Knowledge-Based Control Systems (SC4081)

Lecture 2: Fuzzy Sets and Systems

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Outline

1. **Fuzzy sets and set-theoretic operations.**

2. **Fuzzy relations.**

3. **Fuzzy systems**

4. **Linguistic model, approximate reasoning**

Classical Set Theory

A set is a collection of objects with a common property.

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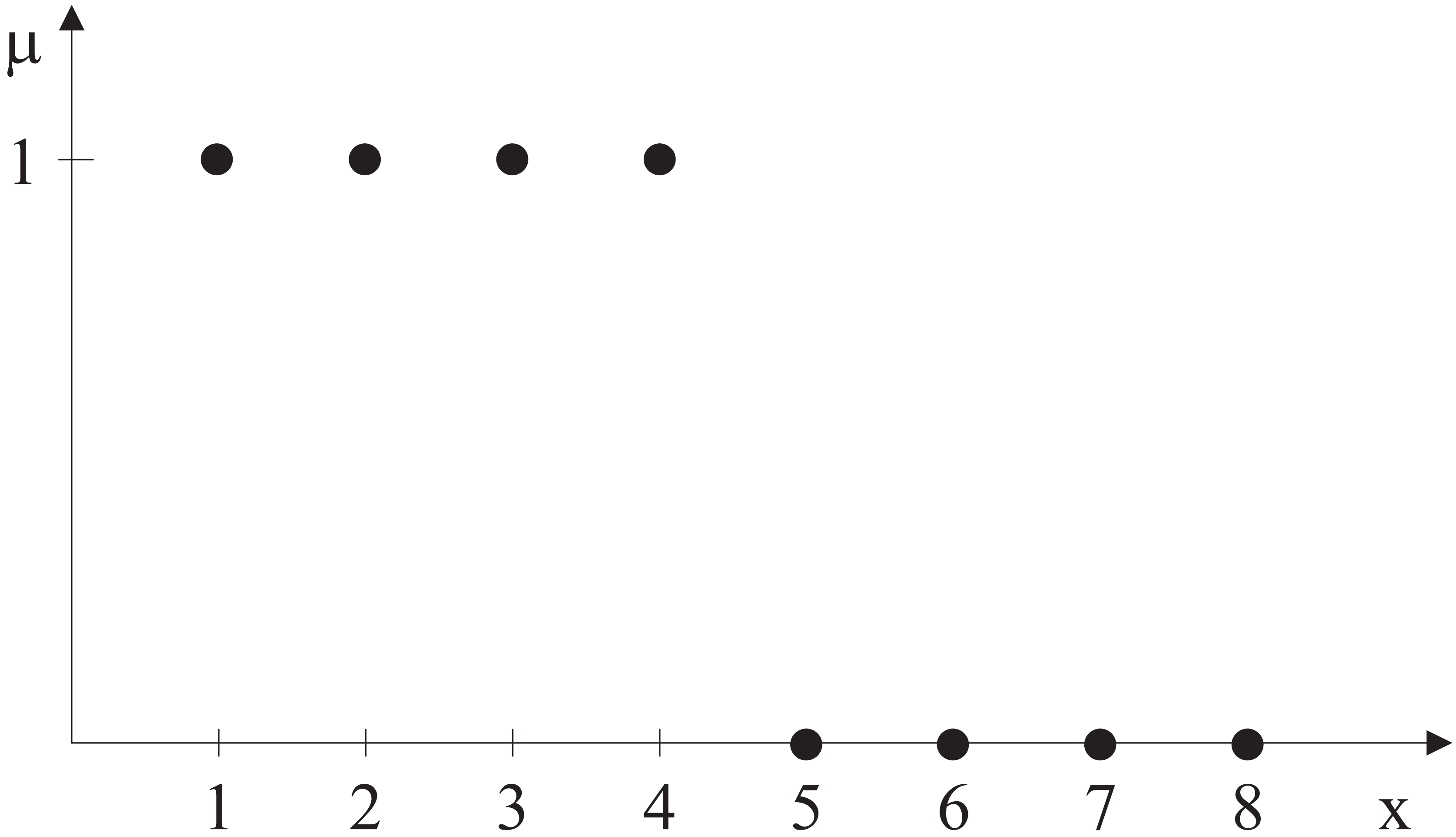
- **Unit disk in the complex plane:** $A = \{z \mid z \in \mathbb{C}, |z| \leq 1\}$
- **A line in $\mathbb{R}^2$:** $A = \{(x, y) \mid ax + by + c = 0, (x, y, a, b, c) \in \mathbb{R}\}$

Representation of Sets

- **Enumeration of elements:** $A = \{x_1,\ x_2, \ldots,\ x_n\}$
- **Definition by property:** $A = \{x \in X \mid x \text{ has property} P\}$
- **Characteristic function:** $\mu_A(x) : X \to \{0, 1\}$

$$\mu_A(x) = \begin{cases} 1 & x \text{ is member of } A \\ 0 & x \text{ is not member of } A \end{cases}$$

Set of natural numbers smaller than 5



Fuzzy sets

Why Fuzzy Sets?

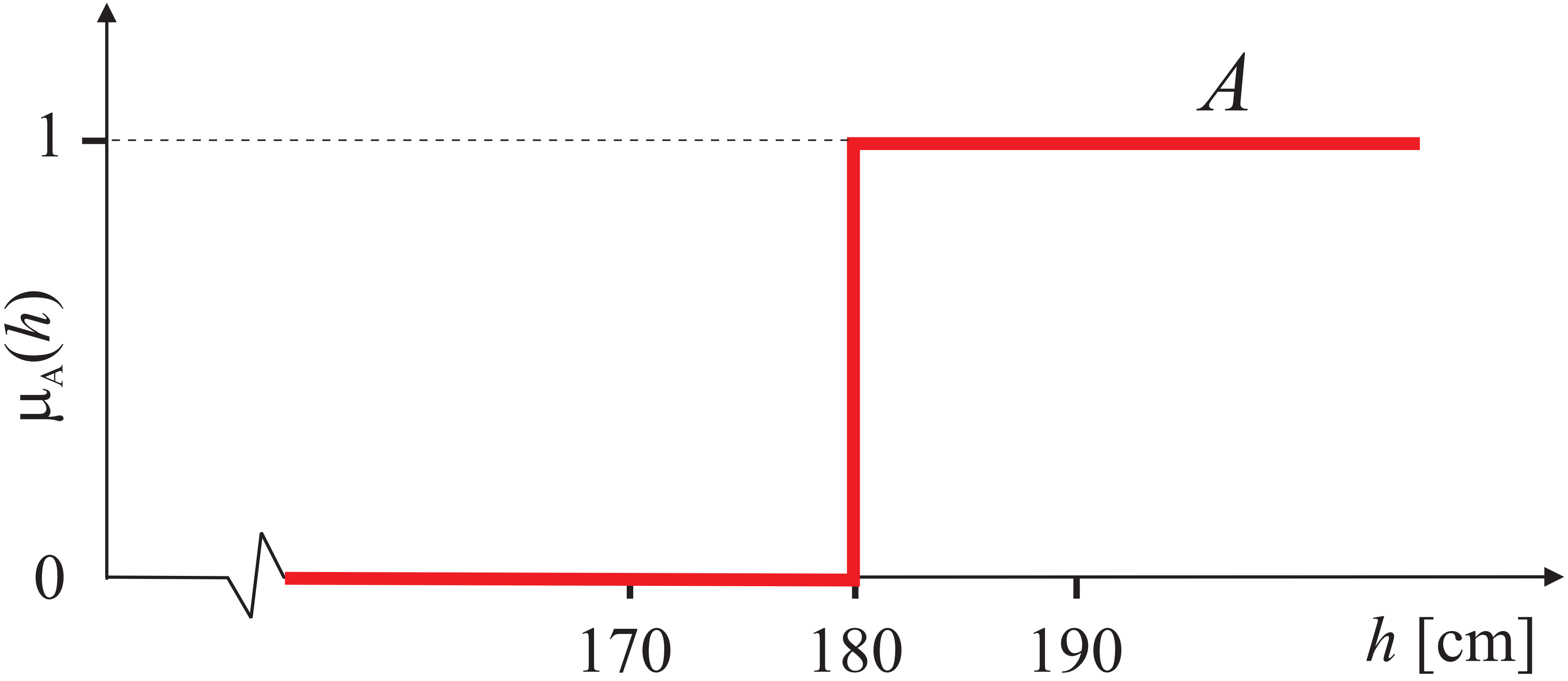
- Classical sets are good for well-defined concepts (maths, programs, etc.)

- Less suitable for representing commonsense knowledge in terms of vague concepts such as:

  - a **tall** person, **slippery** road, **nice** weather, . . .
  - want to buy a **big** car with **moderate** consumption
  - If the temperature is **too low**, increase heating **a lot**

Classical Set Approach

set of tall people $A = \{h \mid h \geq 180\}$

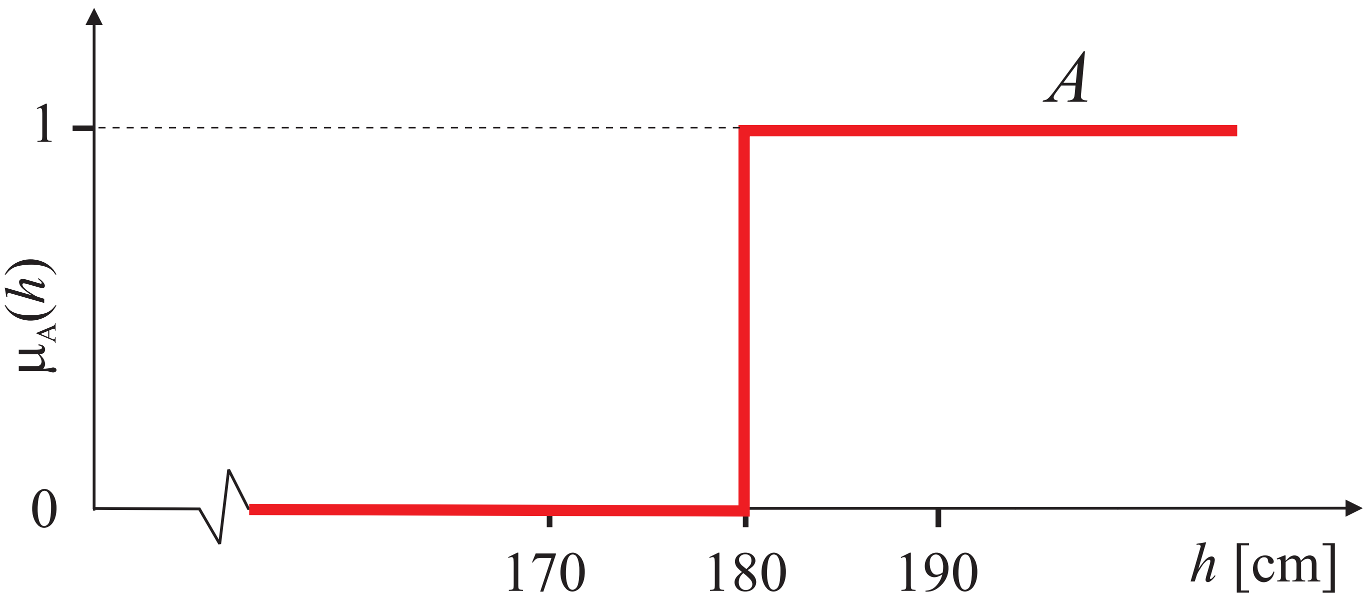


Logical Propositions

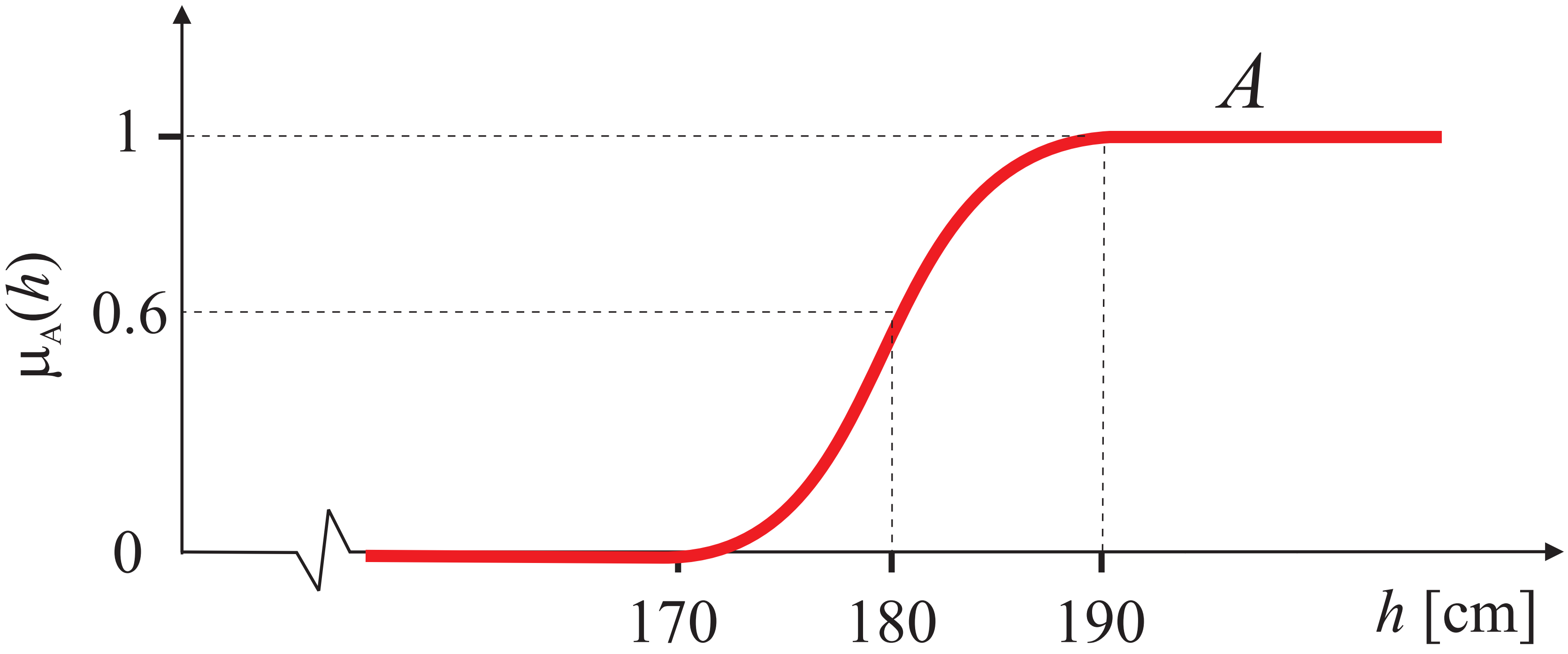
"John is tall" . . . true or false

John's height: $h_{John} = 180.0$ $\mu_A(180.0) = 1$ **(true)**

$h_{John} = 179.5$ $\mu_A(179.5) = 0$ **(false)**



Fuzzy Set Approach



$$\mu_A(h) = \begin{cases} 1 & h \textbf{ is full member of } A \quad (h \geq 190) \\ (0,1) & h \textbf{ is partial member of } A \quad (170 < h < 190) \\ 0 & h \textbf{ is not member of } A \quad (h \leq 170) \end{cases}$$

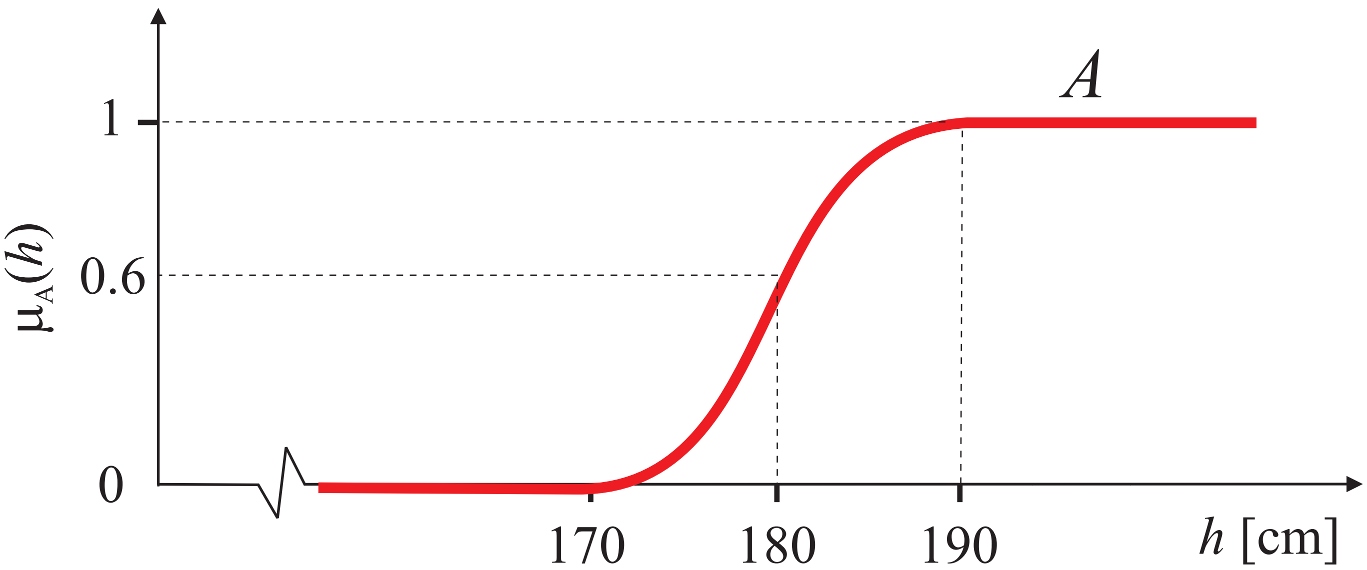
Fuzzy Logic Propositions

"John is tall" ...**degree of truth**

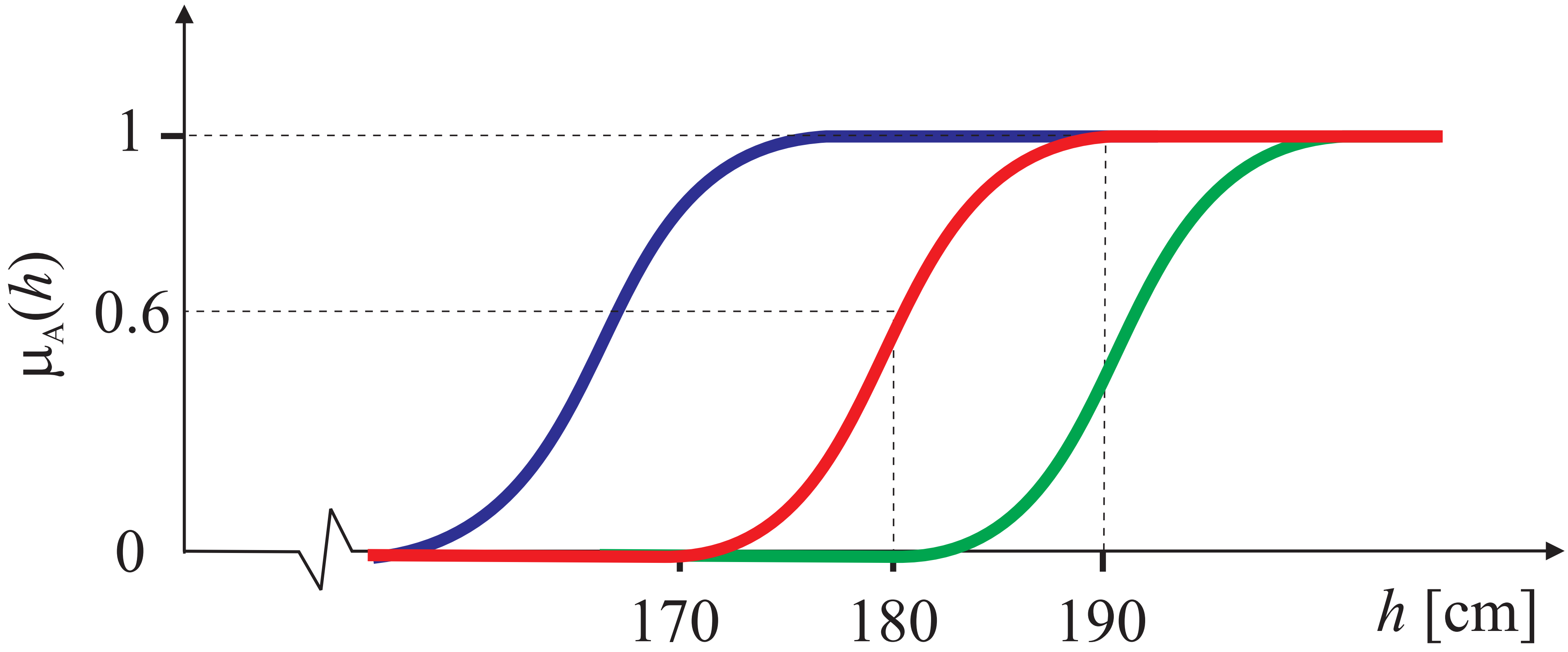
John's height: $h_{John} = 180.0$ $\mu_A(180.0) = 0.6$

