Knowledge-Based Control Systems (SC4081)

Lecture 3: Construction of Fuzzy Systems

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Outline

1. **Singleton and Takagi–Sugeno fuzzy system.**

2. **Dynamic fuzzy systems.**

3. **Knowledge based fuzzy modeling.**

4. **Data-driven construction.**

Singleton Fuzzy Model

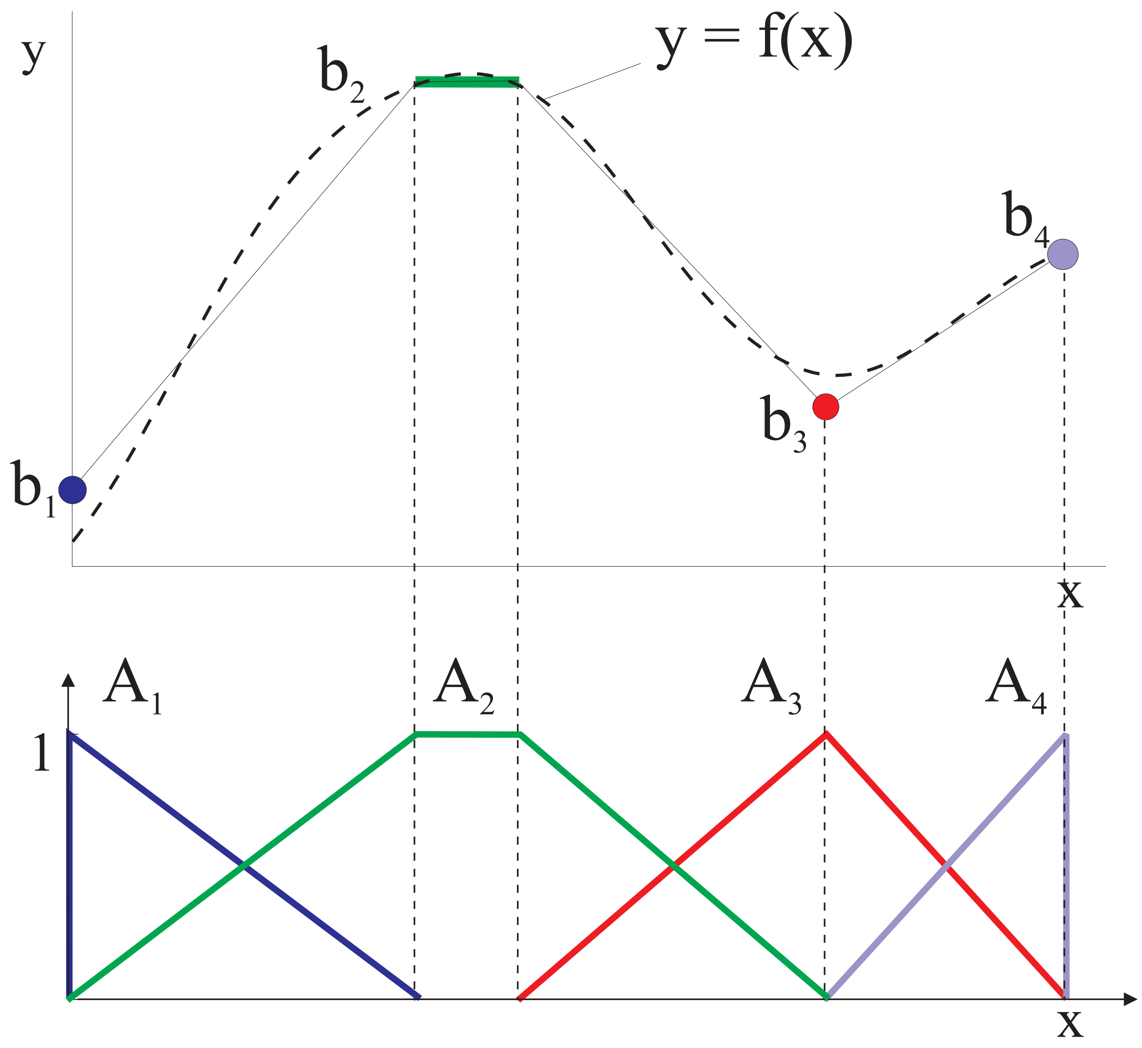
$$\textbf{If } x \textbf{ is } A_i \textbf{ then } y = b_i$$

