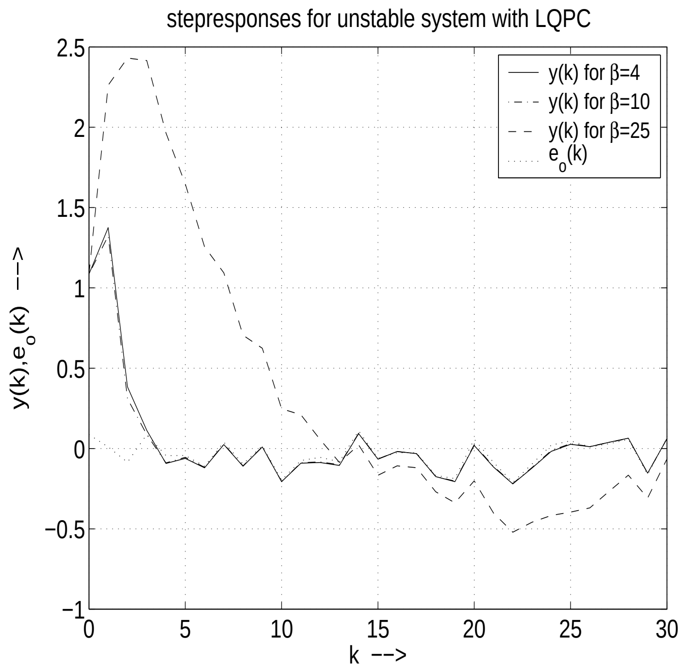


Bandwidth $\omega_b = 2\pi 2.89 = 18.15$ rad/s and dead time $d = 0$.

Initial setting

$$\begin{aligned} N_m &= 1 + d = 1 \\ N_c &= n = 2 \\ N &= \text{int}(\beta_N\, \omega_s/\omega_b) = \text{int}(\beta_N\, 5.51) \end{aligned}$$

Tune β_N with step-response:



$\beta_N = 10$ the best response, so

$$N = \text{int}(\beta_N \cdot 5.51) = \text{int}(10 \cdot 5.51) = 55$$

Tuning as optimization problem

Performance specifications:

1. Overshoot	χ_{os}	5. RMS mistracking on ZMWN	χ_{rm}
2. Rise time	χ_{rt}	6. Bandwidth of closed loop	χ_{bw}
3. Settling time	χ_{st}	7. Stability radius	χ_{sr}
4. Input peak value	χ_{pu}		

Parameters:

$$\begin{aligned} \eta &= (N_m, N, N_c) \\ \theta_{GPC} &= (\lambda, p_0, p_1, \ldots, p_{n_p}) \\ \theta_{LQPC} &= (R^{1/2}, Q^{1/2}) \end{aligned}$$

Tuning as optimization problem

$\eta =$ Integer , $\theta =$ Real

Multi criteria: $\chi_1 \leq c_1 \;, \ldots ,\; \chi_{n_\chi} \leq c_{n_\chi}$

Feasibility problem:

Find parameters such that specific criteria are satisfied.

Minimization problem:

Minimize one criterion χ_o subject to all criteria $\chi_i \leq c_i$.

(Mixed-integer optimization problem).