$h_{John} = 179.5$ $\mu_A(179.5) = 0.56$

$h_{Paul} = 201.0$ $\mu_A(201.0) = 1$

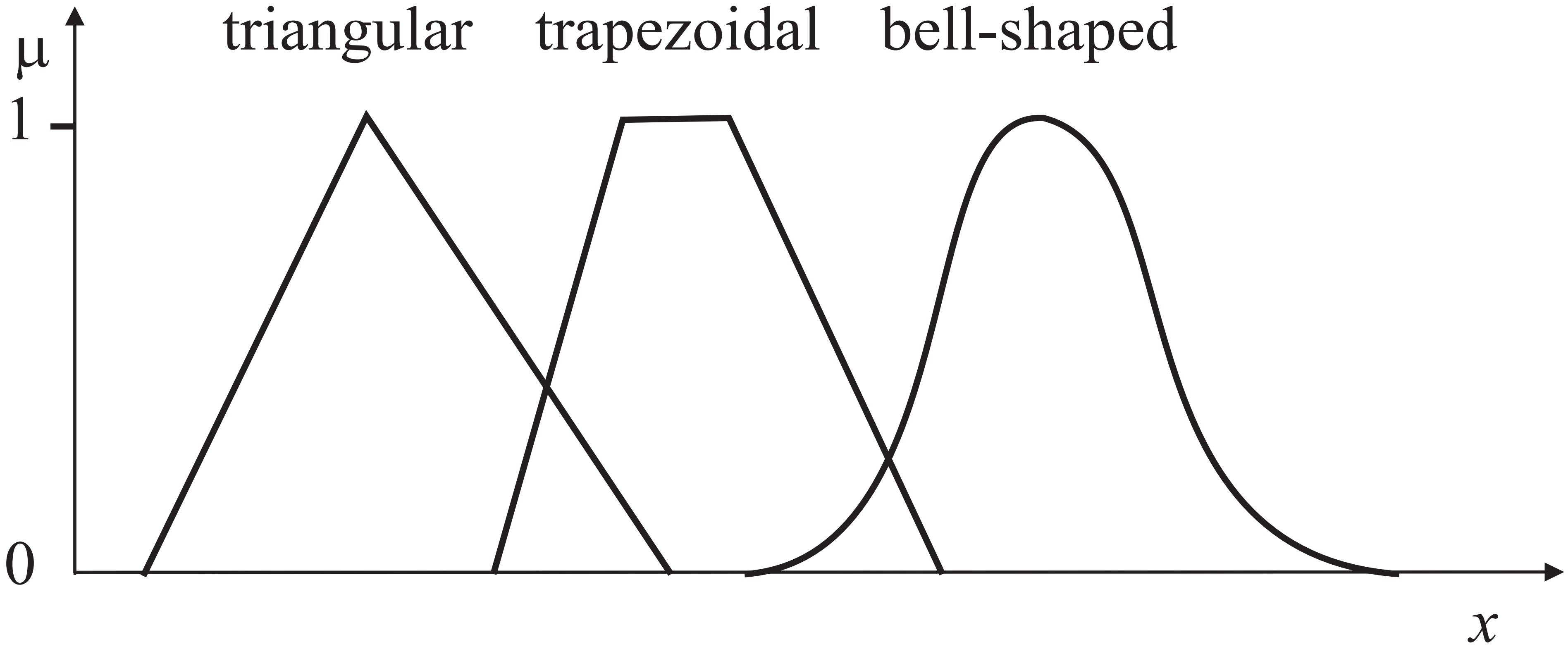


Subjective and Context Dependent



tall in China **tall in Europe** **tall in NBA**

Shapes of Membership Functions



Representation of Fuzzy Sets

- **Pointwise as a list of membership/element pairs:**

$$A = \{\mu_A(x_1)/x_1, \ldots, \mu_A(x_n)/x_n\} = \{\mu_A(x_i)/x_i \mid x_i \in X\}$$

- **As a list of α-level/α-cut pairs:**

$$A = \{\alpha_1/A_{\alpha_1},\ \alpha_2/A_{\alpha_2},\ \ldots, \alpha_n, A_{\alpha_n}\} = \{\alpha_i/A_{\alpha_i} \mid \alpha_i \in (0,1)\}$$

Representation of Fuzzy Sets

- **Analytical formula for the membership function:**

$$\mu_A(x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

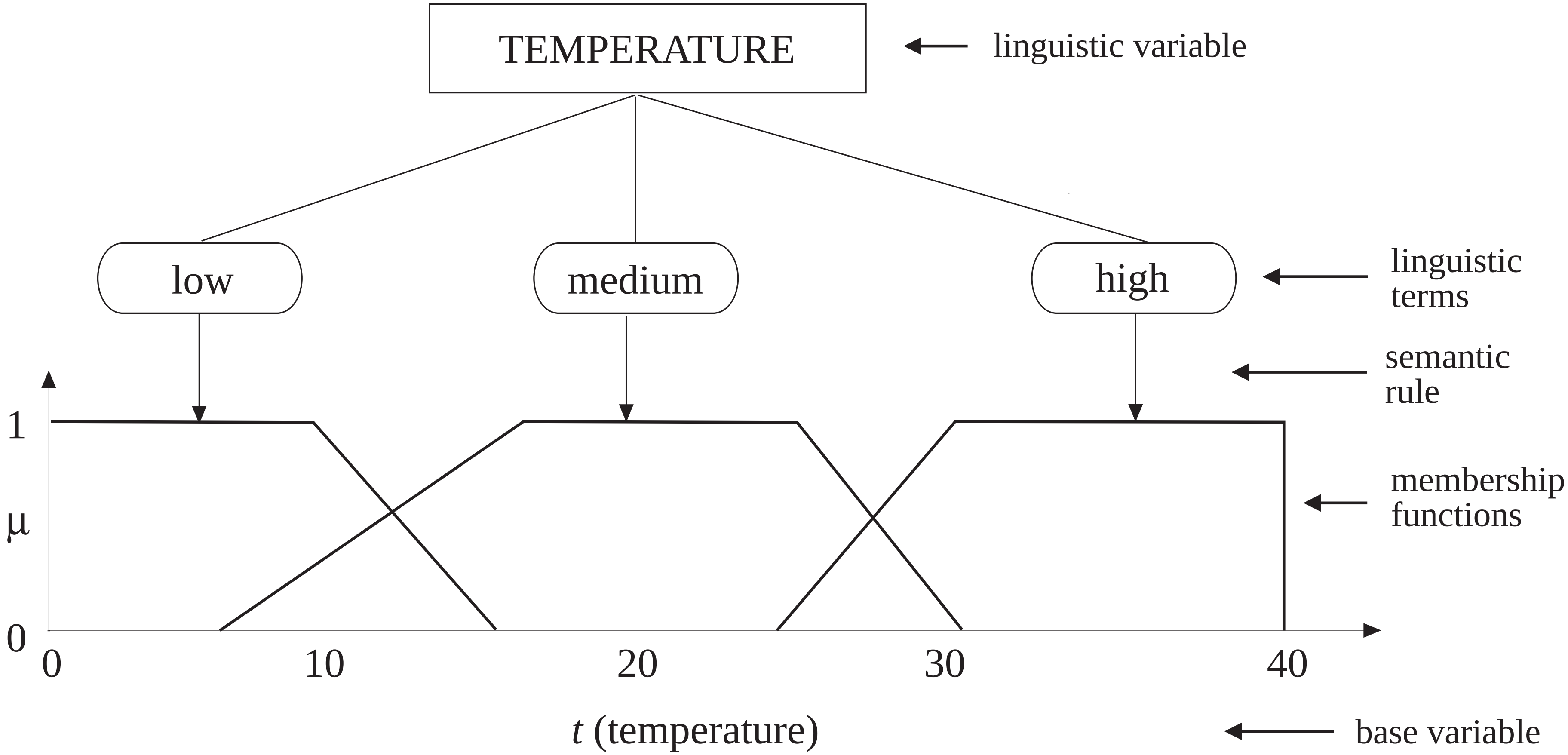
or more generally

$$\mu(x) = \frac{1}{1+d(x,v)}\,.$$

$d(x,v)$ **. . . dissimilarity measure**

Various shorthand notations: $\mu_A(x) \ldots A(x) \ldots a$

Linguistic Variable

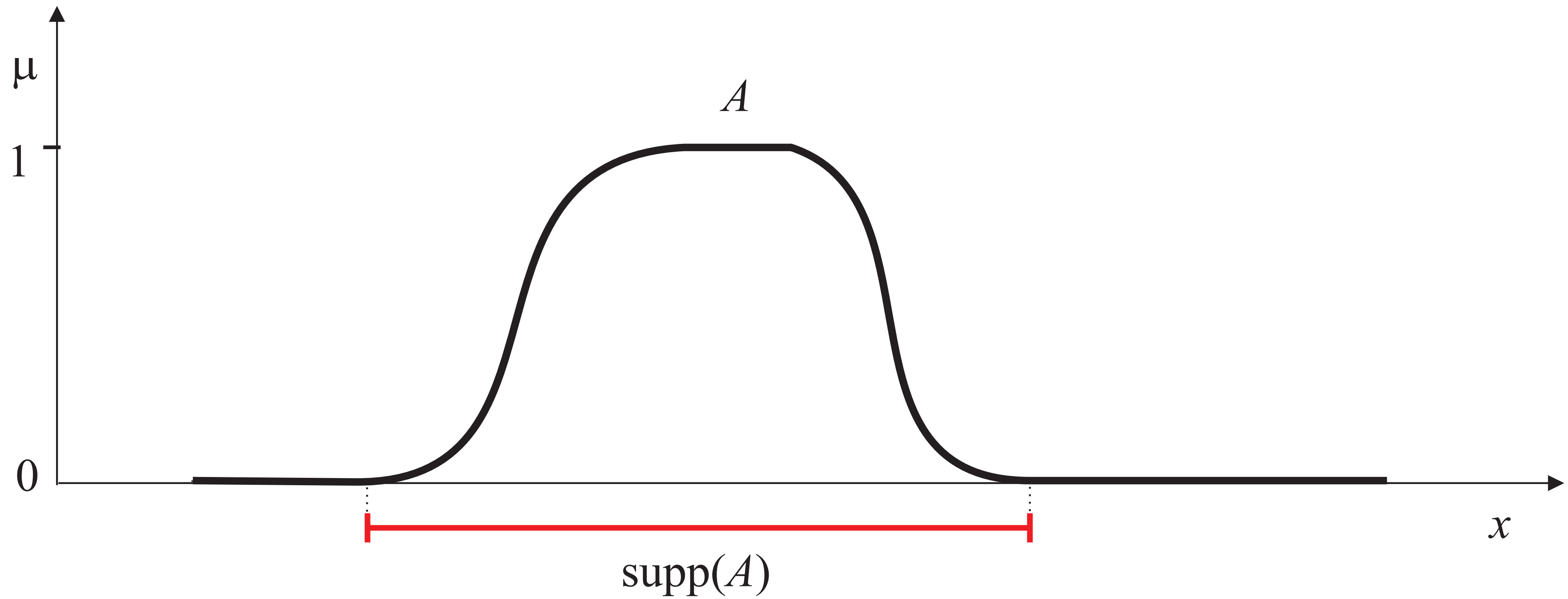


Basic requirements: coverage and semantic soundness

Properties of fuzzy sets

Support of a Fuzzy Set

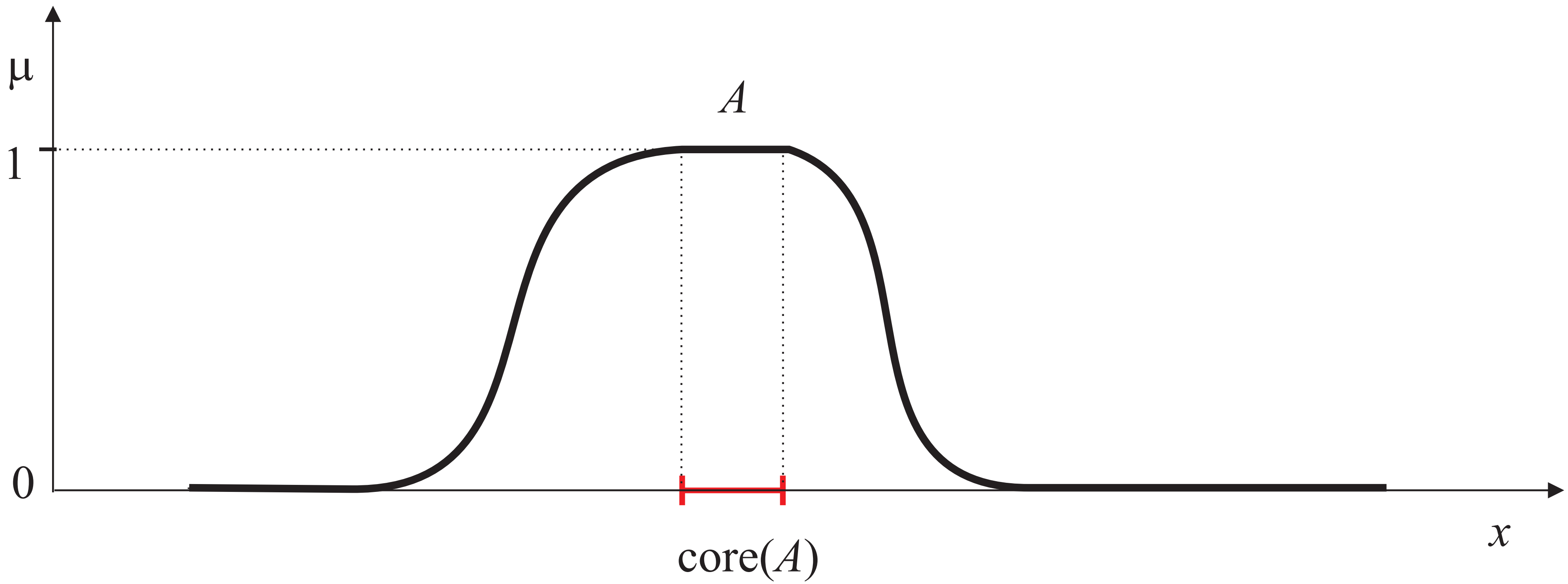
$$\mathrm{supp}(A) = \{x \mid \mu_A(x) > 0\}$$



support is an *ordinary set*

Core (Kernel) of a Fuzzy Set

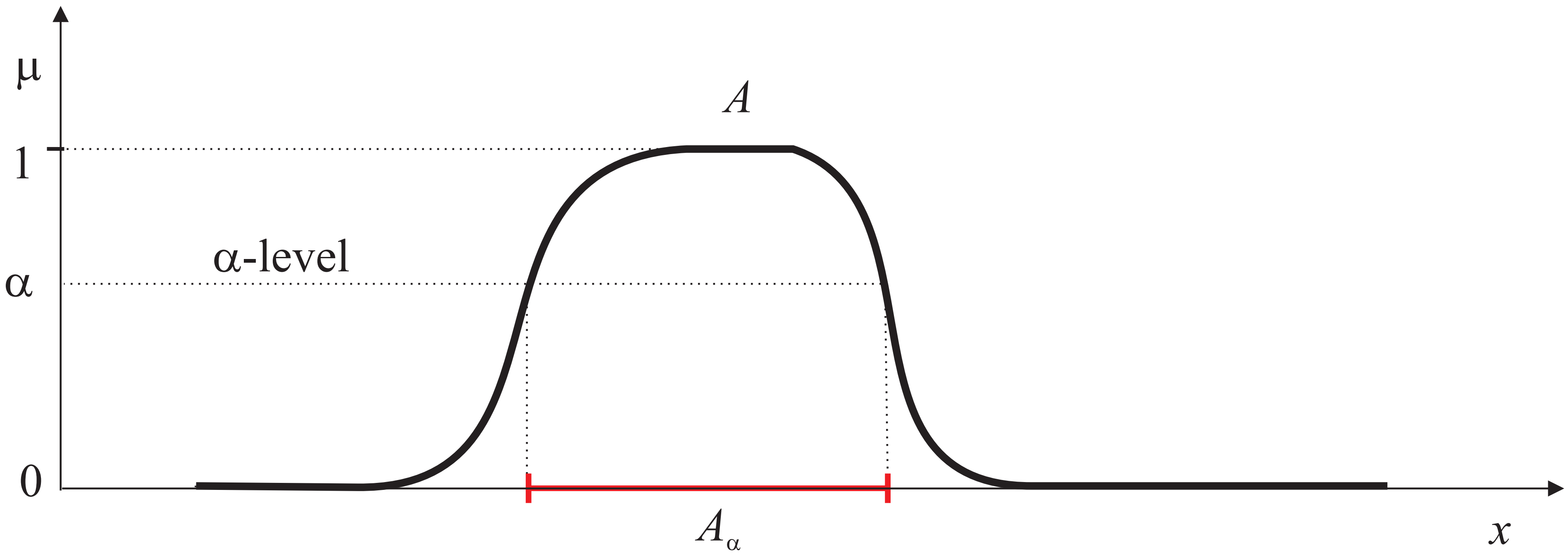
$$\text{core}(A) = \{x \mid \mu_A(x) = 1\}$$



core is an *ordinary set*

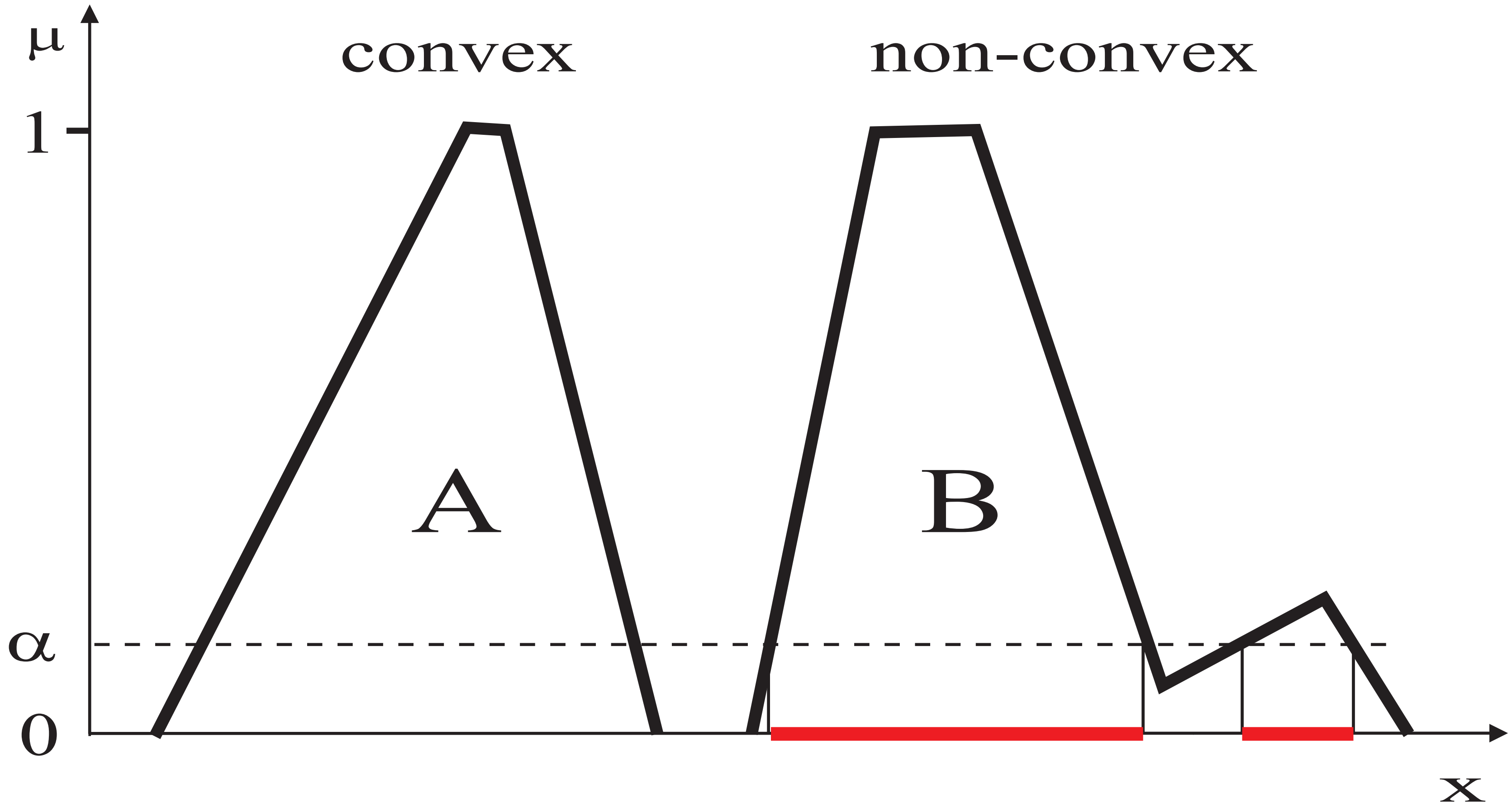
α-cut of a Fuzzy Set

$$A_\alpha = \{x \mid \mu_A(x) > \alpha\} \qquad \textbf{or} \qquad A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}$$