Inference/defuzzification:

$$y = \frac{\sum_{i=1}^{K} \mu_{A_i}(x) b_i}{\sum_{i=1}^{K} \mu_{A_i}(x)}$$

- **well-understood approximation properties**
- **straightforward parameter estimation**

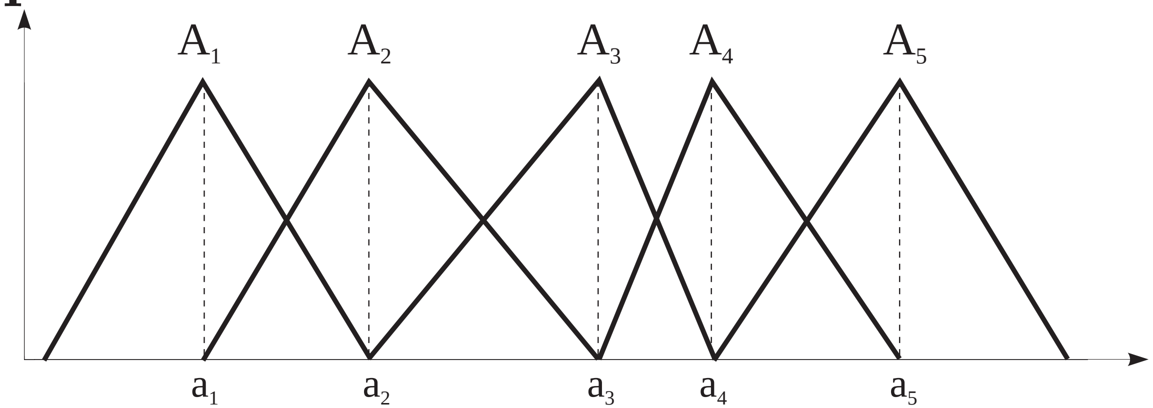
Piece-wise Linear Approximation



Linear Mapping with a Singleton Model

$$y = \mathbf{k}^T \mathbf{x} + q = \sum_{j=1}^{p} k_j x_j + q$$

- **Triangular partition:**



- **Consequent singletons are equal to:**

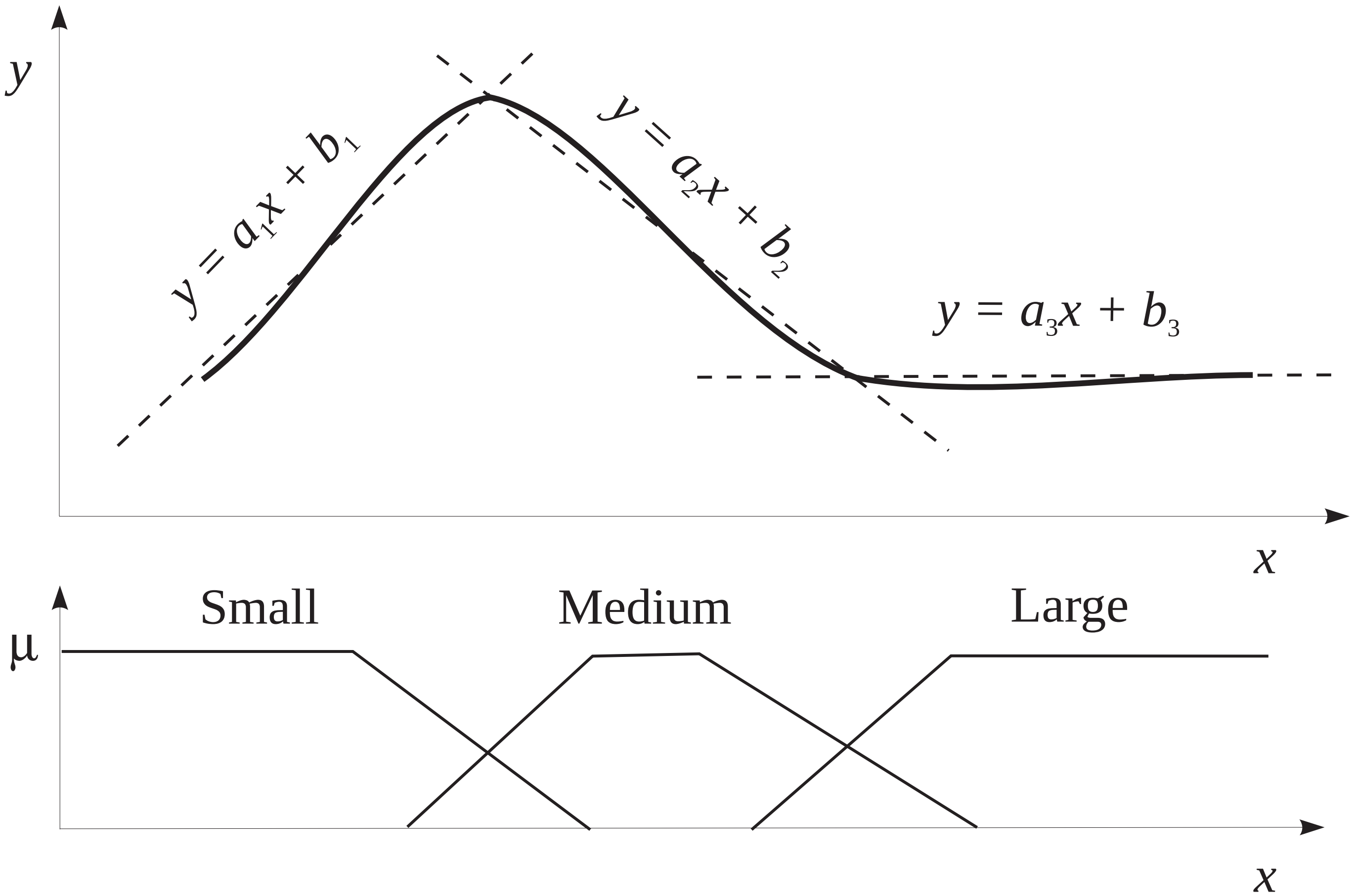
$$b_i = \sum_{j=1}^{p} k_j a_{i,j} + q$$

Takagi–Sugeno (TS) Fuzzy Model

$$\textbf{If } x \textbf{ is } A_i \textbf{ then } y_i = a_i x + b_i$$

$$y = \frac{\sum_{i=1}^{K} \mu_{A_i}(x) y_i}{\sum_{i=1}^{K} \mu_{A_i}(x)} = \frac{\sum_{i=1}^{K} \mu_{A_i}(x)(a_i x + b_i)}{\sum_{i=1}^{K} \mu_{A_i}(x)}$$

Input-Output Mapping of the TS Model



Consequents are approximate local linear models of the system.

TS Model is a Quasi-Linear System

$$y = \frac{\sum_{i=1}^{K} \mu_{A_i}(\mathbf{x}) y_i}{\sum_{j=1}^{K} \mu_{A_j}(\mathbf{x})} = \frac{\sum_{i=1}^{K} \mu_{A_i}(\mathbf{x})(\mathbf{a}_i^T \mathbf{x} + b_i)}{\sum_{j=1}^{K} \mu_{A_j}(\mathbf{x})}$$