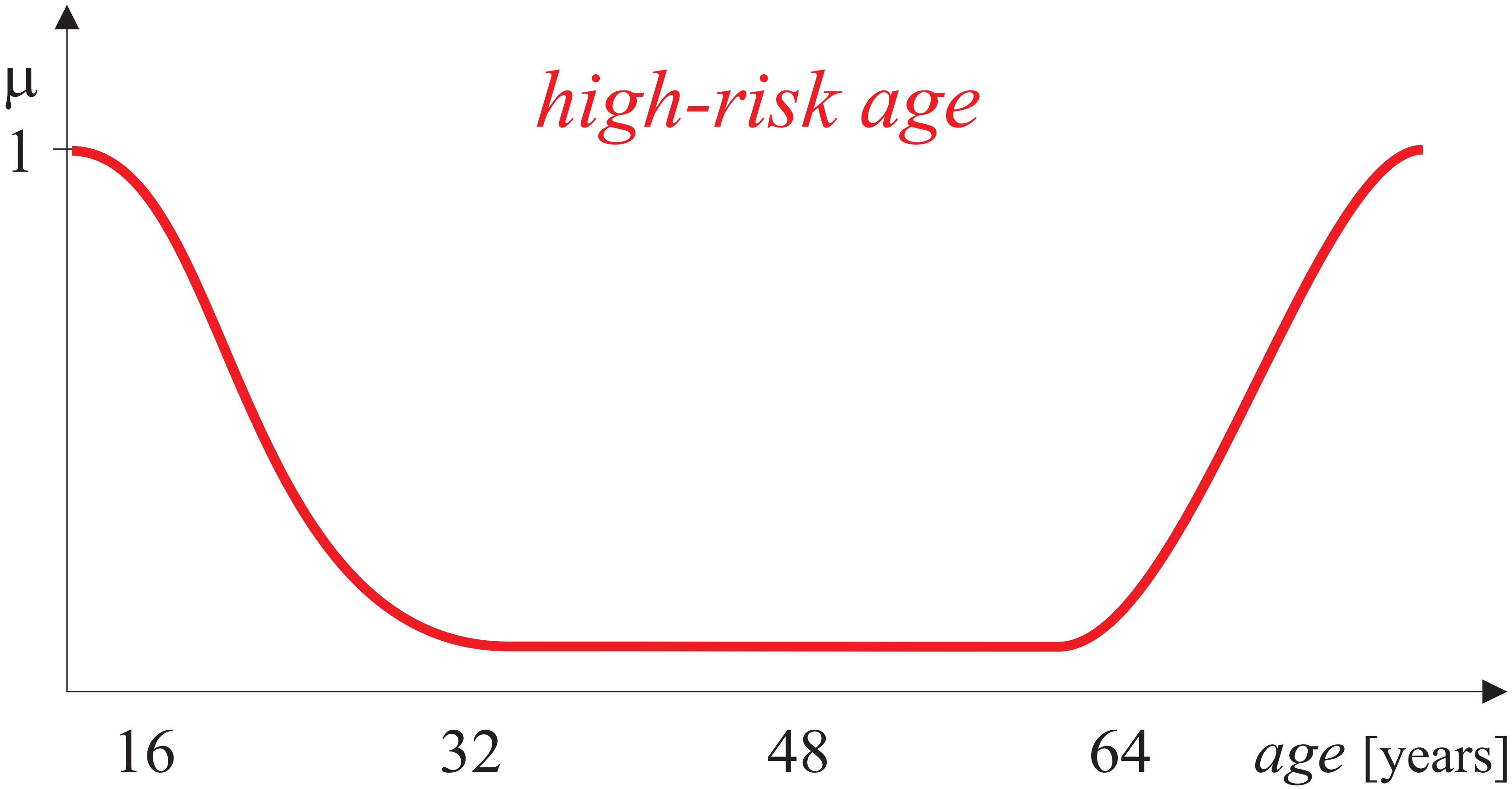
A_α **is an *ordinary set***

Convex and Non-Convex Fuzzy Sets



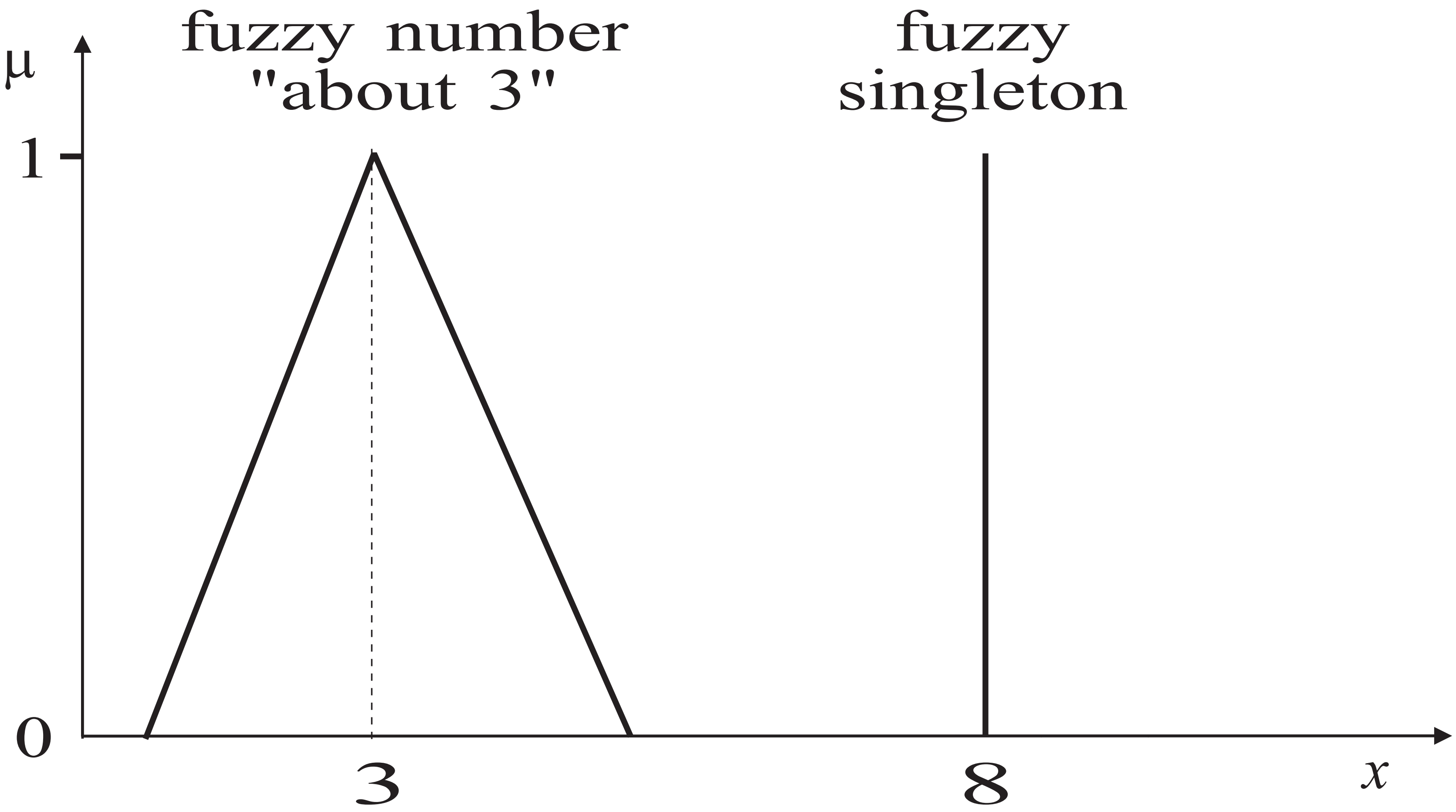
A fuzzy set is convex $\Leftrightarrow$ all its α-cuts are convex sets.

Non-Convex Fuzzy Set: an Example



High-risk age for car insurance policy.

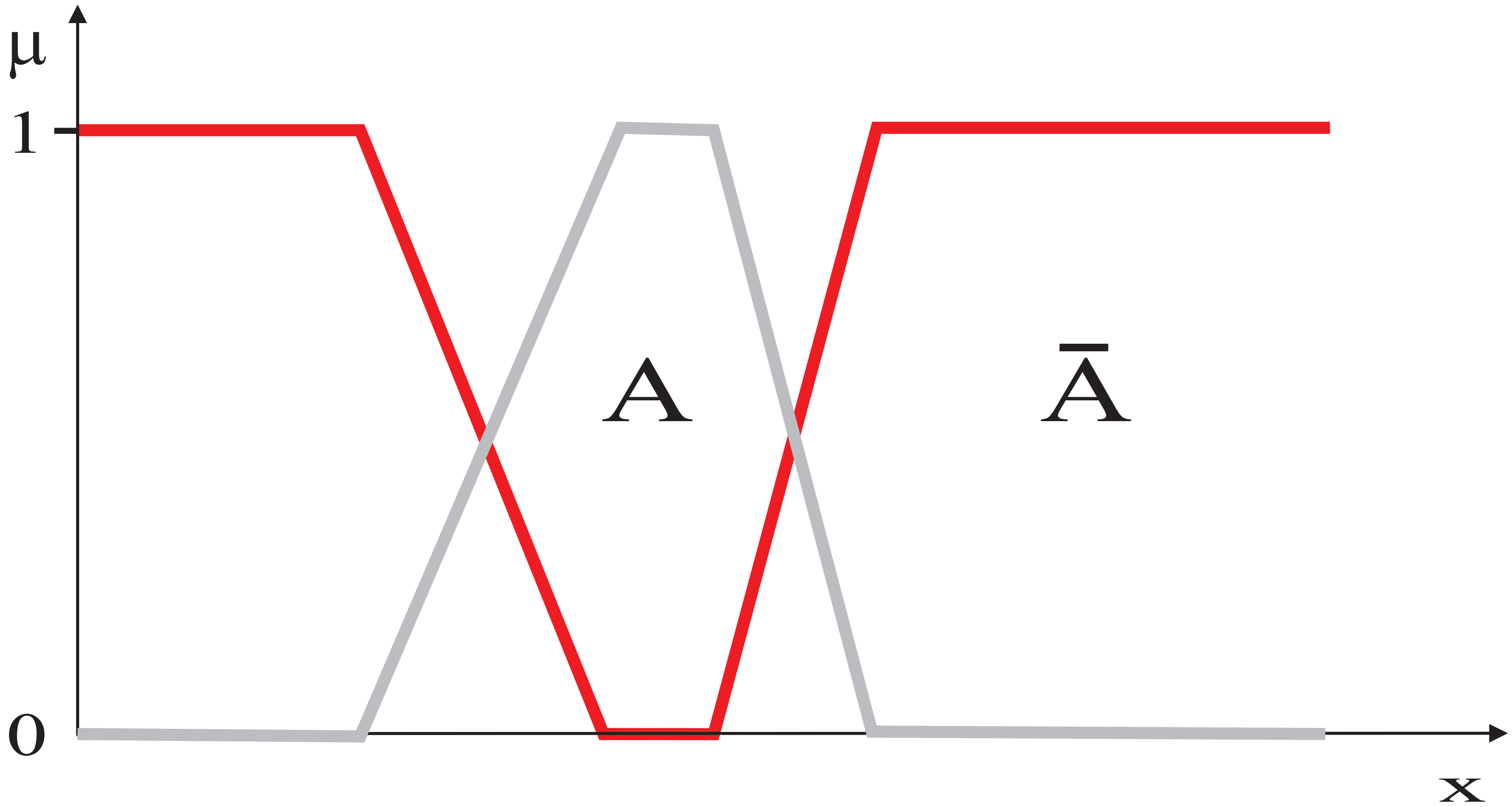
Fuzzy Numbers and Singletons



Fuzzy linear regression: $y = \tilde{3}x_1 + \tilde{5}x_2$

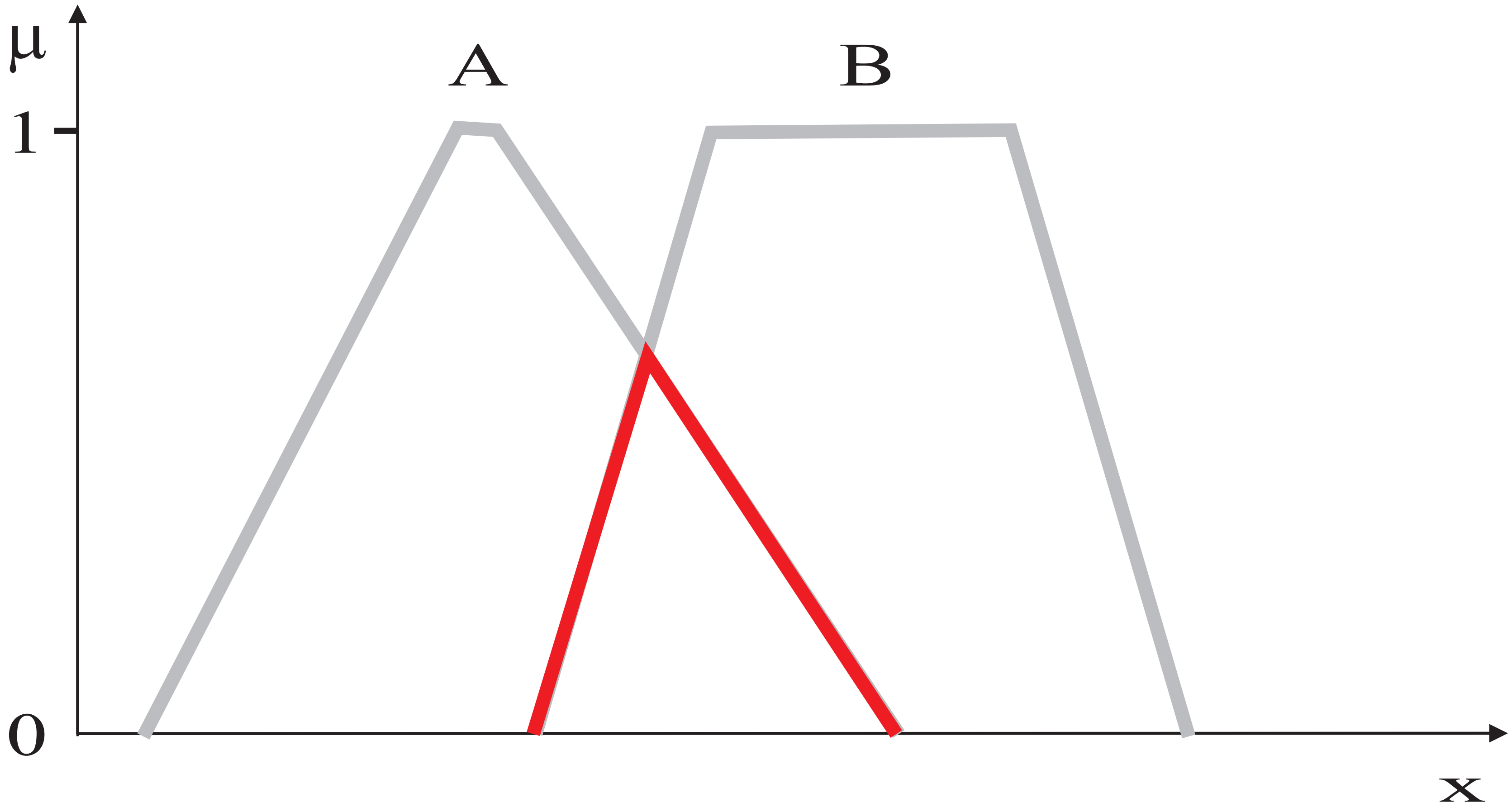
Fuzzy set-theoretic operations

Complement (Negation) of a Fuzzy Set



$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

Intersection (Conjunction) of Fuzzy Sets



$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

Other Intersection Operators (T-norms)

Probabilistic “and” (product operator):

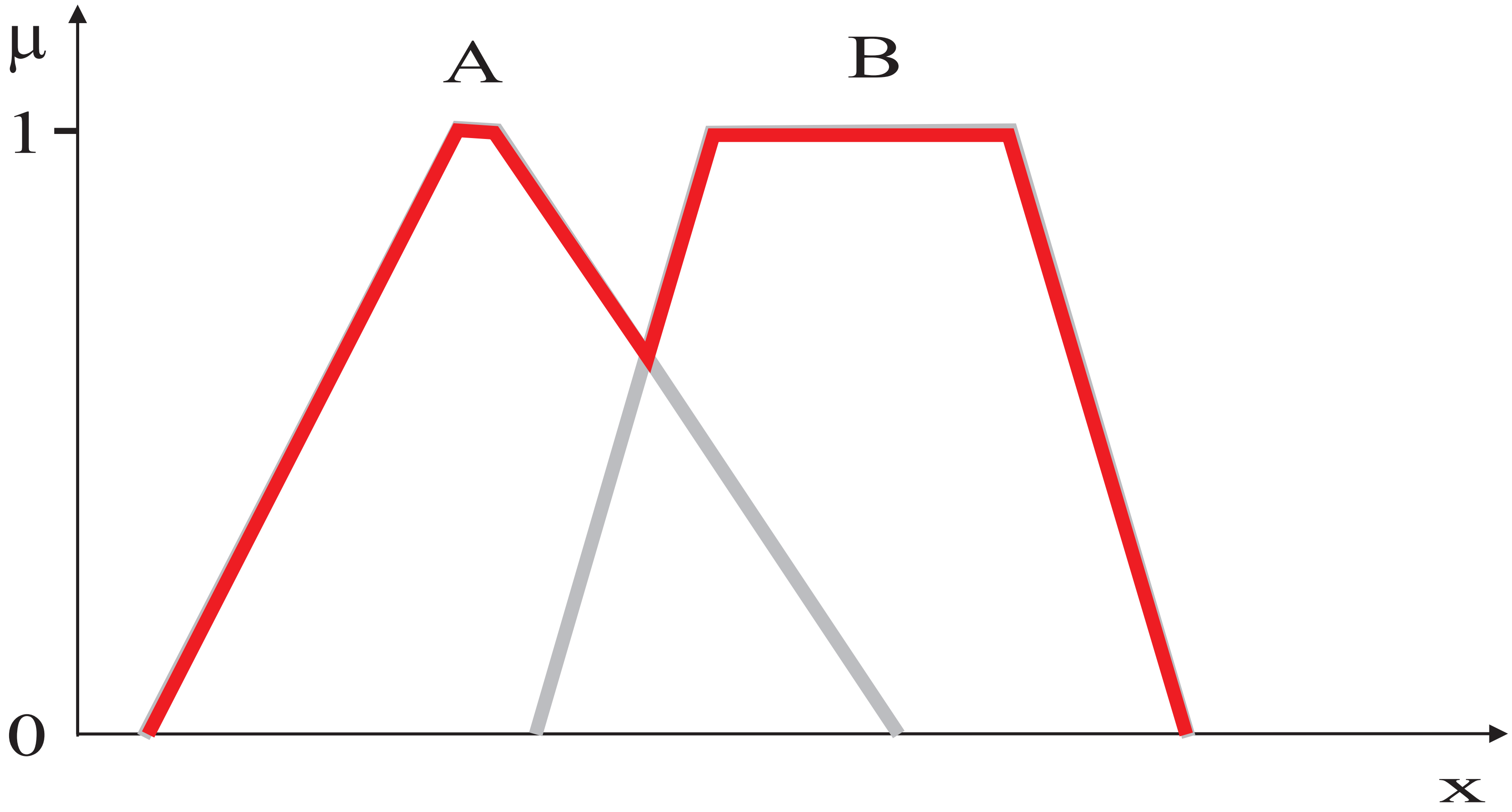
$$\mu_{A\cap B}(x) = \mu_A(x) \cdot \mu_B(x)$$

Łukasiewicz “and” (bounded difference):

$$\mu_{A\cap B}(x) = \max(0, \mu_A(x) + \mu_B(x) - 1)$$

Many other t-norms . . . $[0,1] \times [0,1] \rightarrow [0,1]$

Union (Disjunction) of Fuzzy Sets



$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

Other Union Operators (T-conorms)

Probabilistic "or":

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

Łukasiewicz "or" (bounded sum):

$$\mu_{A\cup B}(x) = \min(1, \mu_A(x) + \mu_B(x))$$

Many other t-conorms . . . $[0,1] \times [0,1] \to [0,1]$

Demo of a Matlab tool

Linguistic Modifiers (Hedges)

Modify the meaning of a fuzzy set.

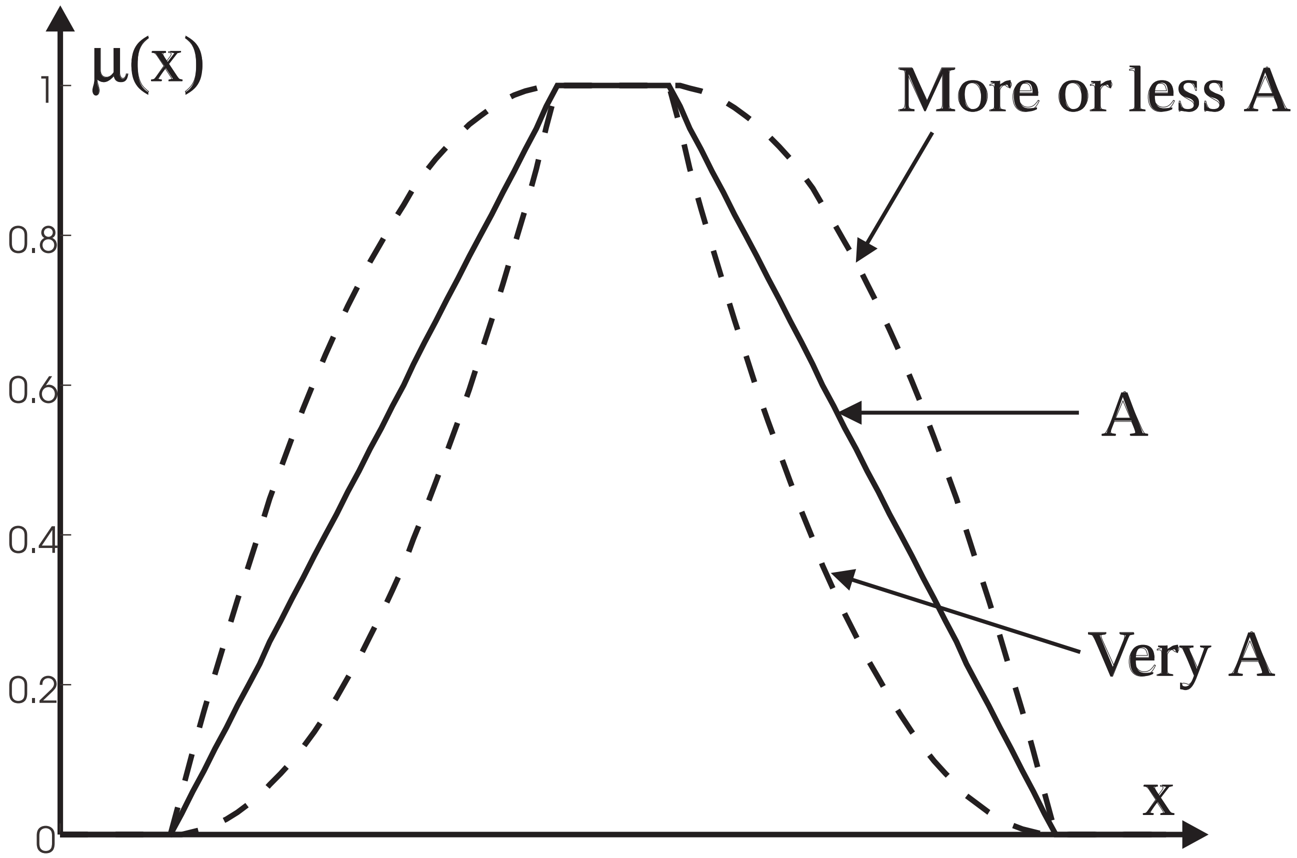
For instance, *very* can change the meaning of the fuzzy set *tall* to *very tall*.

Other common hedges: *slightly*, *more or less*, *rather*, etc.

Usual approach: *powered hedges*:

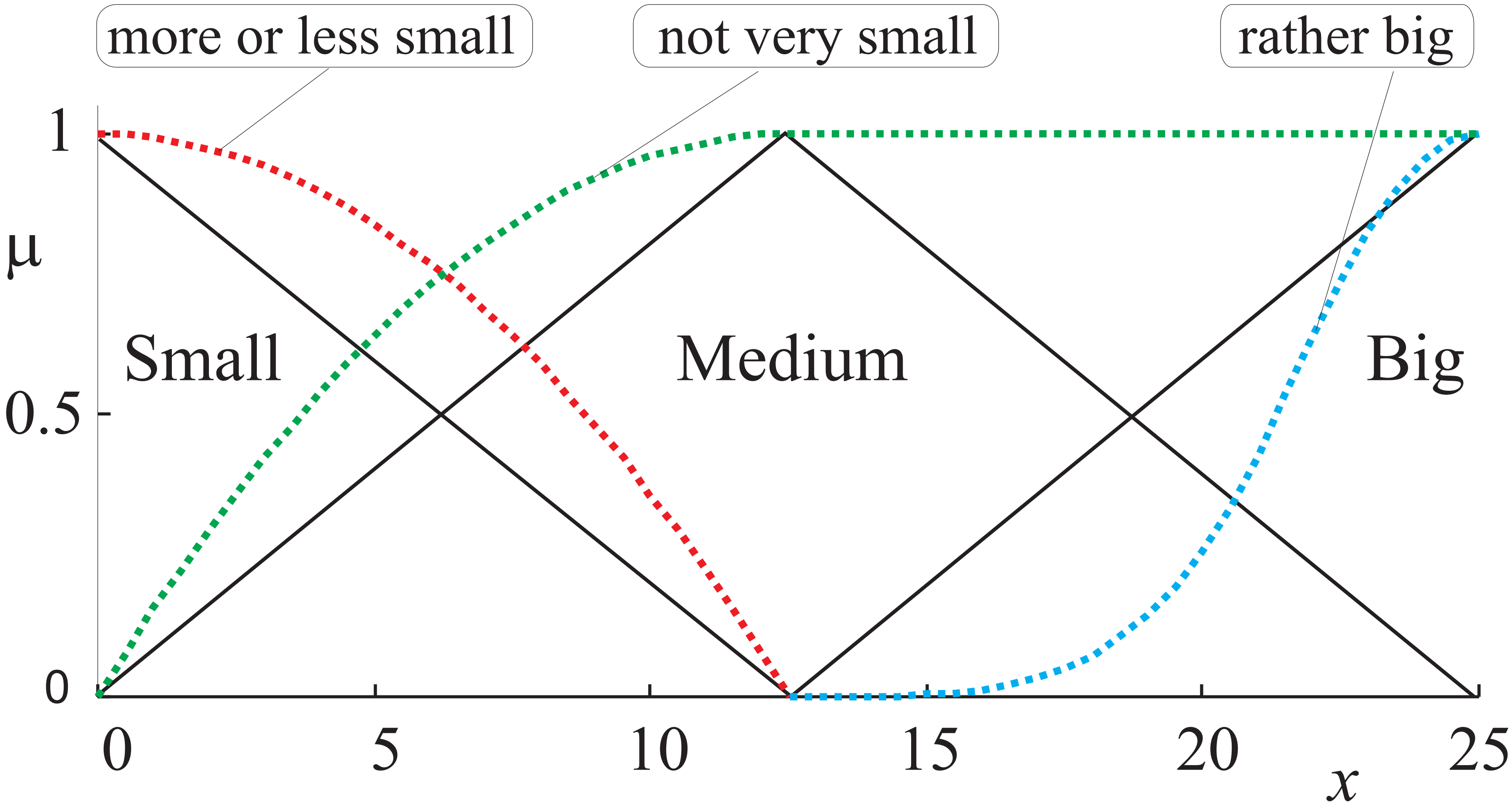
$$\mu_{M_p(A)} = \mu_A^P$$

Linguistic Modifiers: Example

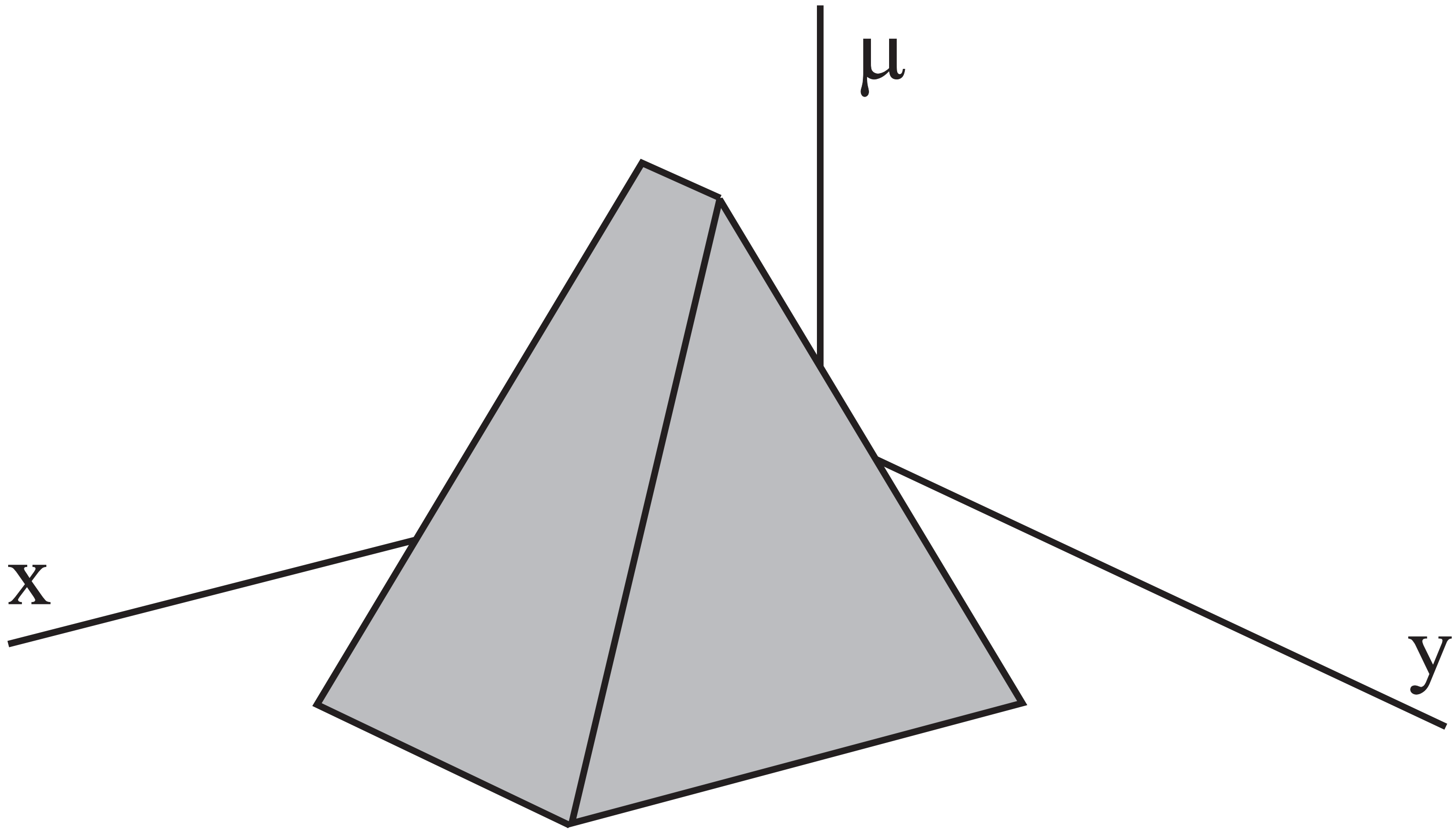


$$\mu_{\mathrm{very}(A)} = \mu_A^2 \qquad\qquad \mu_{\mathrm{More\ or\ less}(A)} = \sqrt{\mu_A}$$

Linguistic Modifiers

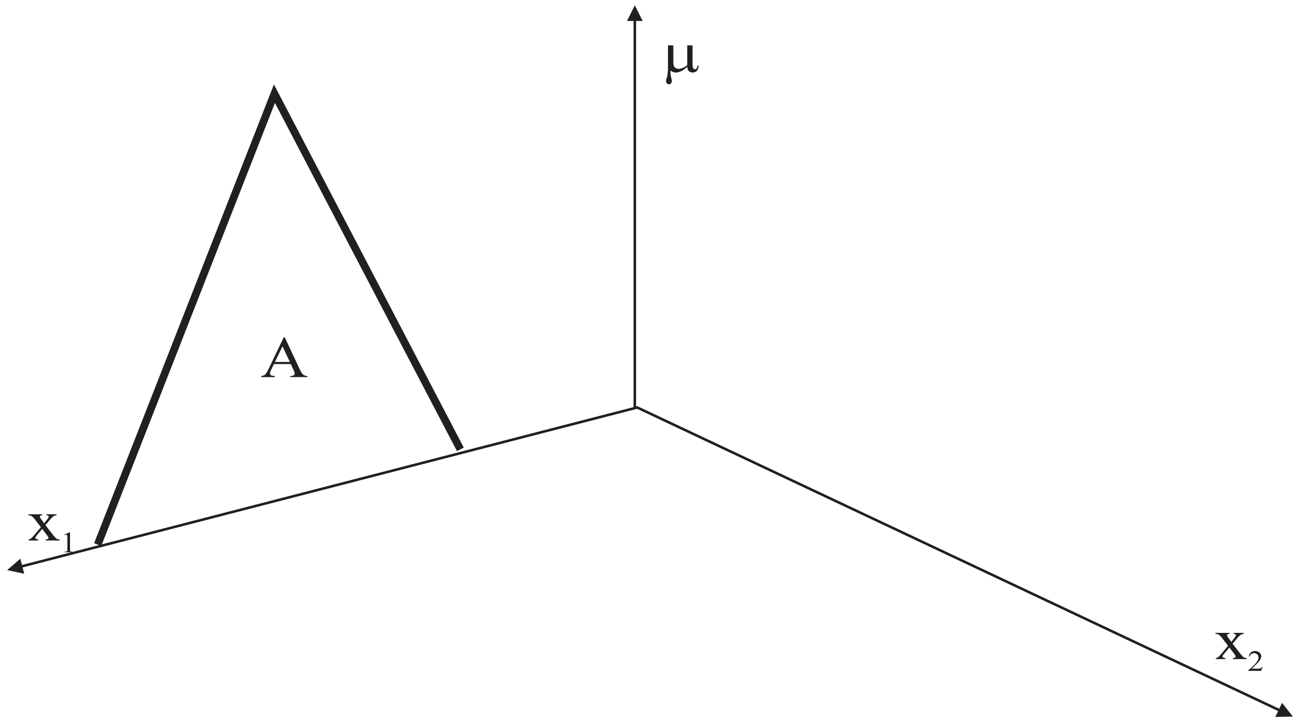


Fuzzy Set in Multidimensional Domains

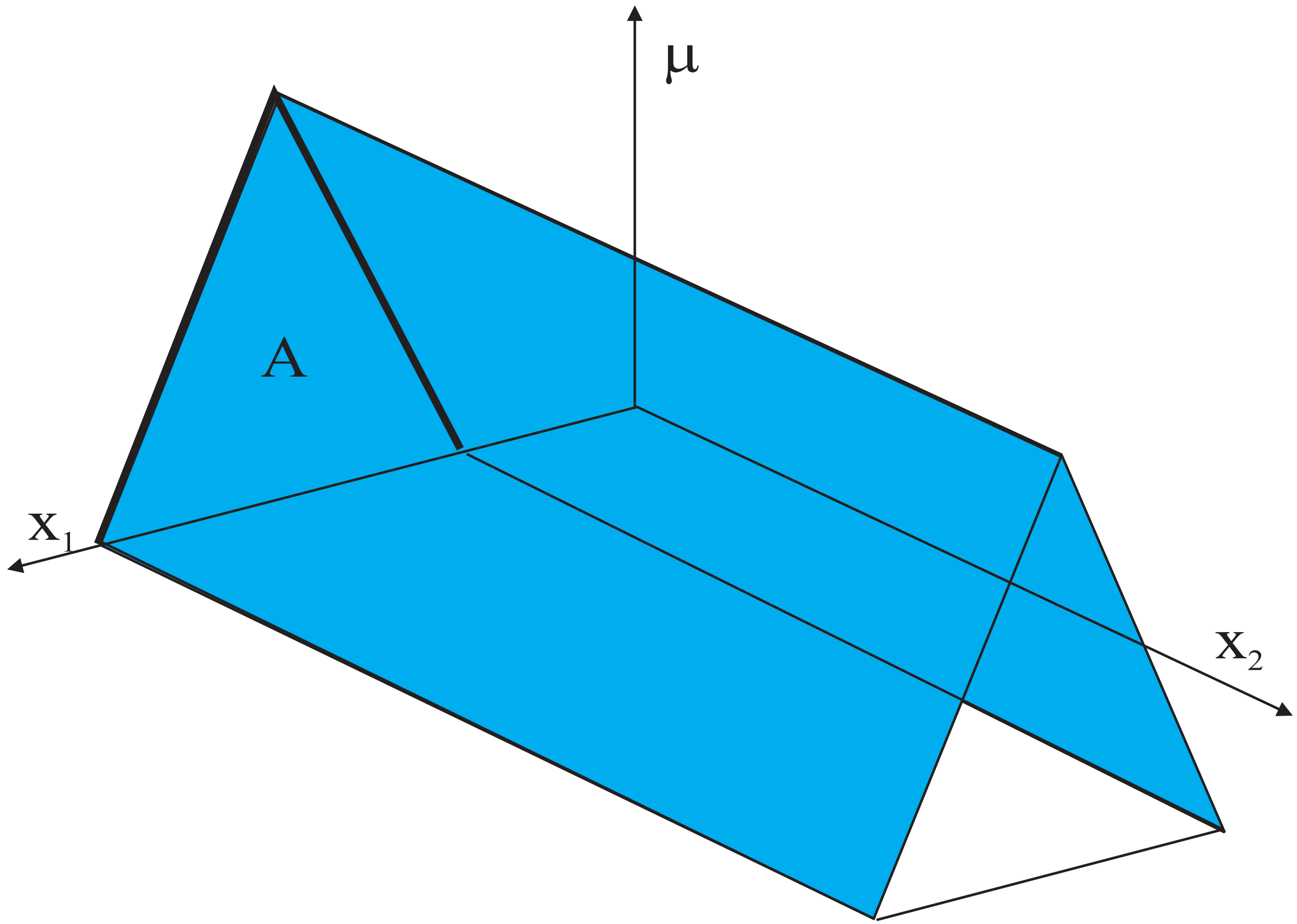


$$A = \{\mu_A(x,y)/(x,y) \mid (x,y) \in X \times Y\}$$

Cylindrical Extension



Cylindrical Extension



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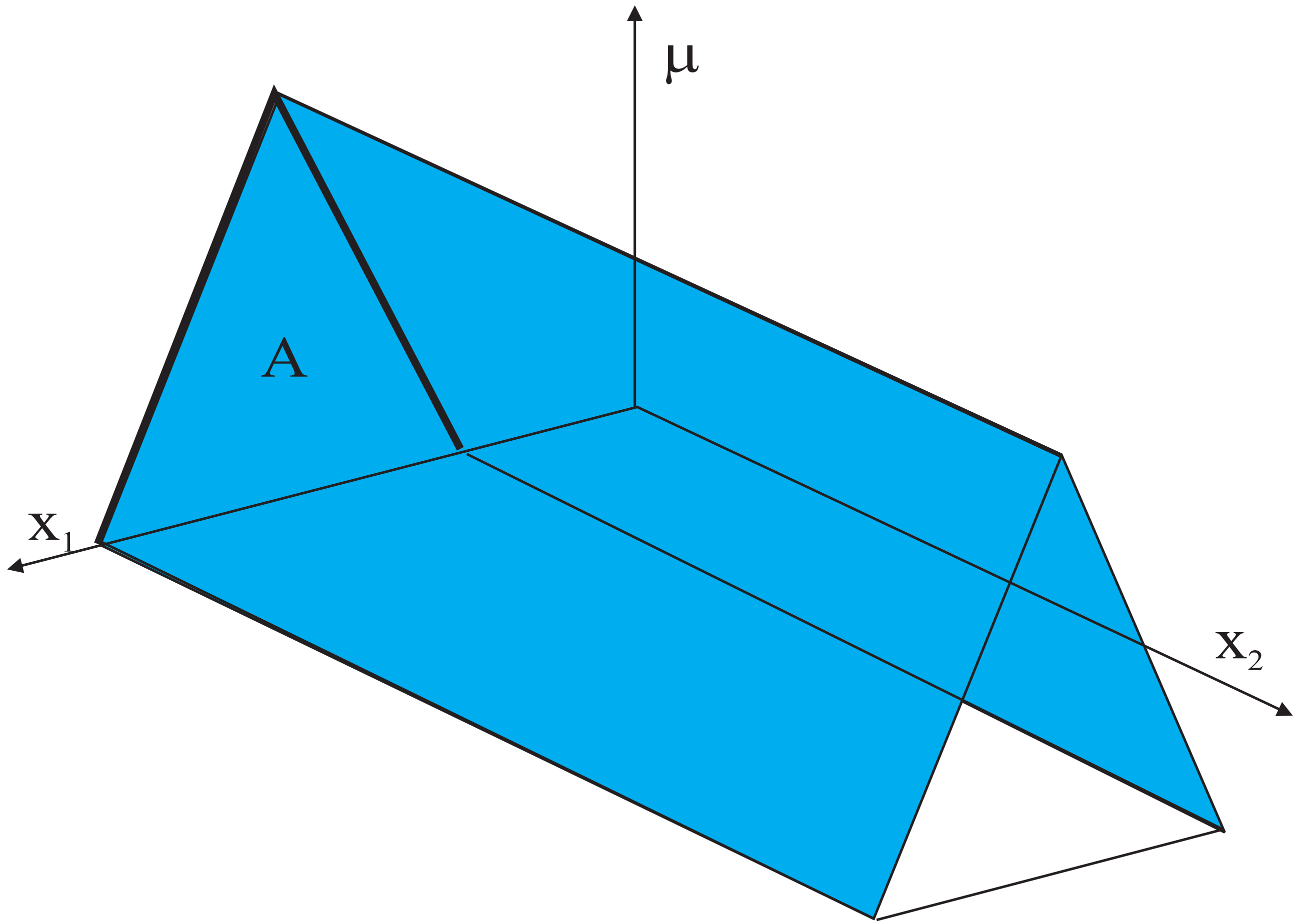
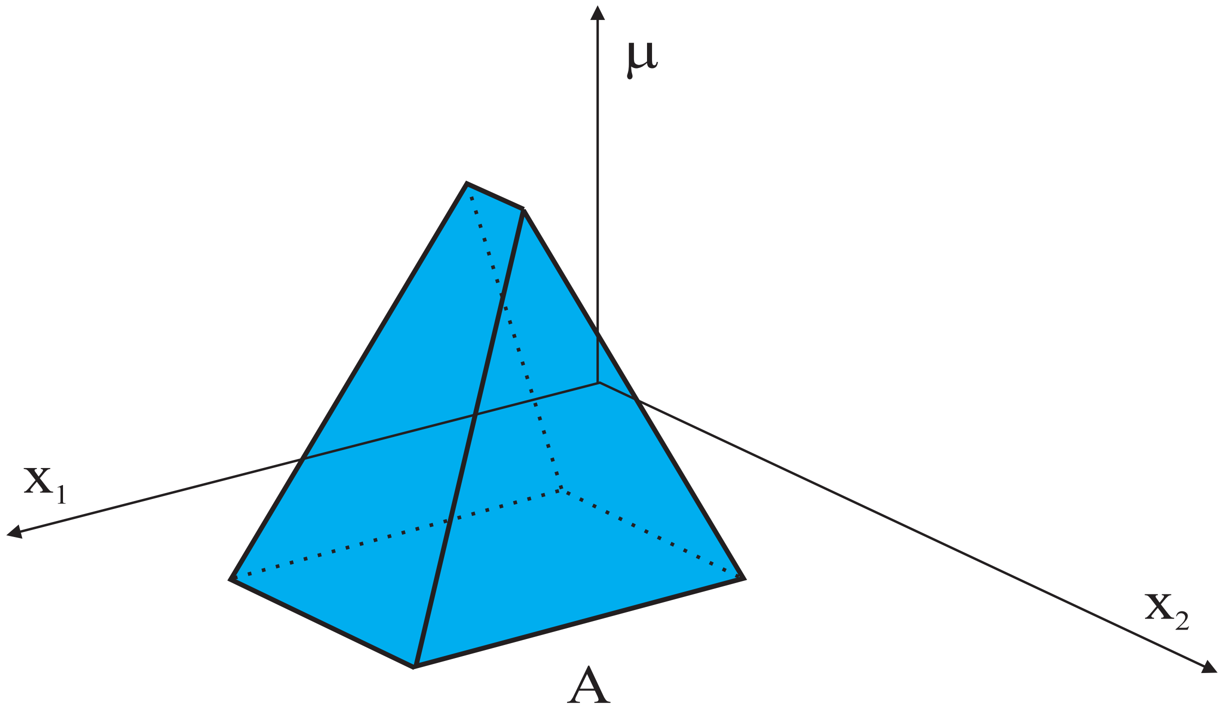
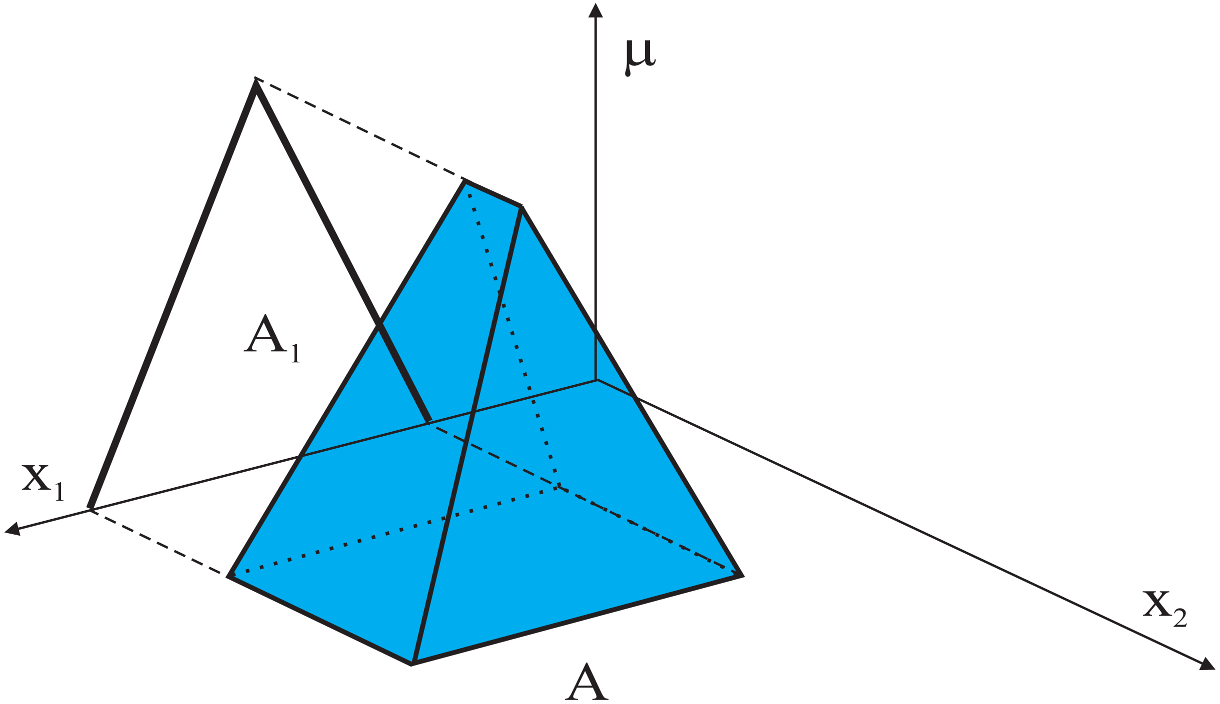


$$\mathrm{ext}_{x_2}(A) = \{\mu_A(x_1)/(x_1, x_2) \mid (x_1, x_2) \in X_1 \times X_2\}$$