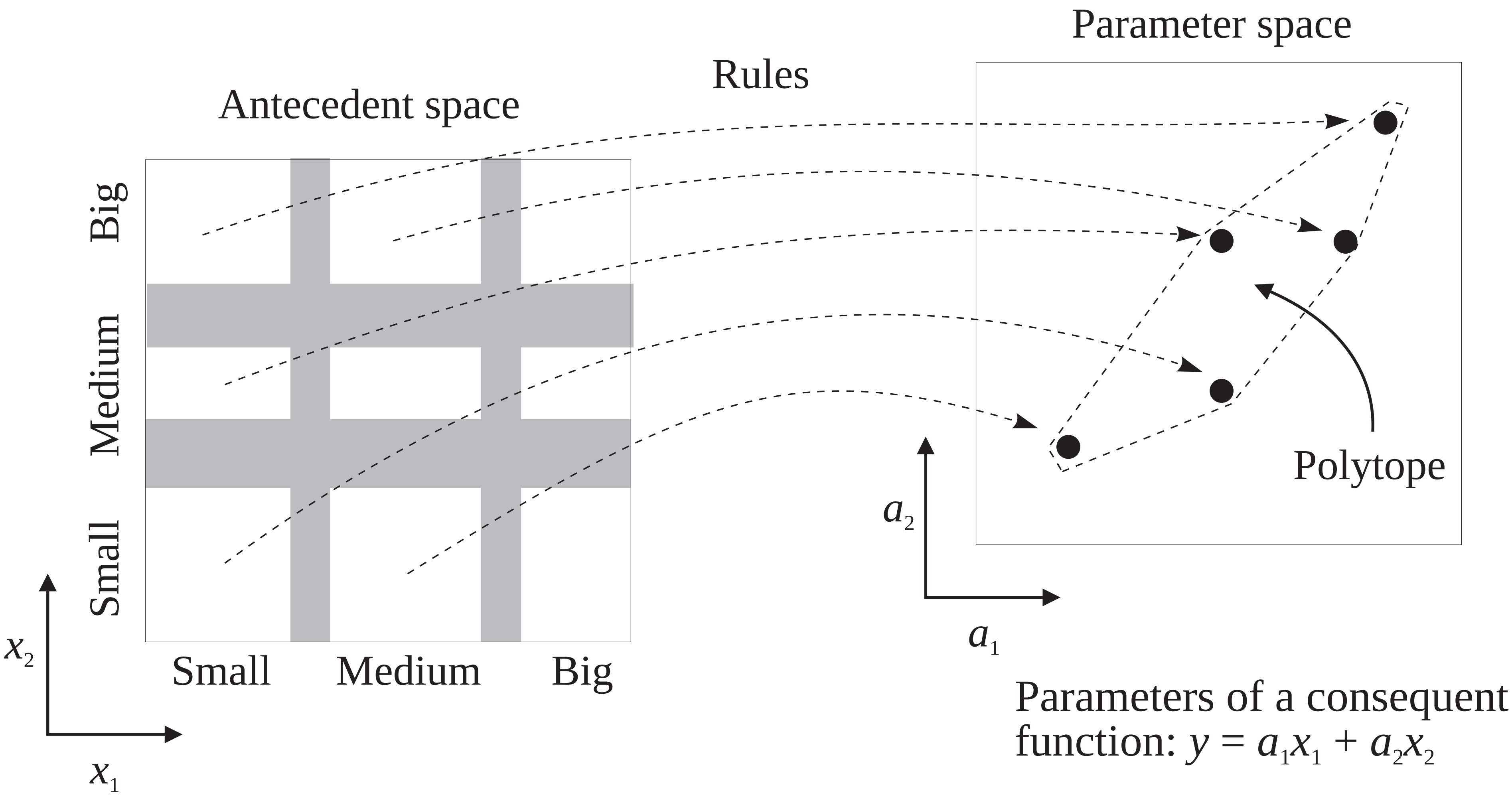
TS Model is a Quasi-Linear System

$$y = \frac{\sum_{i=1}^{K} \mu_{A_i}(\mathbf{x}) y_i}{\sum_{j=1}^{K} \mu_{A_j}(\mathbf{x})} = \frac{\sum_{i=1}^{K} \mu_{A_i}(\mathbf{x})(\mathbf{a}_i^T \mathbf{x} + b_i)}{\sum_{j=1}^{K} \mu_{A_j}(\mathbf{x})}$$

$$y = \underbrace{\left(\sum_{i=1}^{K} \gamma_i(\mathbf{x}) \mathbf{a}_i^T \right)}_{\mathbf{a}(\mathbf{x})} \mathbf{x} + \underbrace{\sum_{i=1}^{K} \gamma_i(\mathbf{x}) b_i}_{\mathbf{b}(\mathbf{x})}$$

linear in parameters a_i and b_i, pseudo-linear in x (LPV)

TS Model is a Polytopic System



Modeling of Dynamic Systems

Nonlinear regression model:

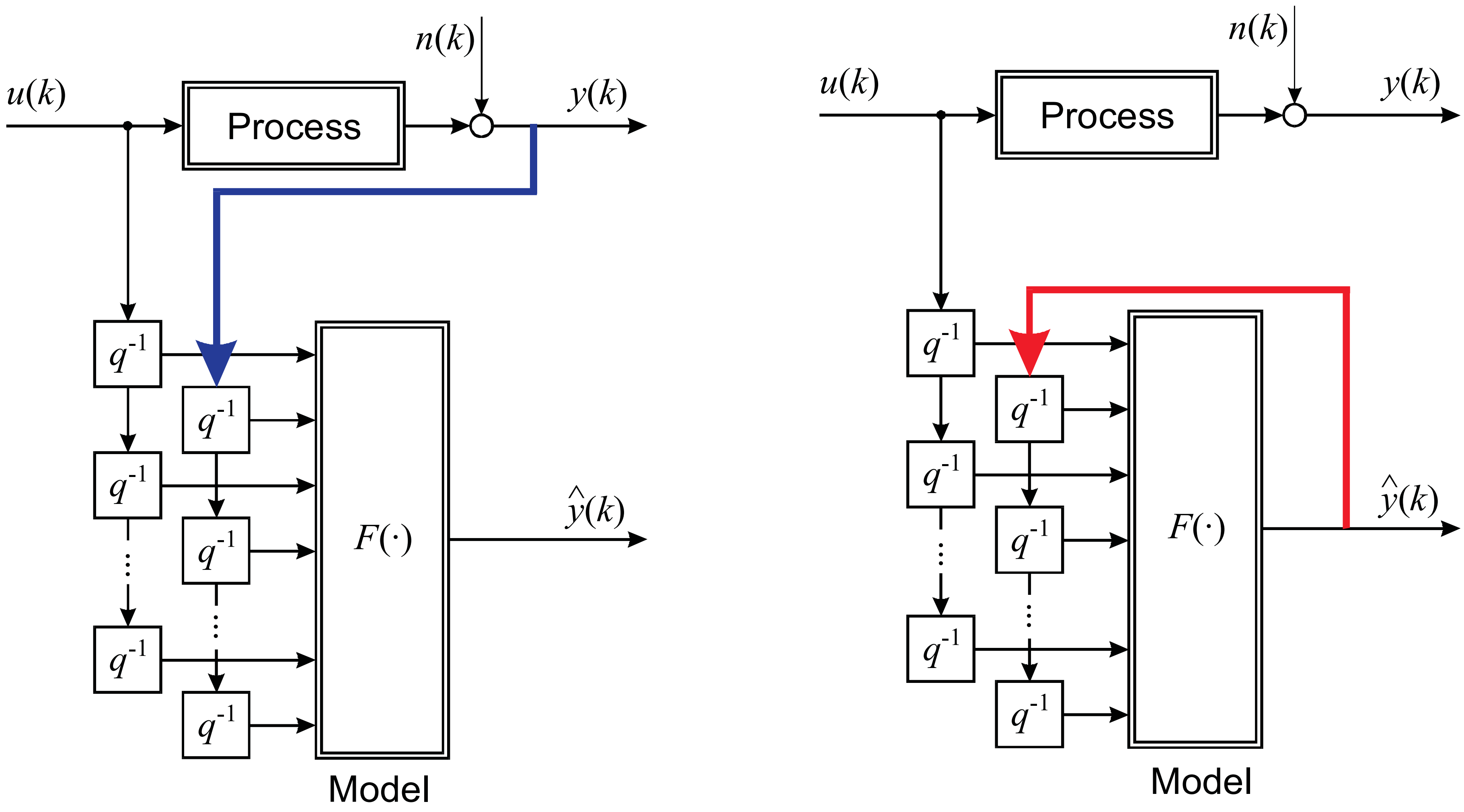
$$\hat{y}(k+1) = F\left[y(k), \ldots, y(k-n_y+1), u(k), \ldots, u(k-n_u+1)\right]$$

In rule-based form (example TS model):