Projection

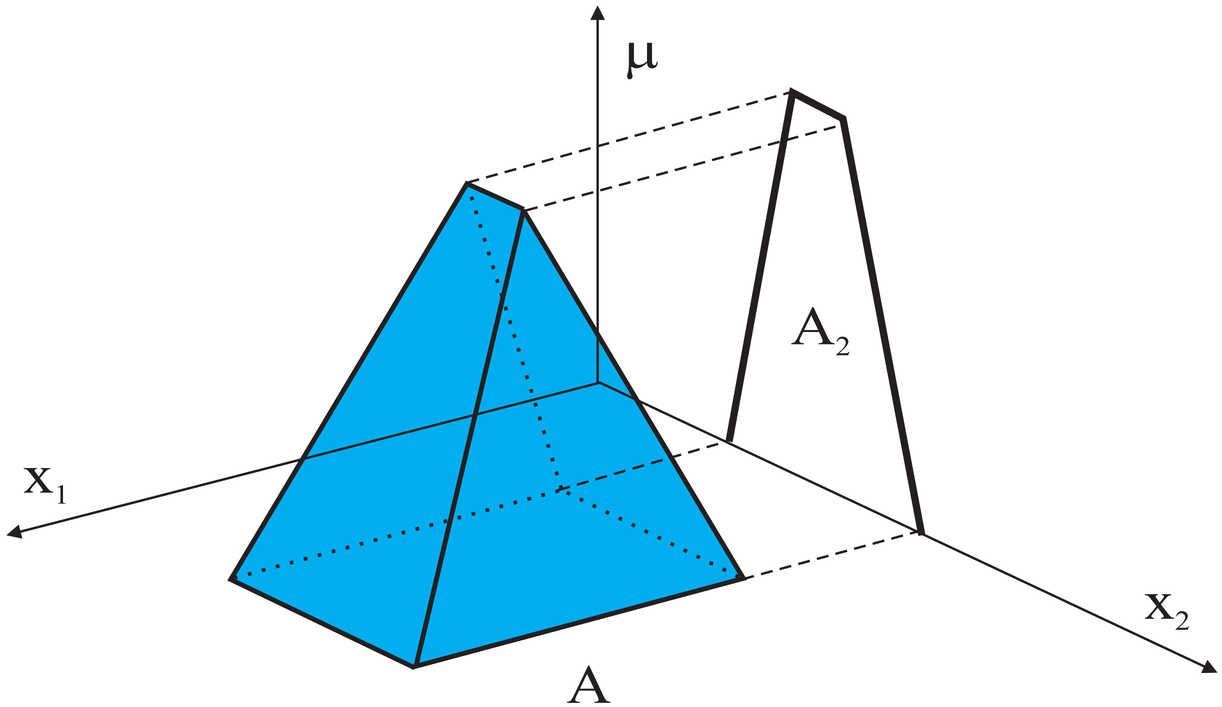


Projection onto x_1



$$\text{proj}_{x_1}(A) = \{ \sup_{x_2 \in X_2} \mu_A(x_1, x_2))/x_1 \mid x_1 \in X_1\}$$

Projection onto x_2

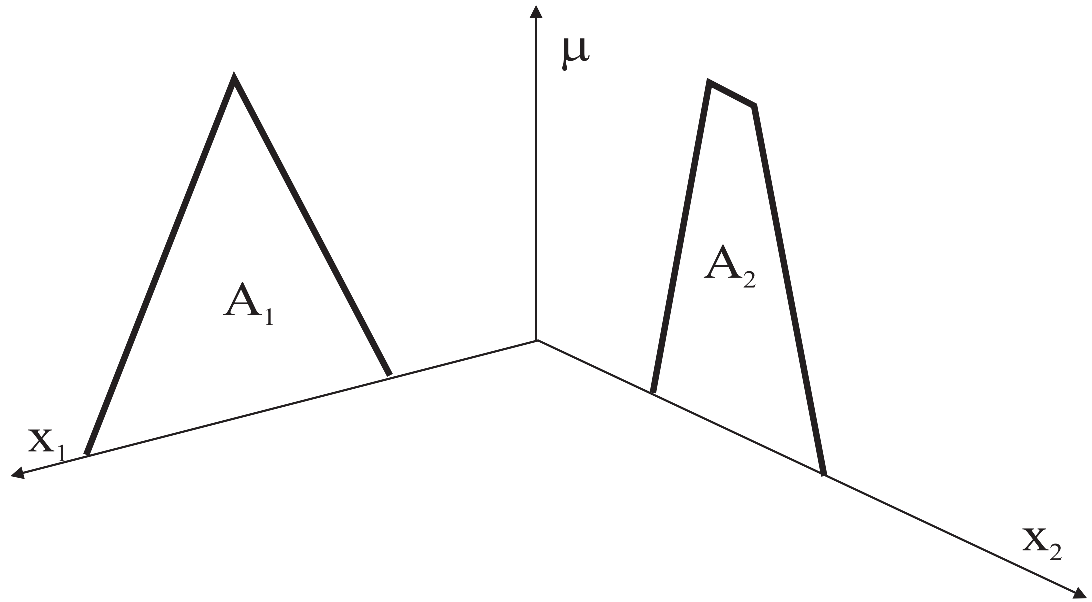


$$\text{proj}_{x_2}(A) = \{ \sup_{x_1 \in X_1} \mu_A(x_1, x_2)/x_2) \mid x_2 \in X_2 \}$$

Intersection on Cartesian Product Space

An operation between fuzzy sets are defined in different domains results in a multi-dimensional fuzzy set.

Example: $A_1 \cap A_2$ **on** $X_1 \times X_2$**:**



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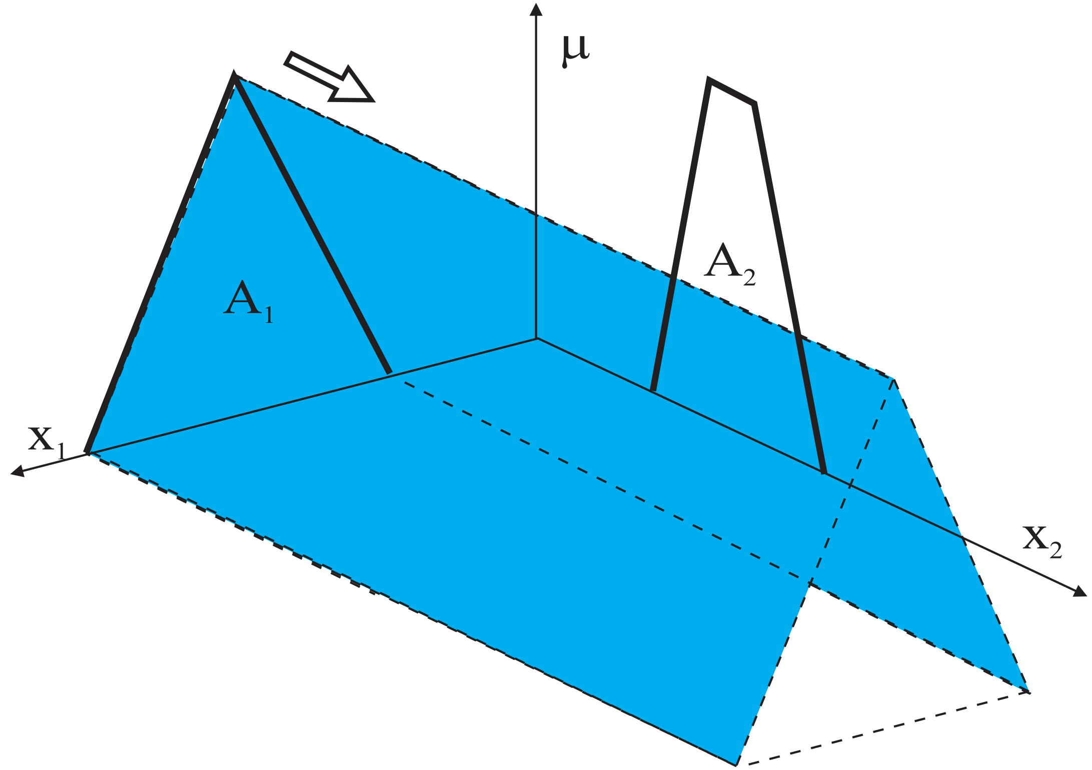

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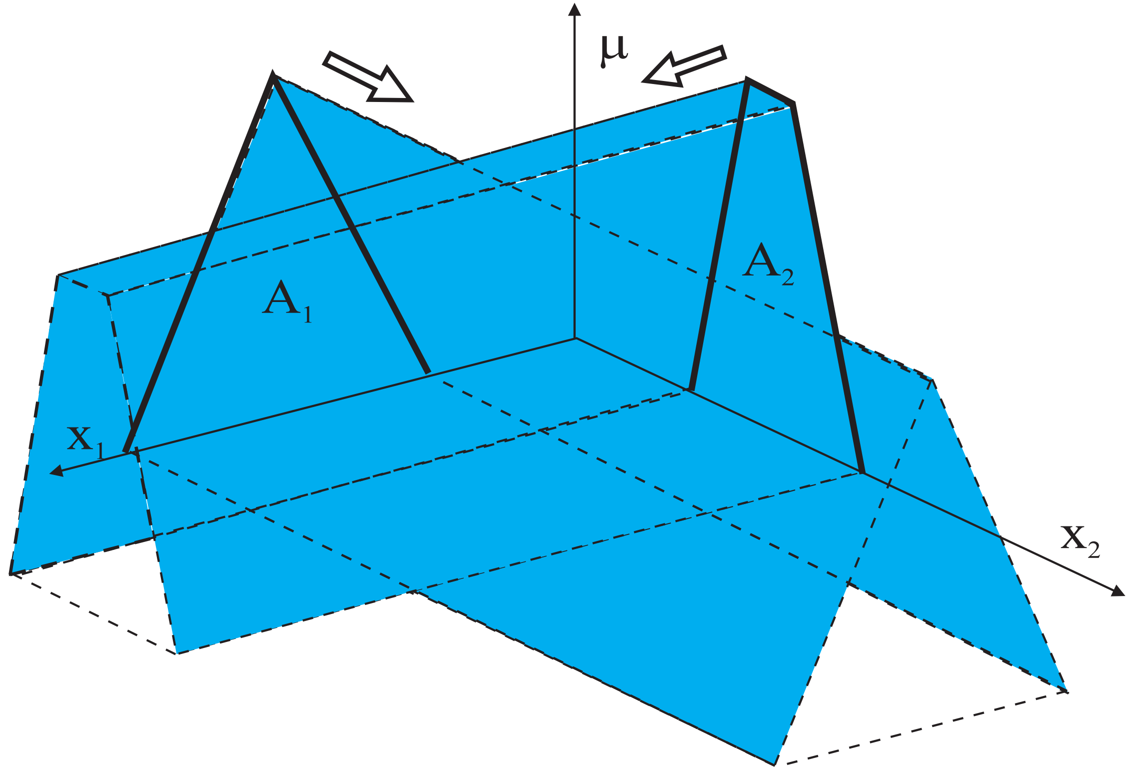

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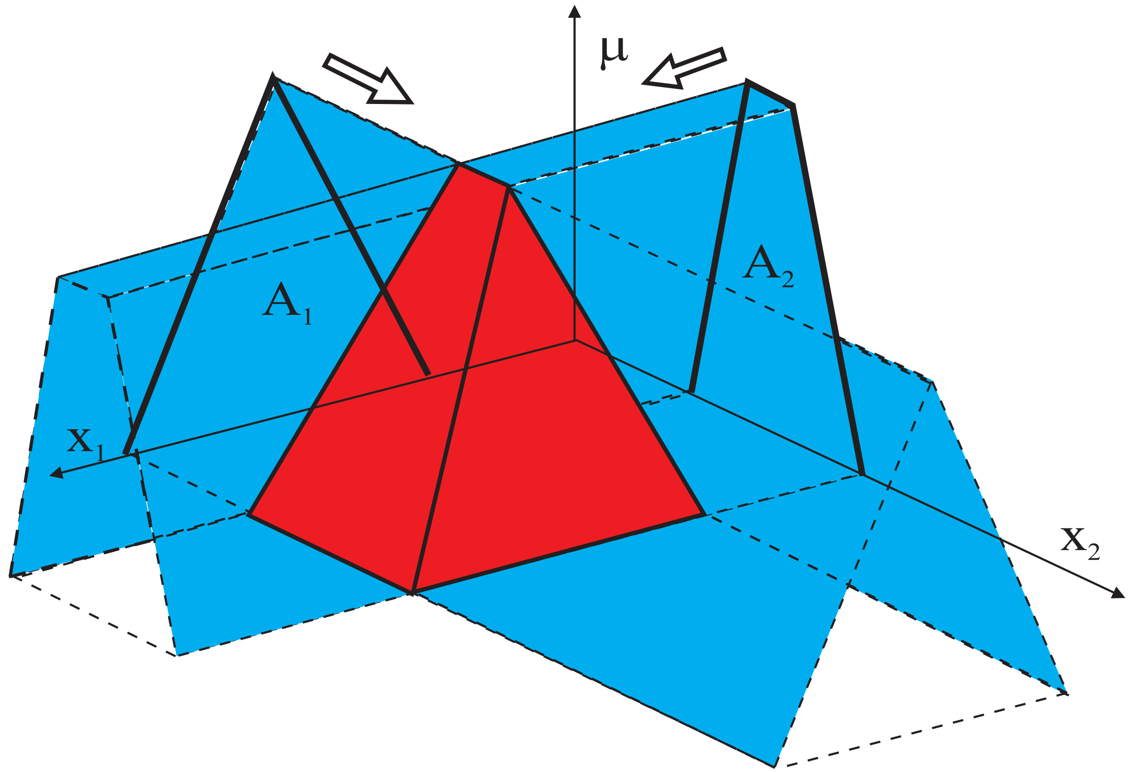


Fuzzy Relations

Classical relation represents the presence or absence of interaction between the elements of two or more sets.

With fuzzy relations, the degree of association (correlation) is represented by membership grades.

An n-dimensional fuzzy relation is a mapping

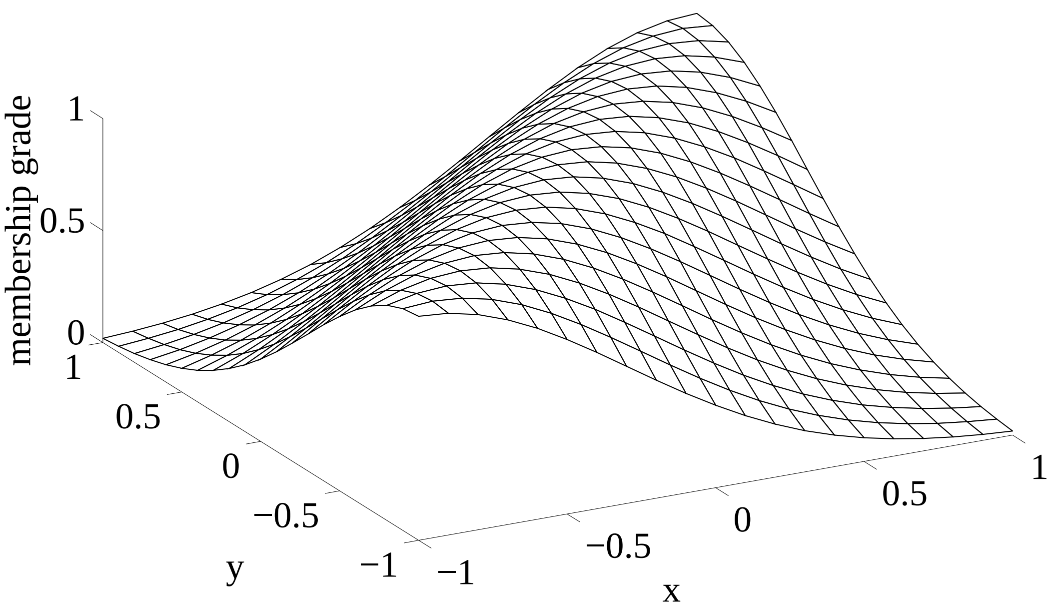
$$R : X_1 \times X_2 \times X_3 \ldots \times X_n \to [0, 1]$$

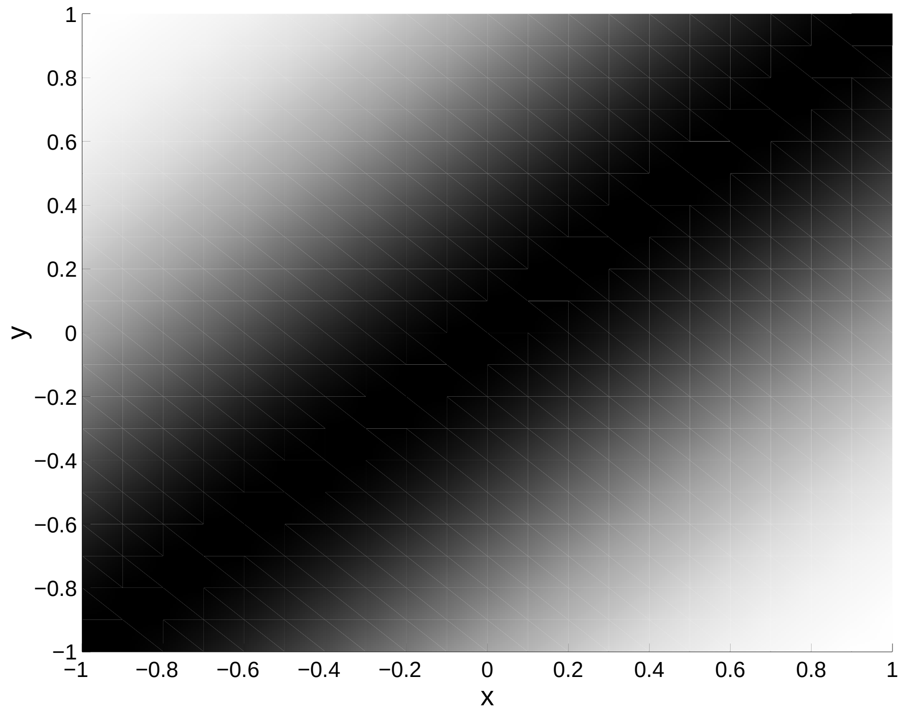
which assigns membership grades to all n-tuples $(x_1, x_2, \ldots, x_n)$ **from the Cartesian product universe.**

Fuzzy Relations: Example

Example: $R: \ x \approx y$ (“x **is approximately equal to** y”)

$$\mu_R(x,y) = e^{-(x-y)^2}$$





Relational Composition

Given fuzzy relation R defined in $X \times Y$ and fuzzy set A defined in X, derive the corresponding fuzzy set B defined in Y:

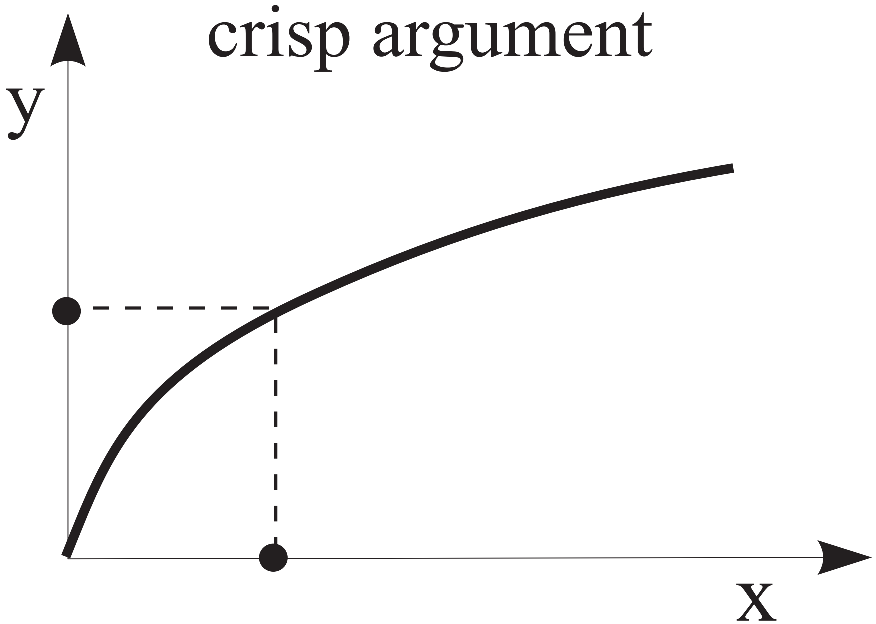
$$B = A \circ R = \mathrm{proj}_Y(\mathrm{ext}_{X \times Y}(A) \cap R)$$

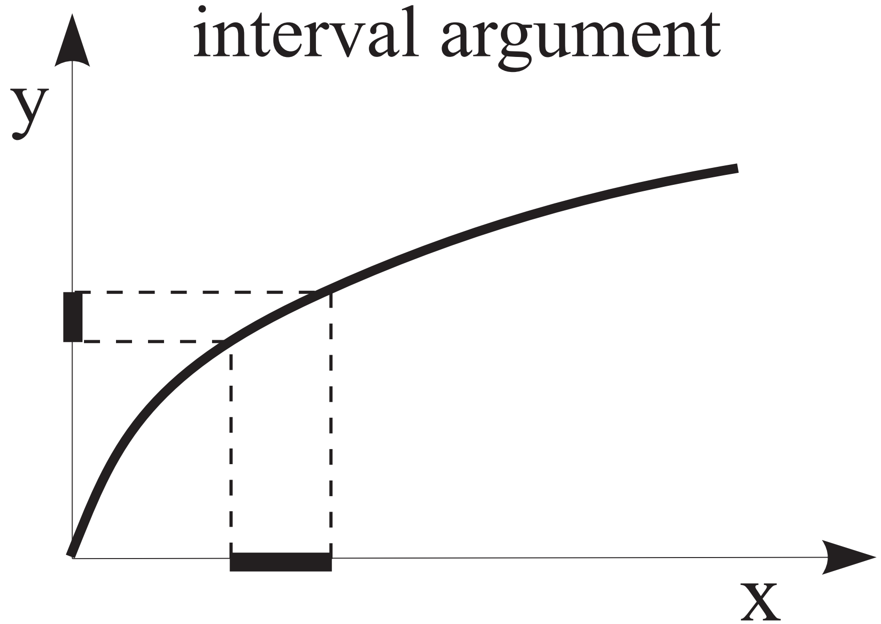
max-min composition:

$$\mu_B(y) = \max_x \; \min\left(\mu_A(x), \mu_R(x, y)\right)$$

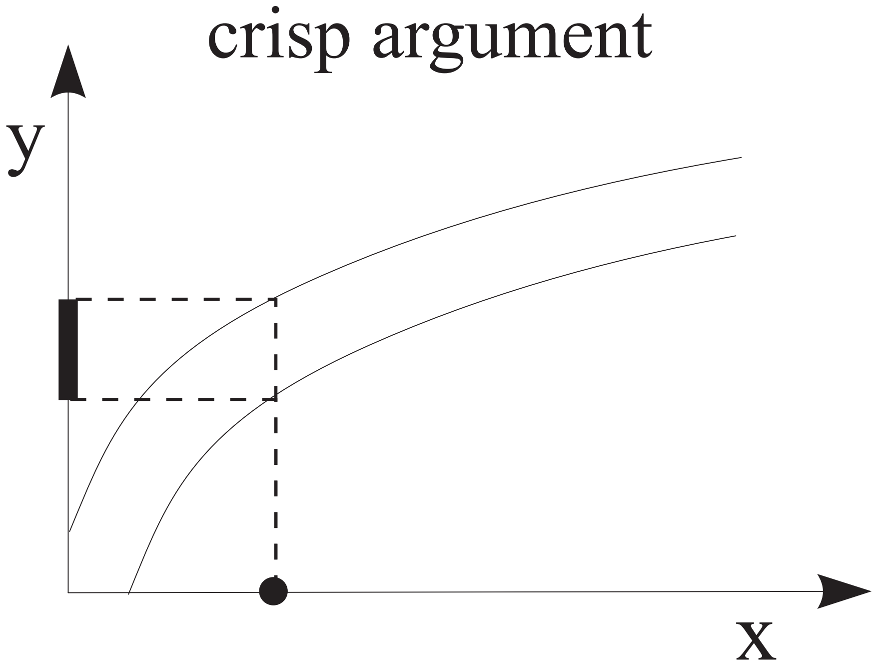
Analogous to evaluating a function.

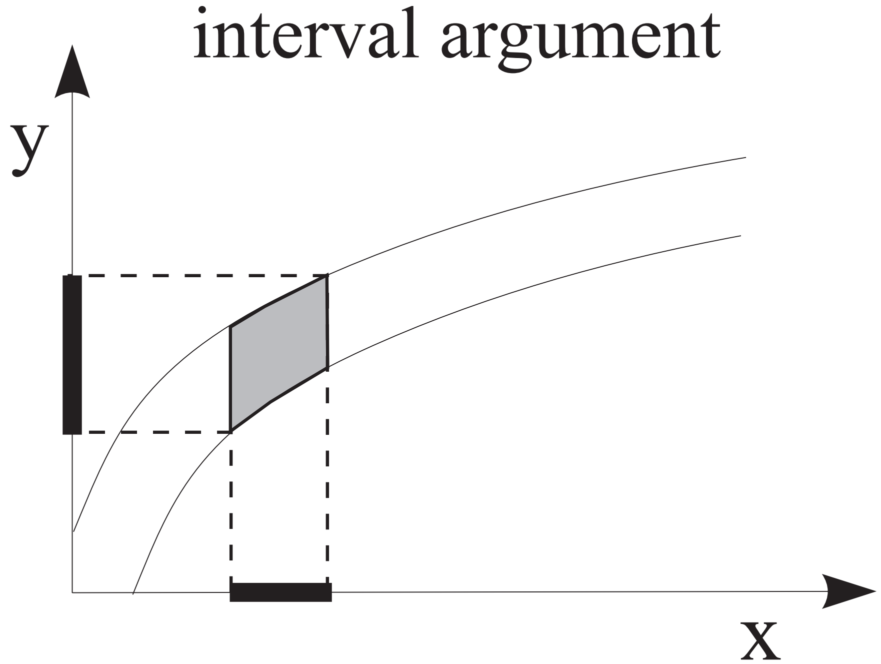
Graphical Interpretation: Crisp Function



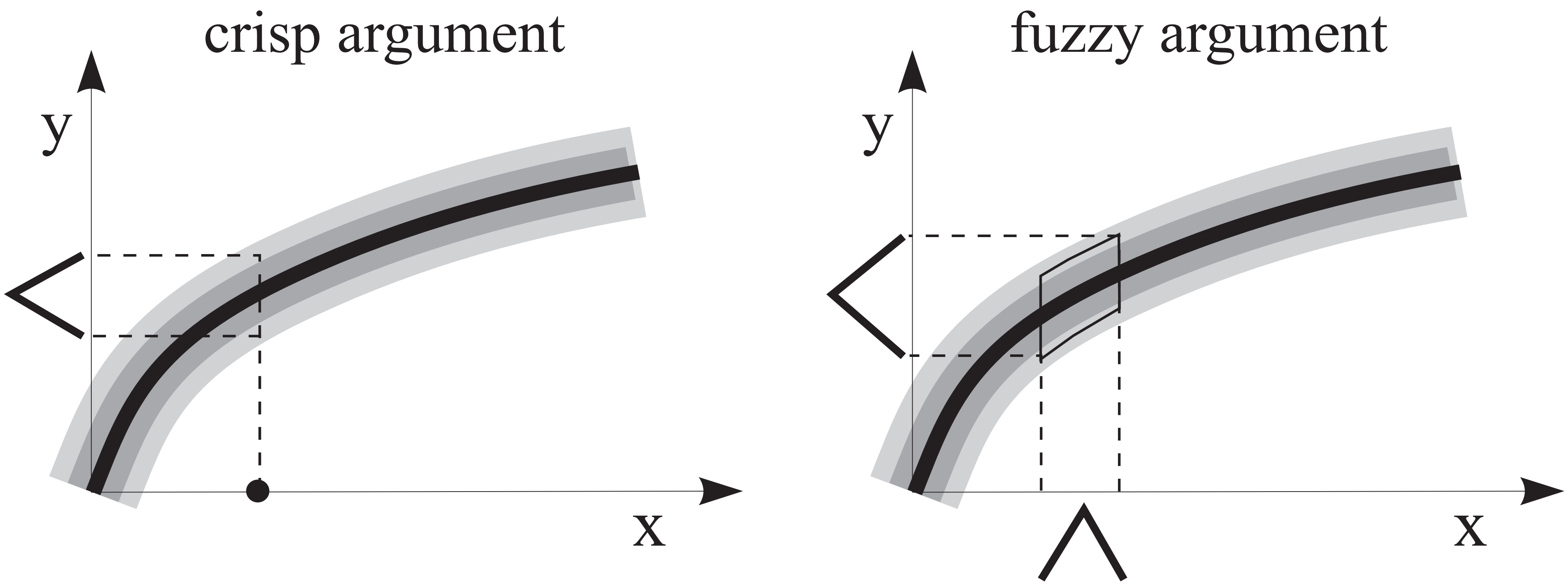


Graphical Interpretation: Interval Function





Graphical Interpretation: Fuzzy Relation



Max-Min Composition: Example

$$\mu_B(y) = \max_x \min\left(\mu_A(x), \mu_R(x,y)\right), \qquad \forall y$$

$$\begin{bmatrix} 1.0 & 0.4 & 0.1 & 0.0 & 0.0 \end{bmatrix} \circ \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.4 & 0.8 \\ 0.0 & 0.1 & 1.0 & 0.2 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.9 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.8 & 0.3 & 0.0 \end{bmatrix} = \begin{bmatrix} 0.0 & 0.1 & 0.4 & 0.4 & 0.8 \end{bmatrix}$$

Fuzzy Systems

Fuzzy Systems

- Systems with **fuzzy parameters**

$$y = \tilde{3}x_1 + \tilde{5}x_2$$

- **Fuzzy inputs and states**

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = \tilde{2}$$

- **Rule-based systems**

If *the heating power is high*

then *the temperature will increase fast*

Rule-based Fuzzy Systems

- **Linguistic (Mamdani) fuzzy model**

$$\textbf{If } x \textbf{ is } A \textbf{ then } y \textbf{ is } B$$

- **Fuzzy relational model**

$$\textbf{If } x \textbf{ is } A \textbf{ then } y \textbf{ is } B_1(0.1), B_2(0.8)$$

- **Takagi–Sugeno fuzzy model**

$$\textbf{If } x \textbf{ is } A \textbf{ then } y = f(x)$$

Linguistic Model

If x **is** A **then** y **is** B

x **is** A – **antecedent (fuzzy proposition)**

y **is** B – **consequent (fuzzy proposition)**

Linguistic Model

If x **is** A **then** y **is** B

x **is** A – **antecedent (fuzzy proposition)**

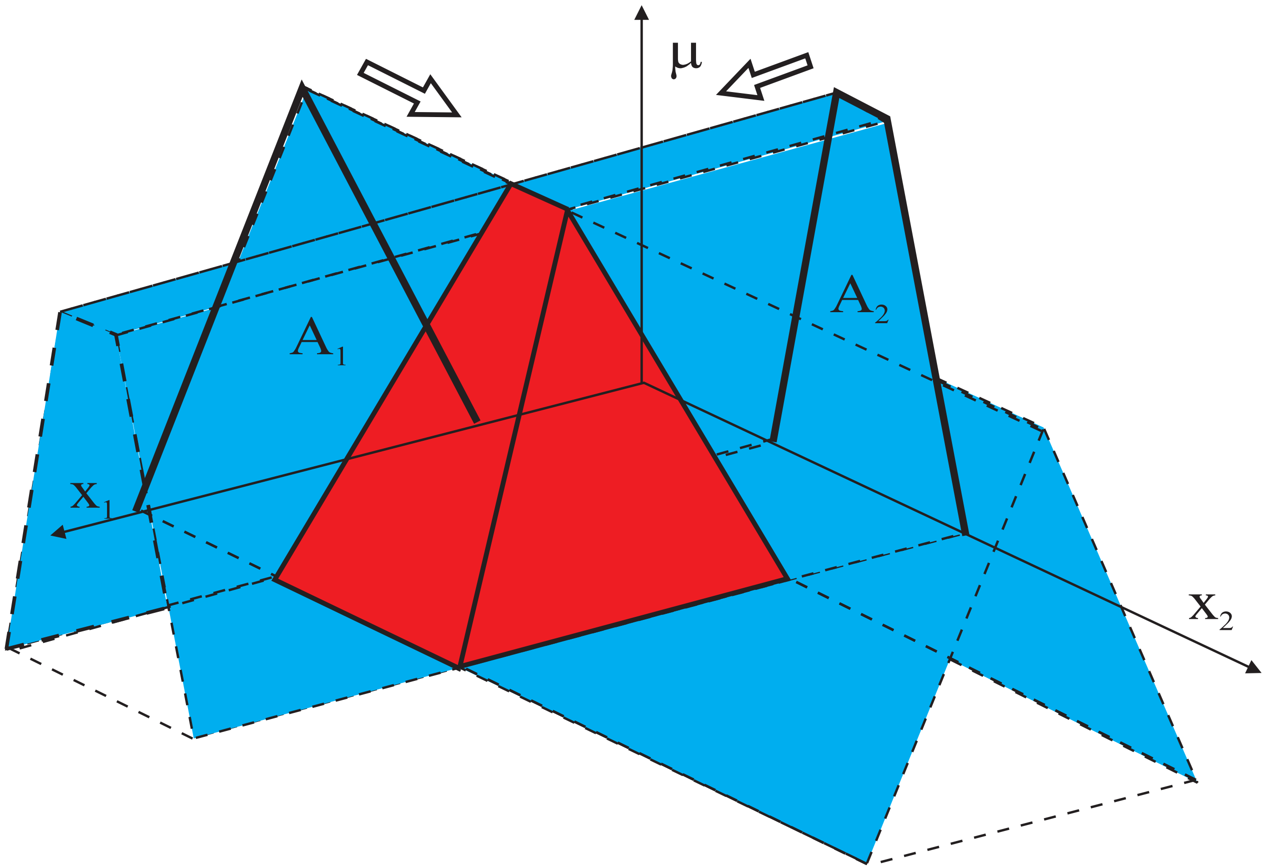
y **is** B – **consequent (fuzzy proposition)**

Compound propositions (logical connectives, hedges):

If x_1 **is very big and** x_2 **is not small**

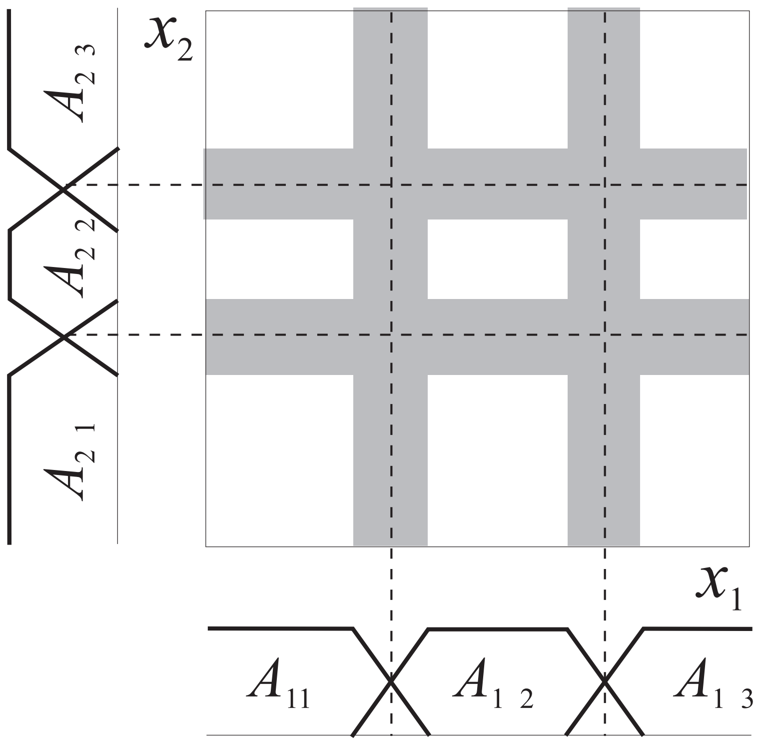
Multidimensional Antecedent Sets

$A_1 \cap A_2$ **on** $X_1 \times X_2$**:**

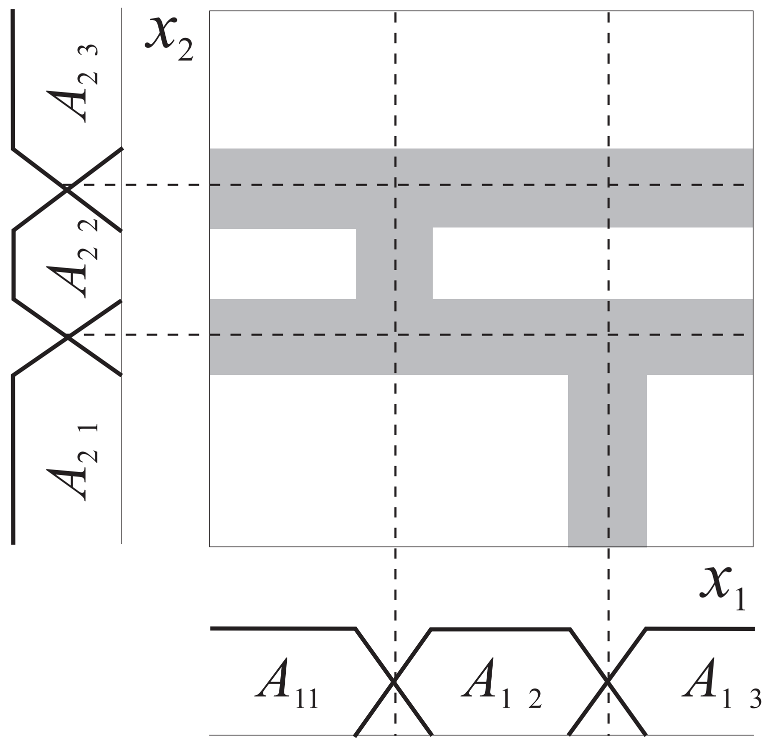


Partitioning of the Antecedent Space

conjunctive



other connectives



Inference Mechanism

Given the if-then rules and an input fuzzy set, deduce the corresponding output fuzzy set.

- Formal approach based on fuzzy relations.
- Simplified approach (Mamdani inference).
- Interpolation (additive fuzzy systems).

Formal Approach

1. Represent each if–then rule as a fuzzy relation.

2. Aggregate these relations in one relation representative for the entire rule base.

3. Given an input, use *relational composition* to derive the corresponding output.

Modus Ponens Inference Rule

Classical logic

if x **is** A **then** y **is** B

x **is** A

---

y **is** B

Fuzzy logic

if x **is** A **then** y **is** B

x **is** A'

---

y **is** B'

Relational Representation of Rules

If–then rules can be represented as a *relation*, using implications or conjunctions.

Classical implication

A	B	$A \to B$ $(\neg A \vee B)$
0	0	1
0	1	1
1	0	0
1	1	1

$A \backslash B$	0	1
0	1	1
1	0	1

$$R \colon \{0,1\} \times \{0,1\} \to \{0,1\}$$

Relational Representation of Rules

If–then rules can be represented as a *relation*, using implications or conjunctions.

Conjunction

A	B	$A \wedge B$
0	**0**	**0**
0	**1**	**0**
1	**0**	**0**
1	**1**	**1**

$A \backslash B$	**0**	**1**
0	**0**	**0**
1	**0**	**1**

$$R\colon \{0,1\} \times \{0,1\} \to \{0,1\}$$

Fuzzy Implications and Conjunctions

Fuzzy implication is represented by a fuzzy relation:

$$R\colon [0,1] \times [0,1] \rightarrow [0,1]$$

$$\mu_R(x,y) = \mathrm{I}(\mu_A(x), \mu_B(y))$$

$\mathrm{I}(a,b)$ – **implication function**

“classical”	**Kleene–Diene**	$\mathrm{I}(a,b) = \max(1-a, b)$
	Lukasiewicz	$\mathrm{I}(a,b) = \min(1, 1-a+b)$
T-norms	**Mamdani**	$\mathrm{I}(a,b) = \min(a,b)$
	Larsen	$\mathrm{I}(a,b) = a \cdot b$

Inference With One Rule

1. Construct implication relation:

$$\mu_R(x, y) = \mathrm{I}(\mu_A(x), \mu_B(y))$$

Inference With One Rule

1. Construct implication relation:

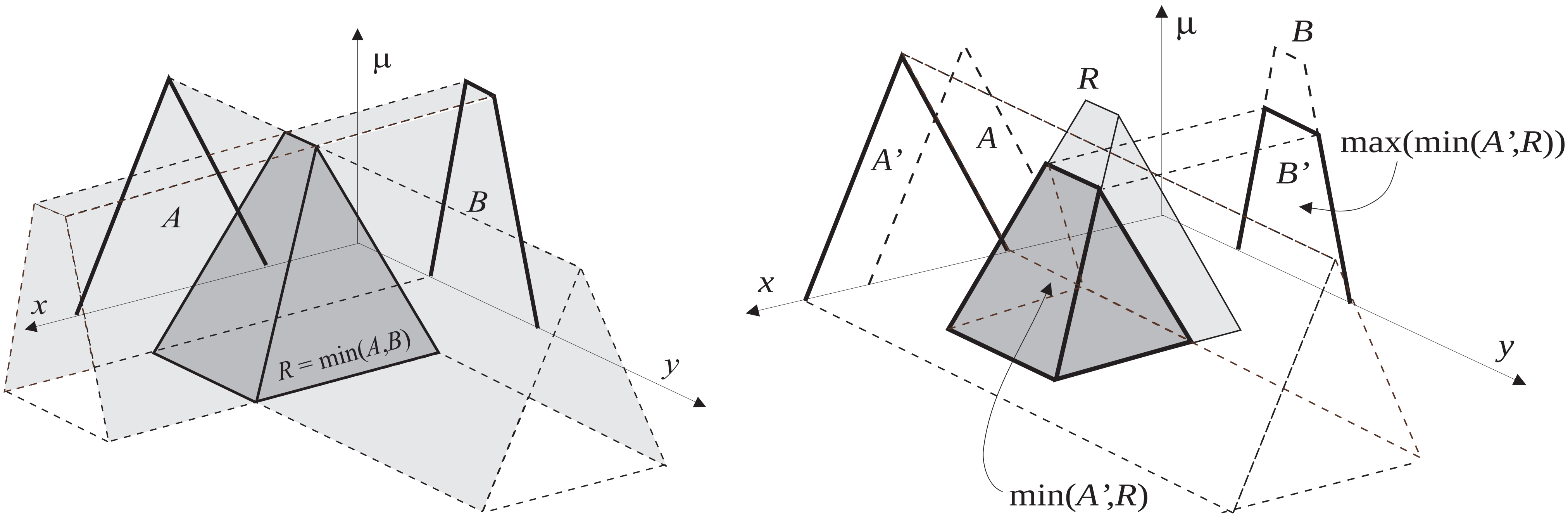
$$\mu_R(x, y) = \mathrm{I}(\mu_A(x), \mu_B(y))$$

2. Use relational composition to derive B' from A':

$$B' = A' \circ R$$

Graphical Illustration

$$\mu_R(x, y) = \min(\mu_A(x), \mu_B(y)) \qquad \mu_{B'}(y) = \max_x \min\left(\mu_{A'}(x), \mu_R(x, y)\right)$$



Inference With Several Rules

1. **Construct implication relation for each rule i:**

$$\mu_{R_i}(x, y) = \mathrm{I}(\mu_{A_i}(x), \mu_{B_i}(y))$$

2. **Aggregate relations R_i into one:**

$$\mu_R(x, y) = \mathrm{aggr}(\mu_{A_i}(x))$$

The aggr operator is the minimum for implications and the maximum for conjunctions.