If $y(k)$ **is Small and** $u(k)$ **is Large**

then $\hat{y}(k+1) = \sum_{j=1}^{n_y} a_{ij} y(k-j+1) + \sum_{j=1}^{n_u} b_{ij} u(k-j+1) + c_i$

One-Step-Ahead Prediction vs. Simulation



Construction of Fuzzy Models

Modeling Paradigms

- **Mechanistic** (white-box, physical)

- **Qualitative** (naive physics, knowledge-based)

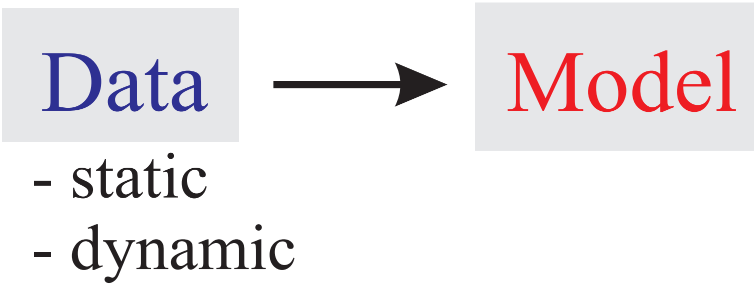
- **Data-driven** (black-box, inductive)

Often combination of different approaches semi-mechanistic, gray-box modeling.

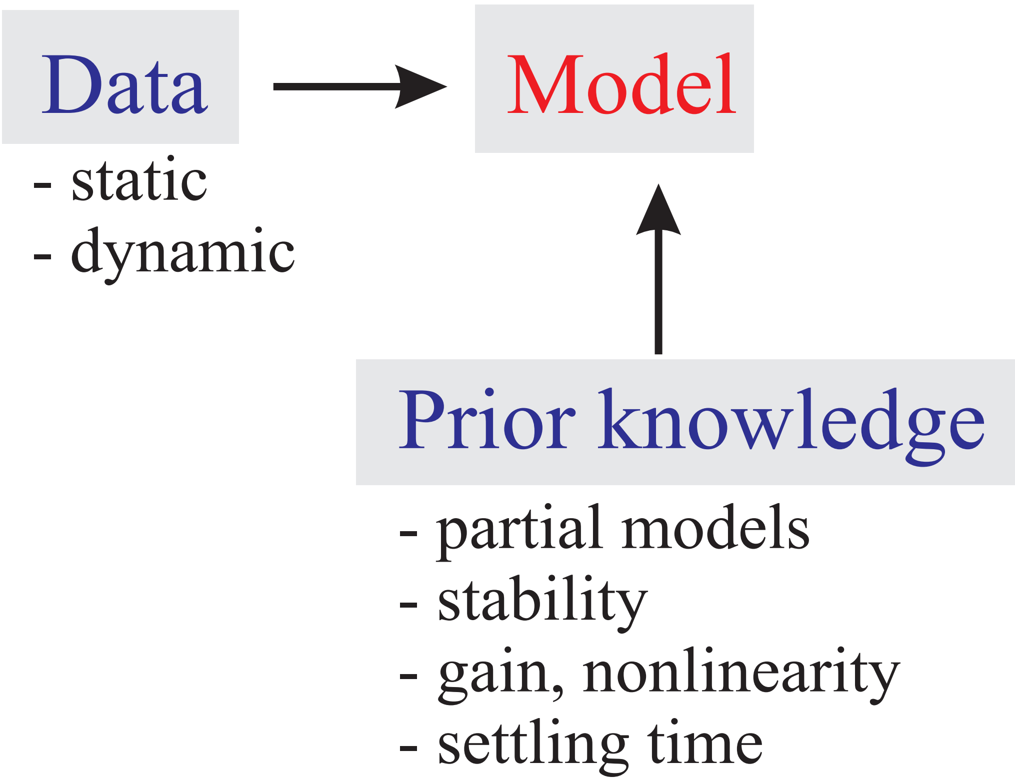
Parameterization of nonlinear models

- polynomials, splines
- look-up tables
- fuzzy systems
- neural networks
- (neuro-)fuzzy systems
- radial basis function networks
- wavelet networks
- . . .