3. **Use relational composition to derive B' from A':**

$$B' = A' \circ R$$

Example: Conjunction

1. **Each rule**

$$\textbf{If } \tilde{x} \textbf{ is } A_i \textbf{ then } \tilde{y} \textbf{ is } B_i$$

is represented as a fuzzy relation on $X \times Y$**:**

$$R_i = A_i \times B_i \qquad \mu_{R_i}(\mathbf{x}, \mathbf{y}) = \mu_{A_i}(\mathbf{x}) \wedge \mu_{B_i}(\mathbf{y})$$

Aggregation and Composition

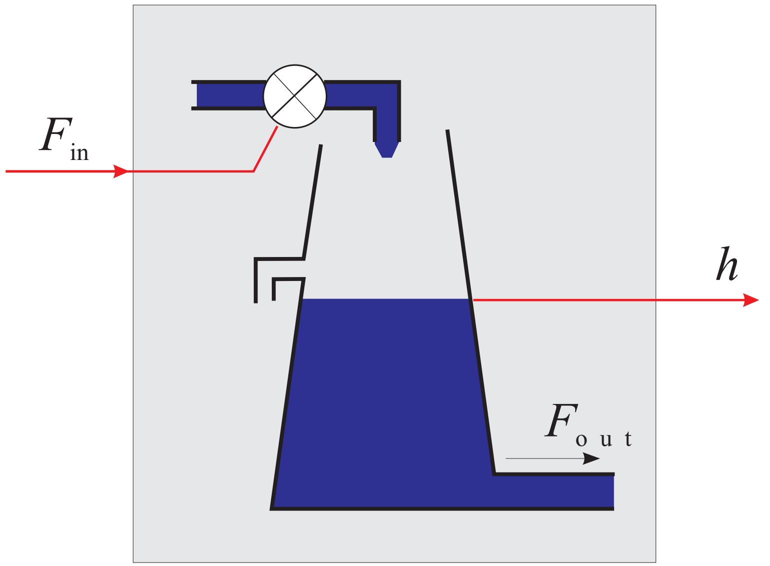
2. The entire rule base's relation is the union:

$$R = \bigcup_{i=1}^{K} R_i \qquad \mu_R(\mathbf{x}, \mathbf{y}) = \max_{1 \le i \le K} [\mu_{R_i}(\mathbf{x}, \mathbf{y})]$$

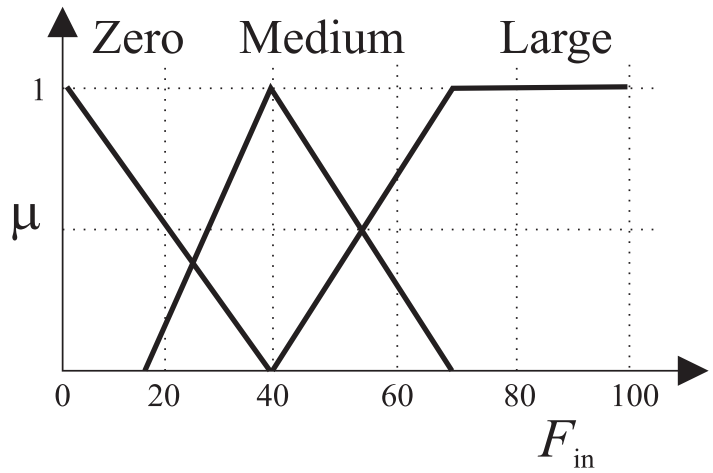
3. Given an input value A' the output value B' is:

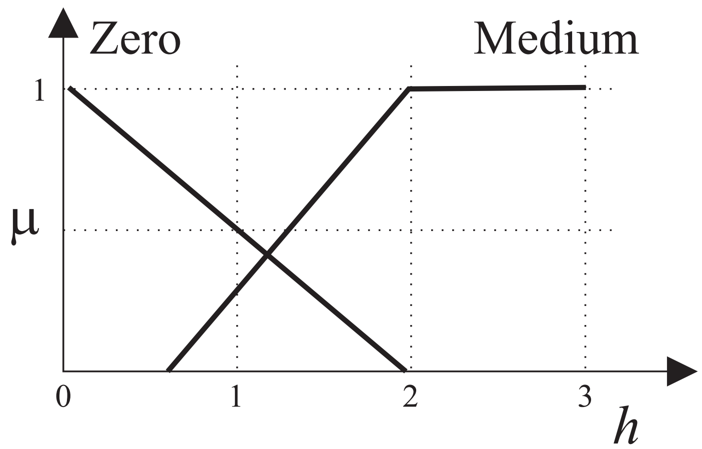
$$B' = A' \circ R \qquad \mu_{B'}(\mathbf{y}) = \max_X [\mu_{A'}(\mathbf{x}) \wedge \mu_R(\mathbf{x}, \mathbf{y})]$$

Example: Modeling of Liquid Level

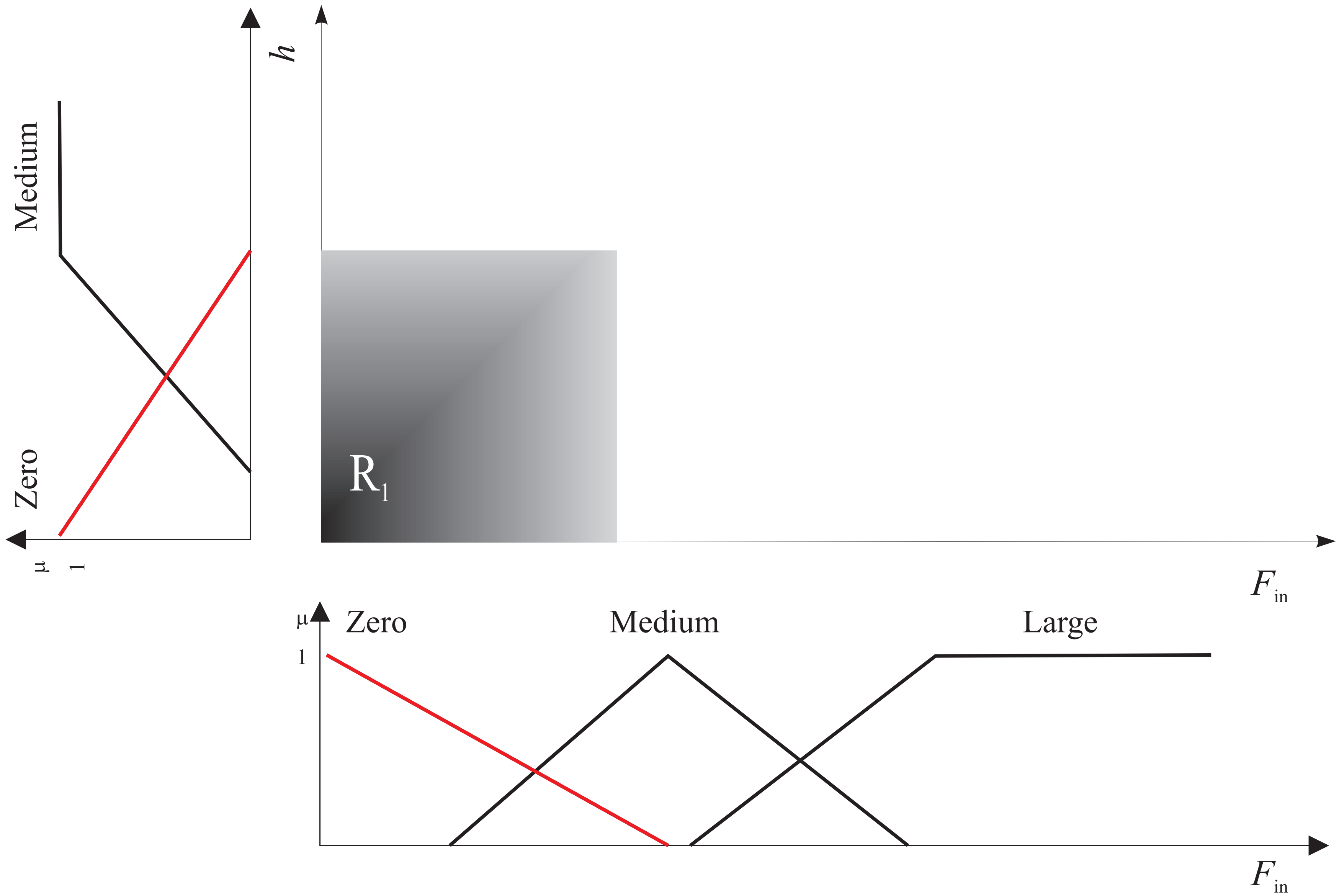


- If F_{in} is **Zero** then h is **Zero**
- If F_{in} is **Med** then h is **Med**
- If F_{in} is **Large** then h is **Med**

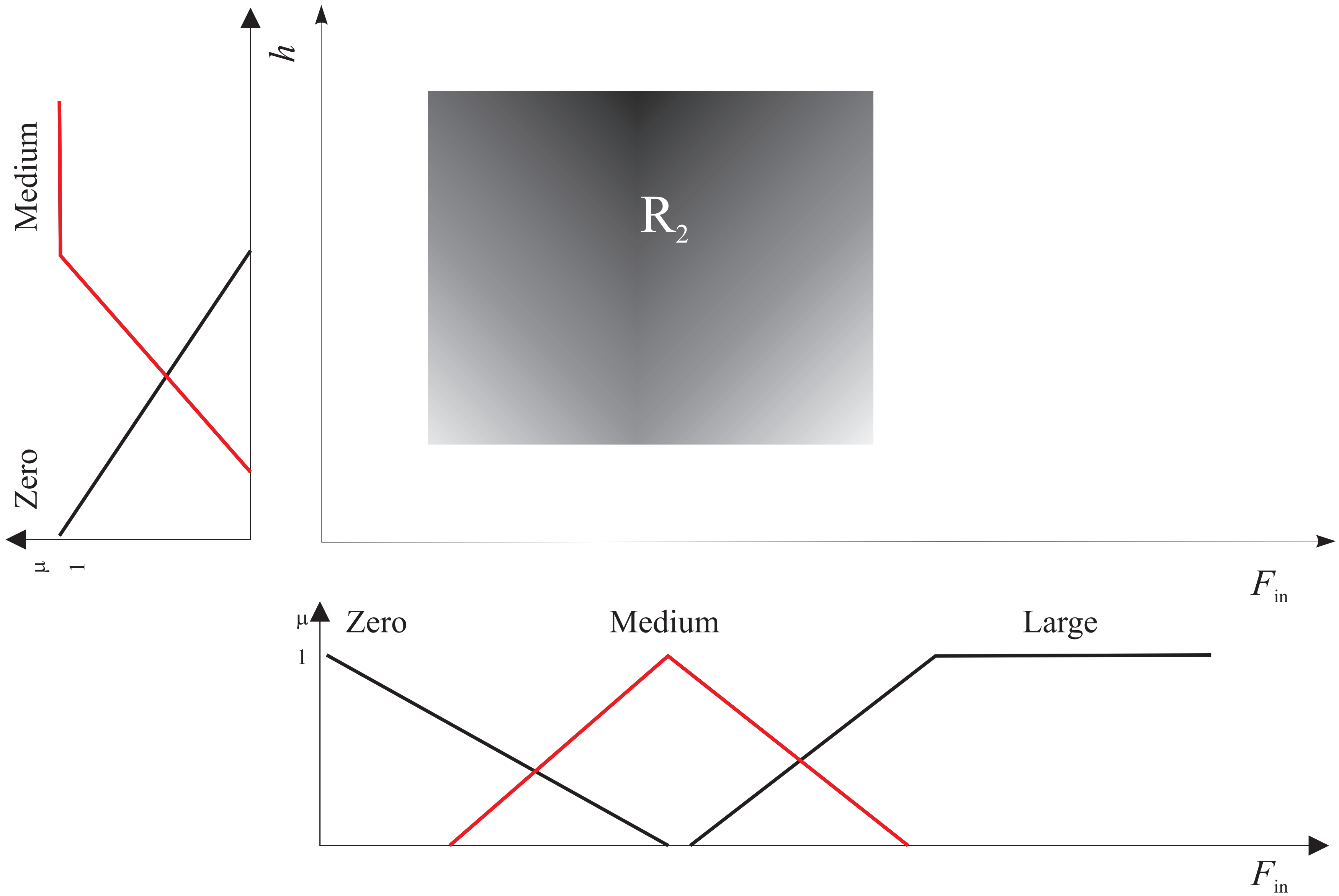




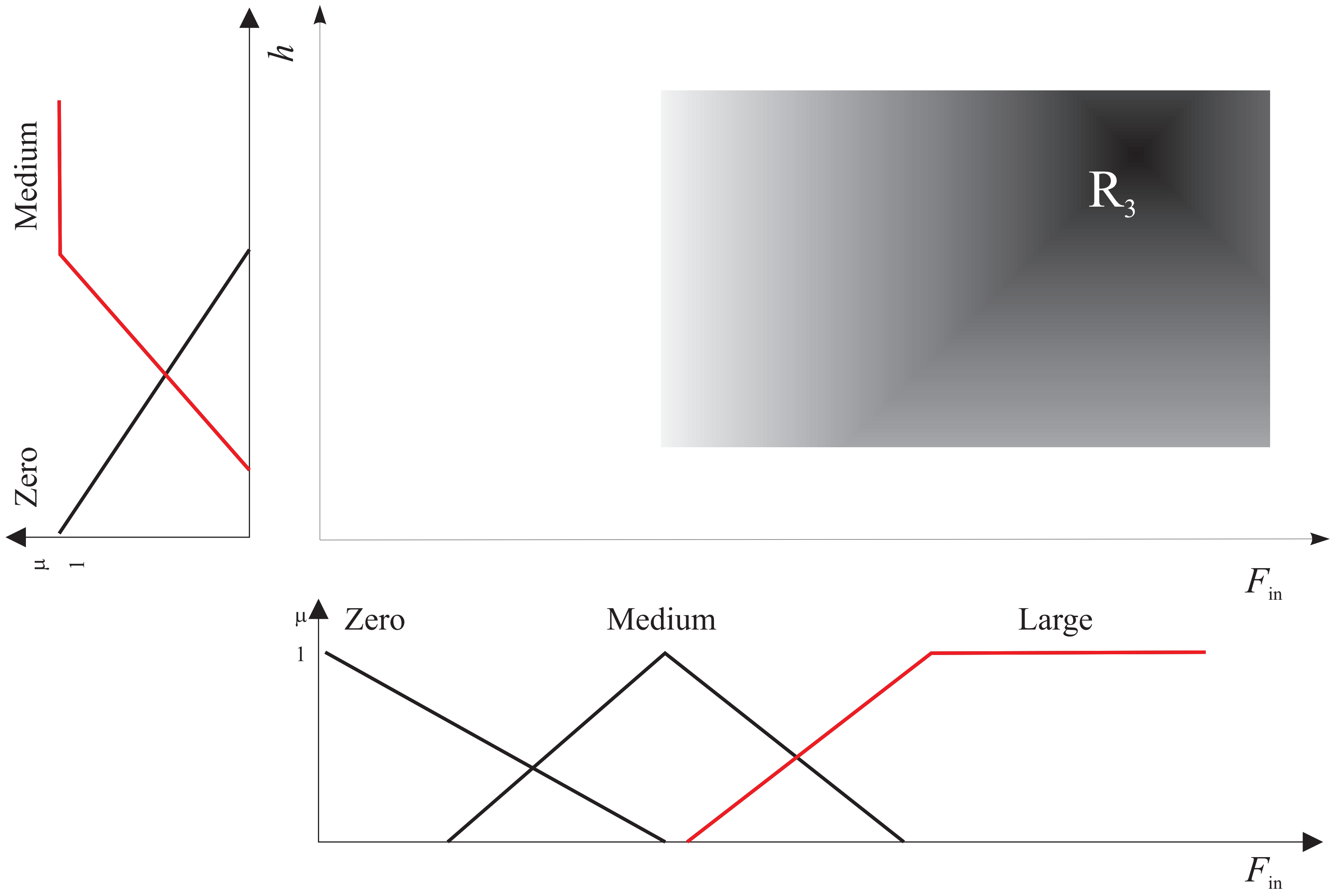
$\mathcal{R}_1$ If Flow is Zero then Level is Zero



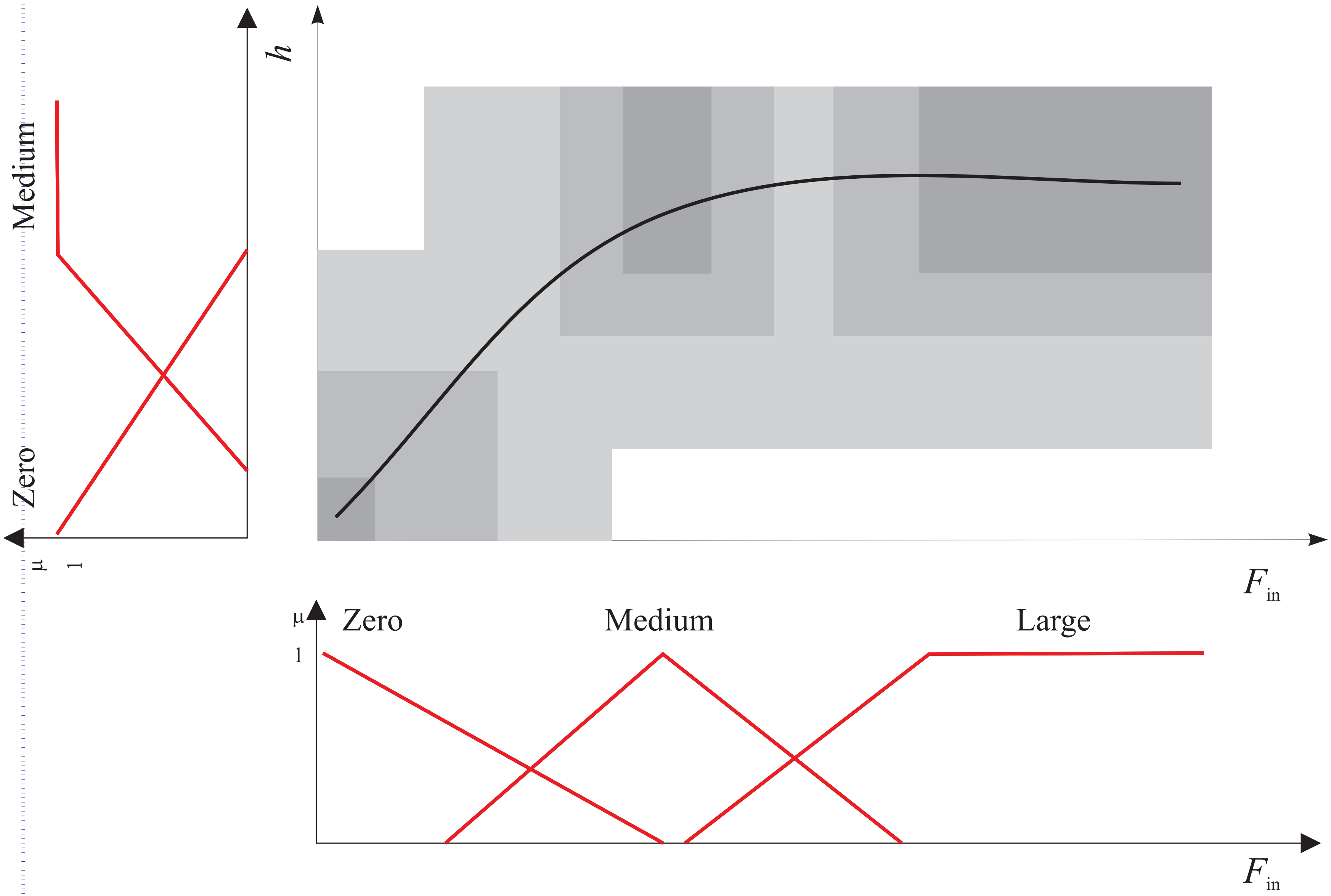
$\mathcal{R}_2$ If Flow is Medium then Level is Medium



$\mathcal{R}_3$ If Flow is Large then Level is Medium



Aggregated Relation



Simplified Approach

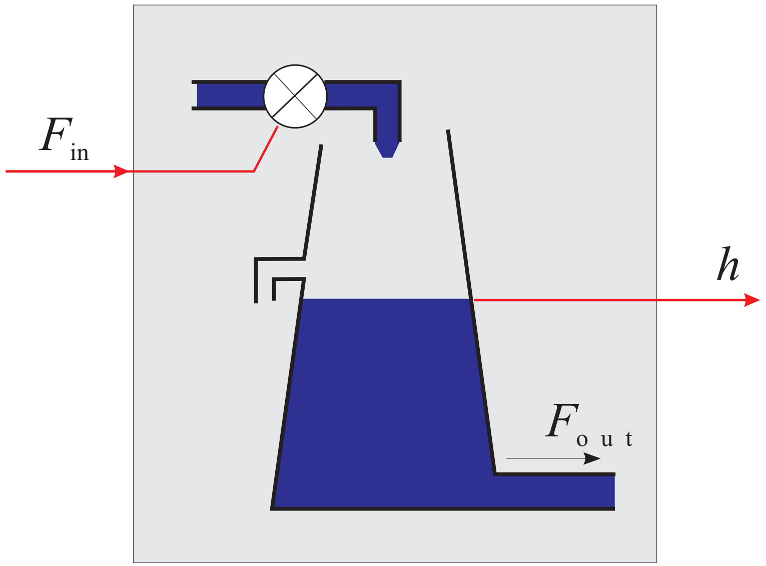
1. Compute the match between the input and the antecedent membership functions (*degree of fulfillment*).

2. Clip the corresponding output fuzzy set for each rule by using the degree of fulfillment.

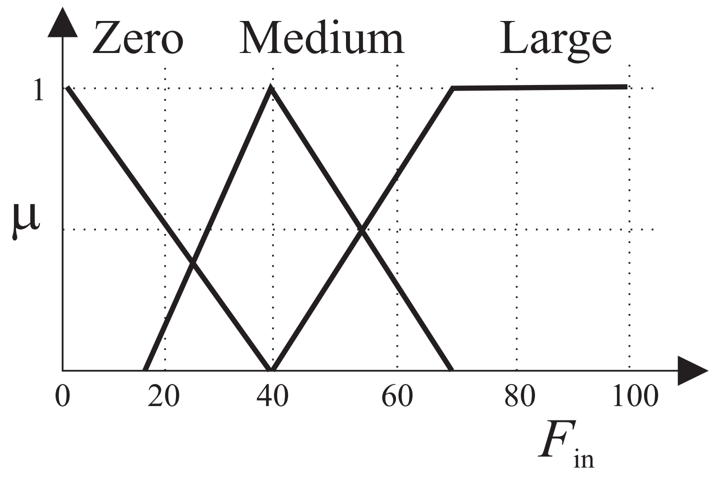
3. Aggregate output fuzzy sets of all the rules into one fuzzy set.

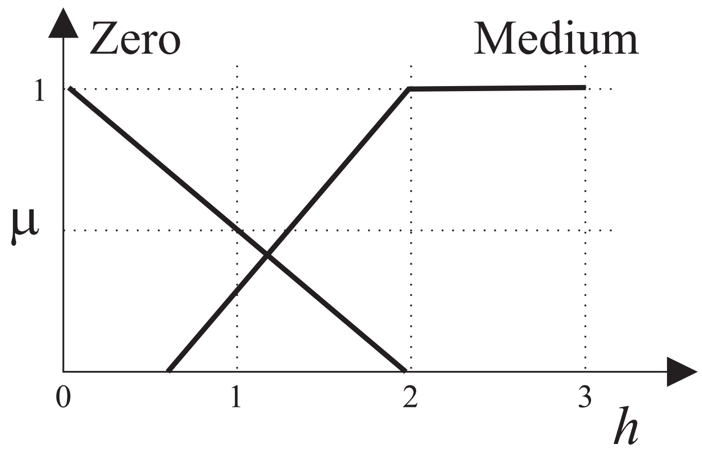
This is called the *Mamdani* or *max-min* inference method.

Water Tank Example

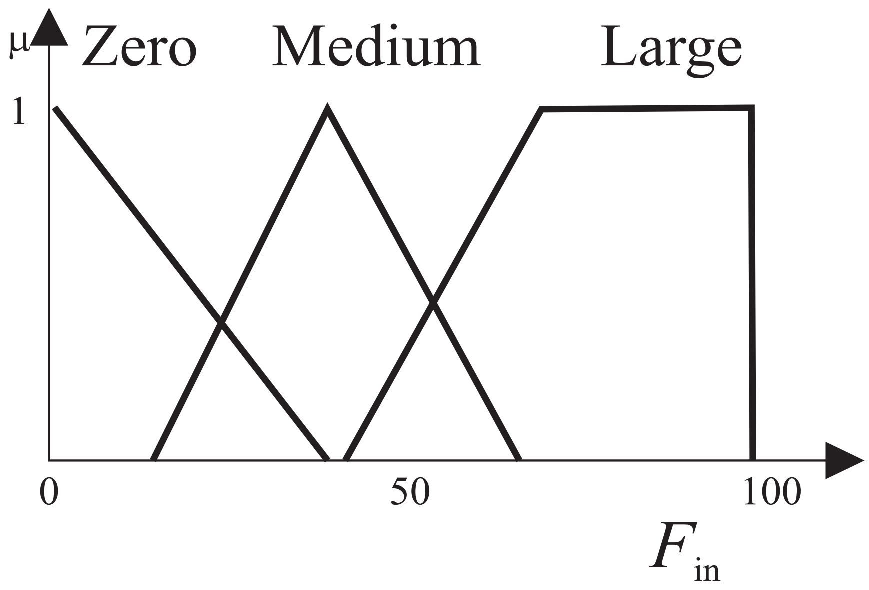


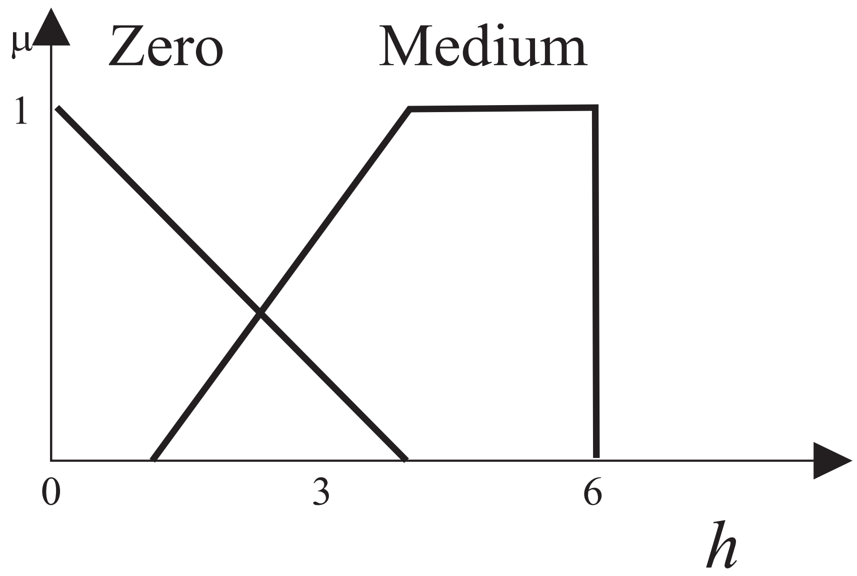
- If F_{in} is **Zero** then h is **Zero**
- If F_{in} is **Med** then h is **Med**
- If F_{in} is **Large** then h is **Med**



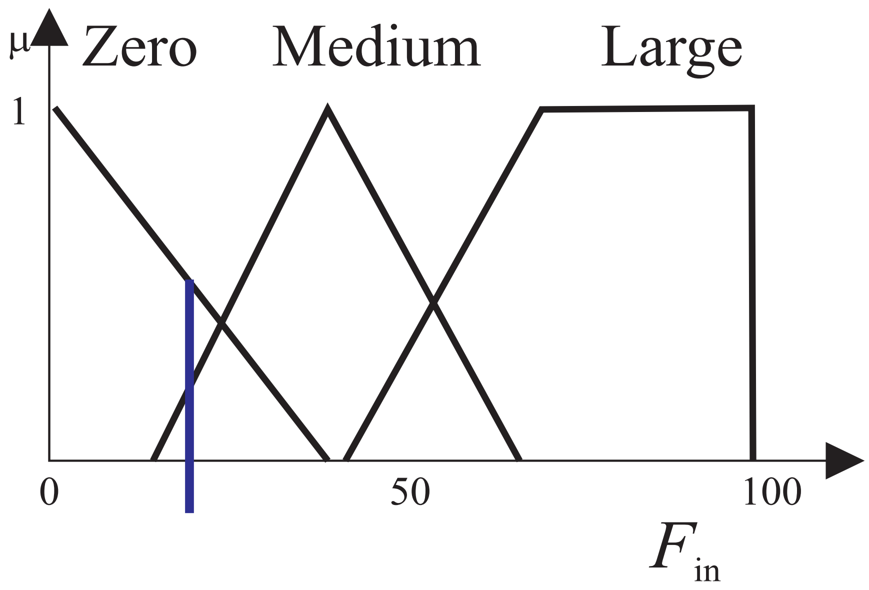


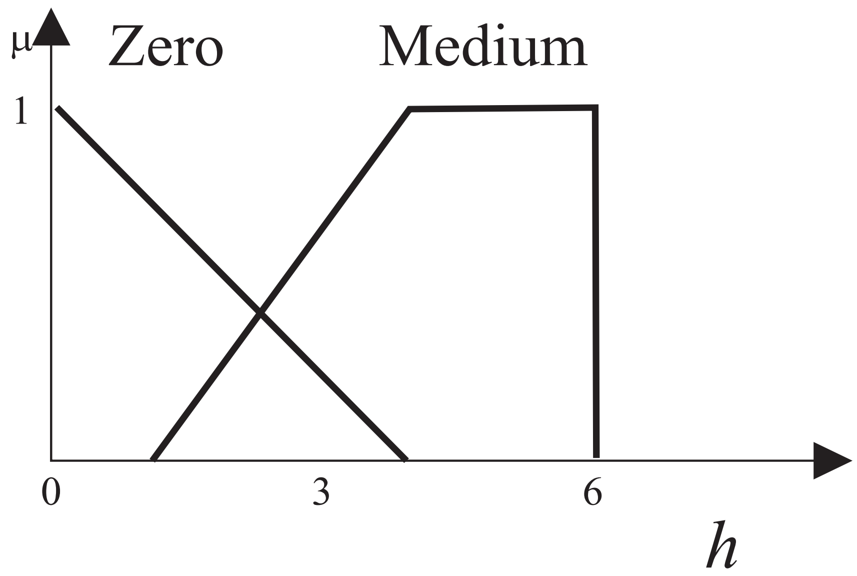
Mamdani Inference: Example





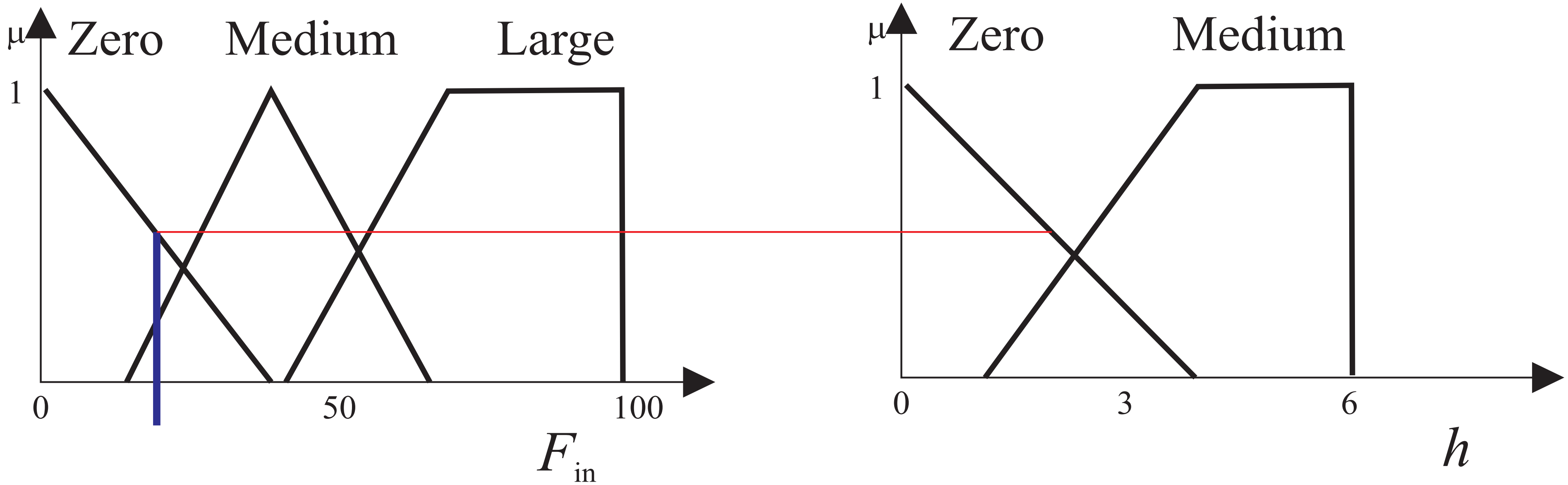
Mamdani Inference: Example





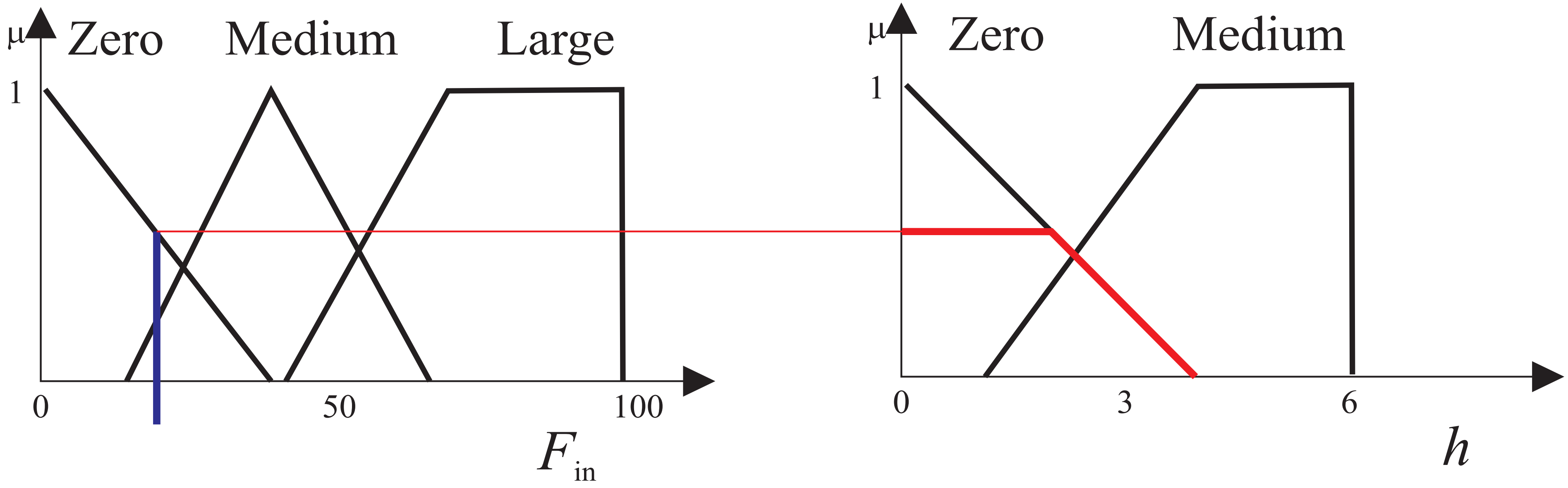
Given a crisp (numerical) input (F_{in}).

If F_{in} is Zero then . . .



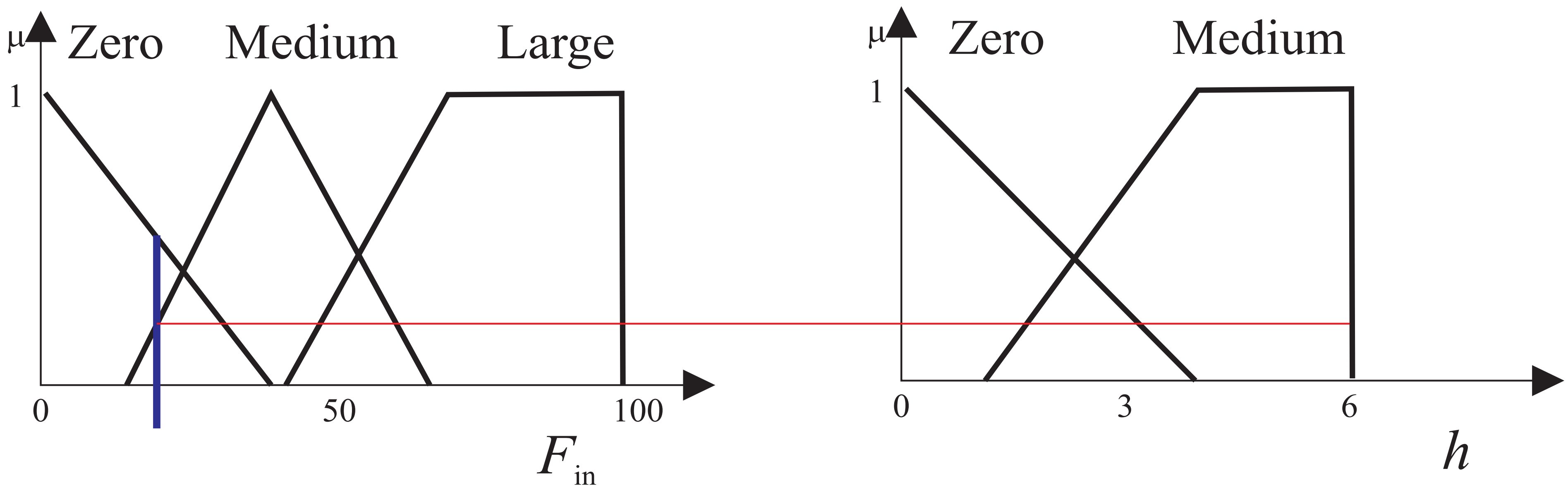
Determine the degree of fulfillment (truth) of the first rule.

If F_{in} is Zero then h is Zero



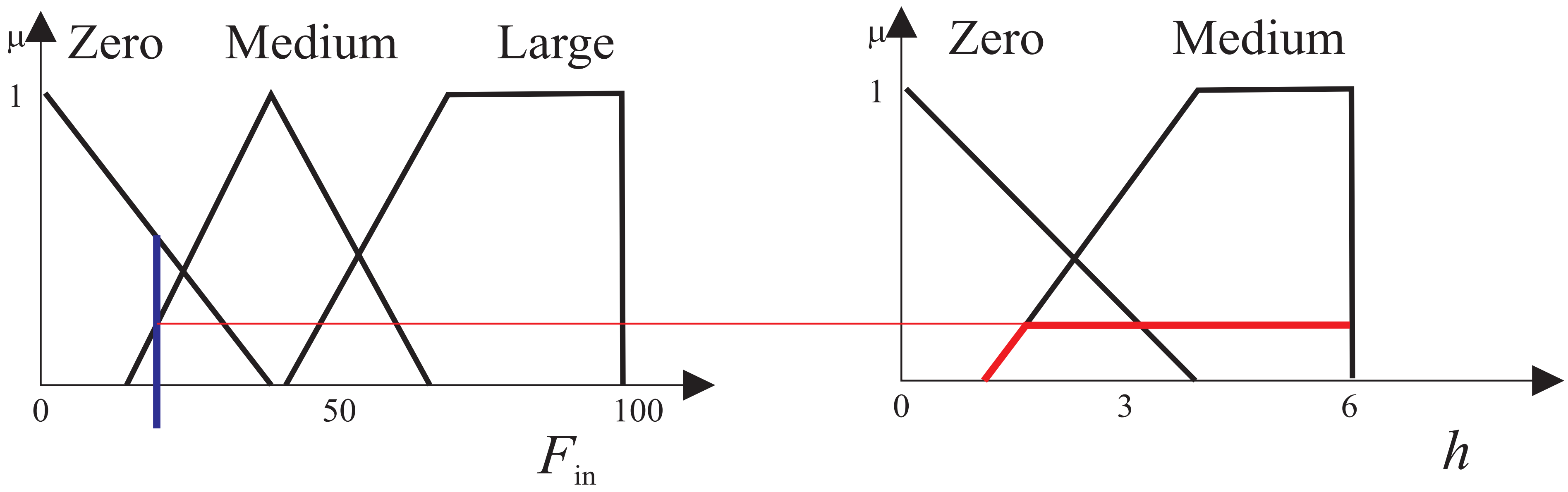
Clip consequent membership function of the first rule.

If F_{in} is Medium then ...



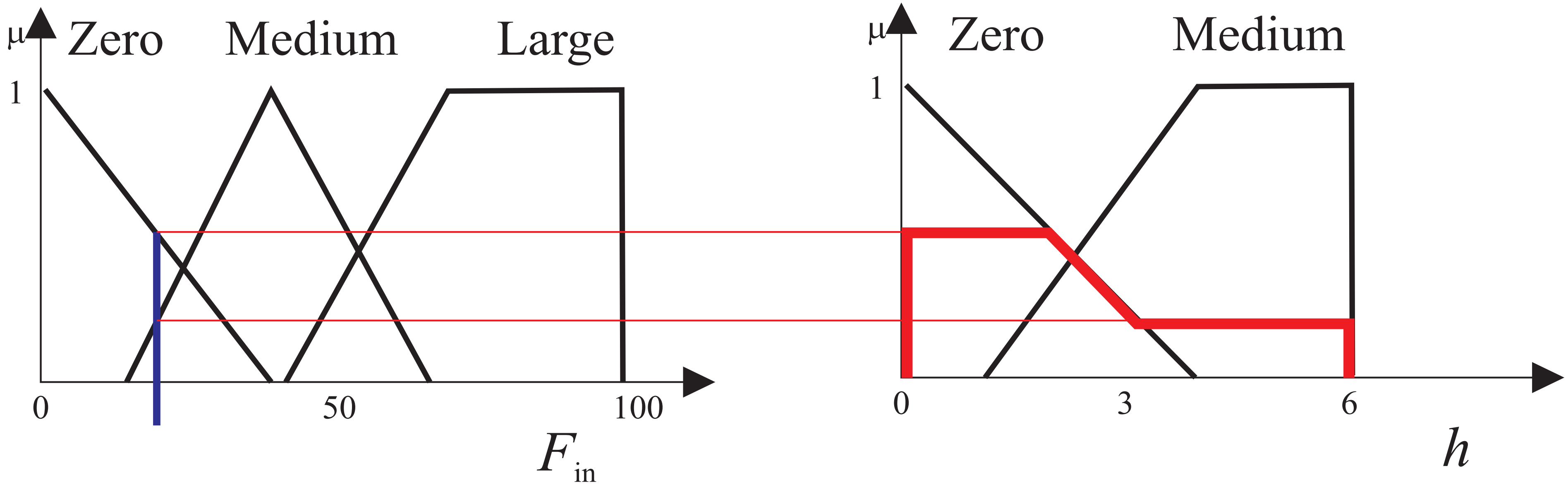
Determine the degree of fulfillment (truth) of the second rule.

If F_{in} is Medium then h is Medium



Clip consequent membership function of the second rule.

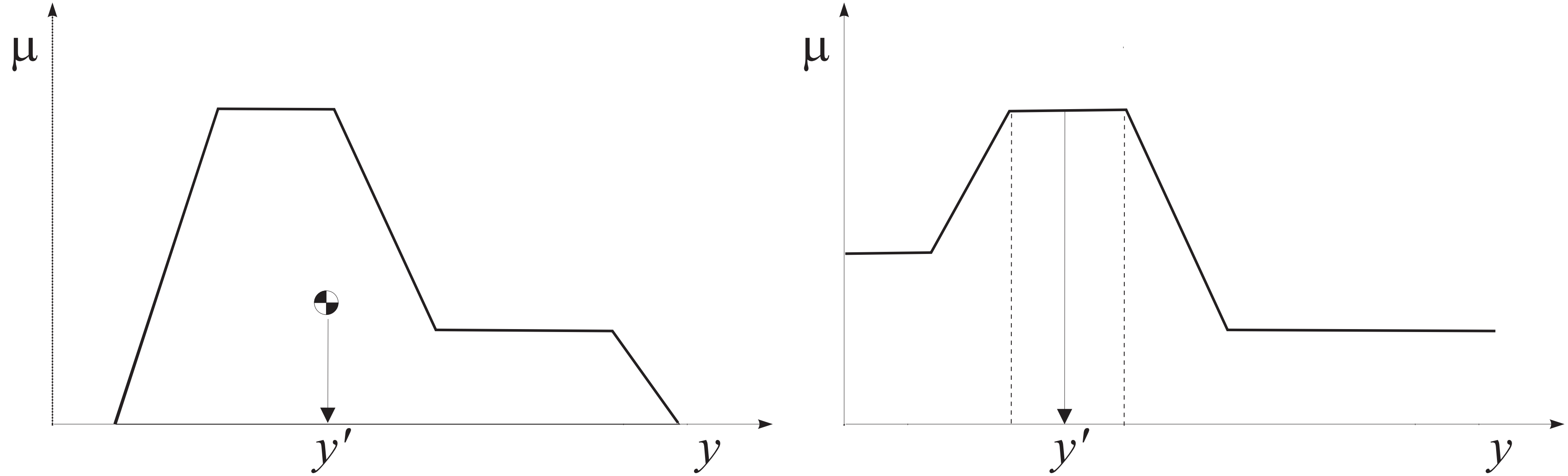
Aggregation



Combine the result of the two rules (union).

Defuzzification

conversion of a fuzzy set to a crisp value

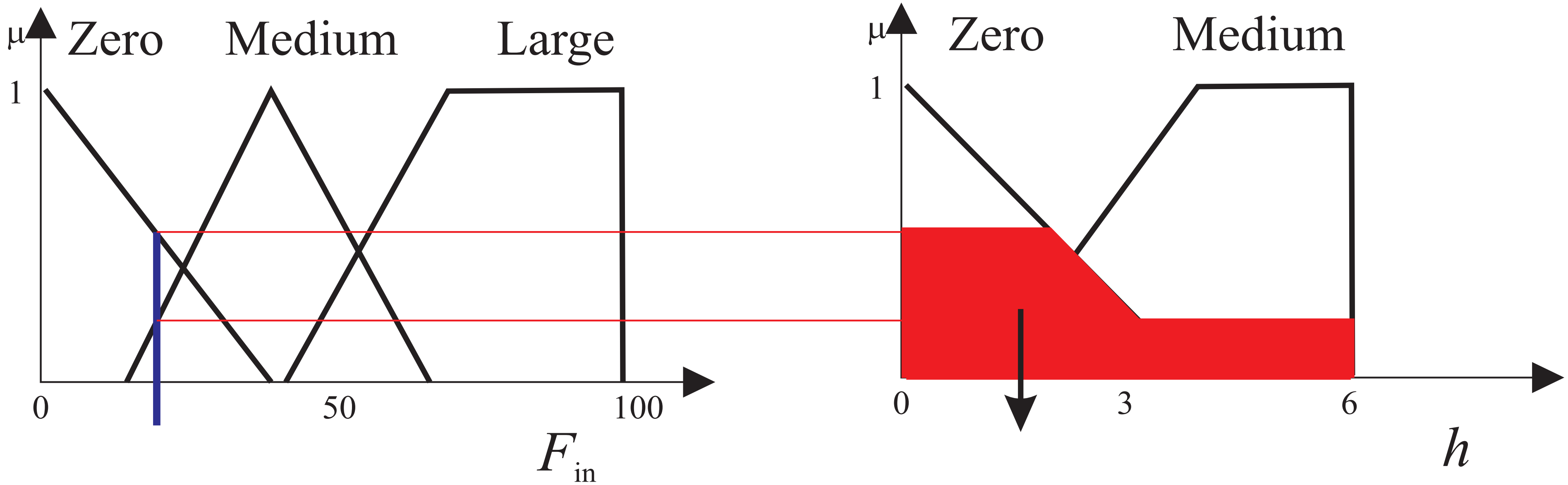


(a) center of gravity (b) mean of maxima

Center-of-Gravity Method

$$y_0 = \frac{\sum_{j=1}^{F} \mu_{B'}(y_j) y_j}{\sum_{j=1}^{F} \mu_{B'}(y_j)}$$

Defuzzification



Compute a crisp (numerical) output of the model (center-of-gravity method).

Fuzzy System Components