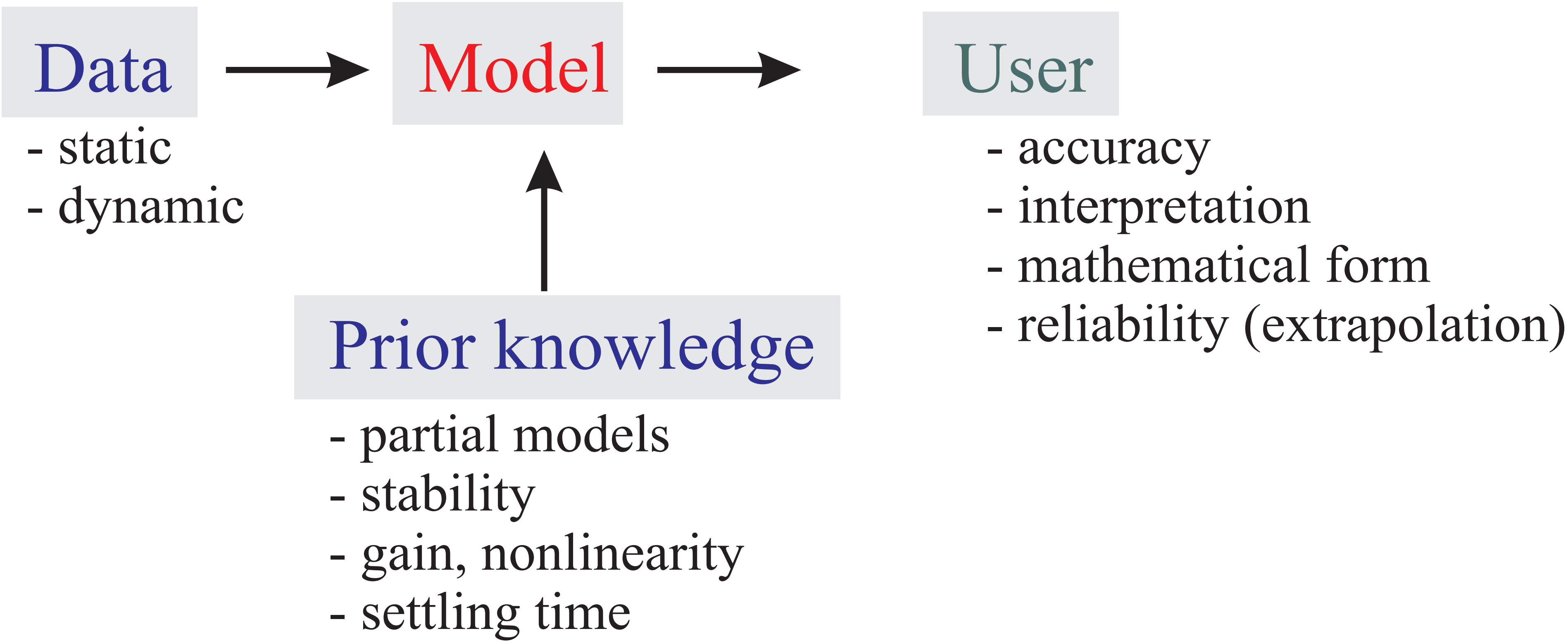
Modeling of Complex Systems



Modeling of Complex Systems



Modeling of Complex Systems



Building Fuzzy Models

Knowledge-based approach:

- expert knowledge $\rightarrow$ rules & membership functions
- fuzzy model of human operator
- linguistic interpretation

Building Fuzzy Models

Knowledge-based approach:

- expert knowledge $\rightarrow$ rules & membership functions
- fuzzy model of human operator
- linguistic interpretation

Data-driven approach:

- nonlinear mapping, universal approximation
- extract rules & membership functions from data

Knowledge-Based Modeling

- Problems where little or no data are available.
- Similar to expert systems.
- Presence of both quantitative and qualitative variables or parameters.

Typical applications: fuzzy control and decision support, but also modeling of poorly understood processes

Wear Prediction for a Trencher

Trencher T-850 (Vermeer)

Chain Detail

Goal: Given the terrain properties, predict bit wear and production rate of trencher.

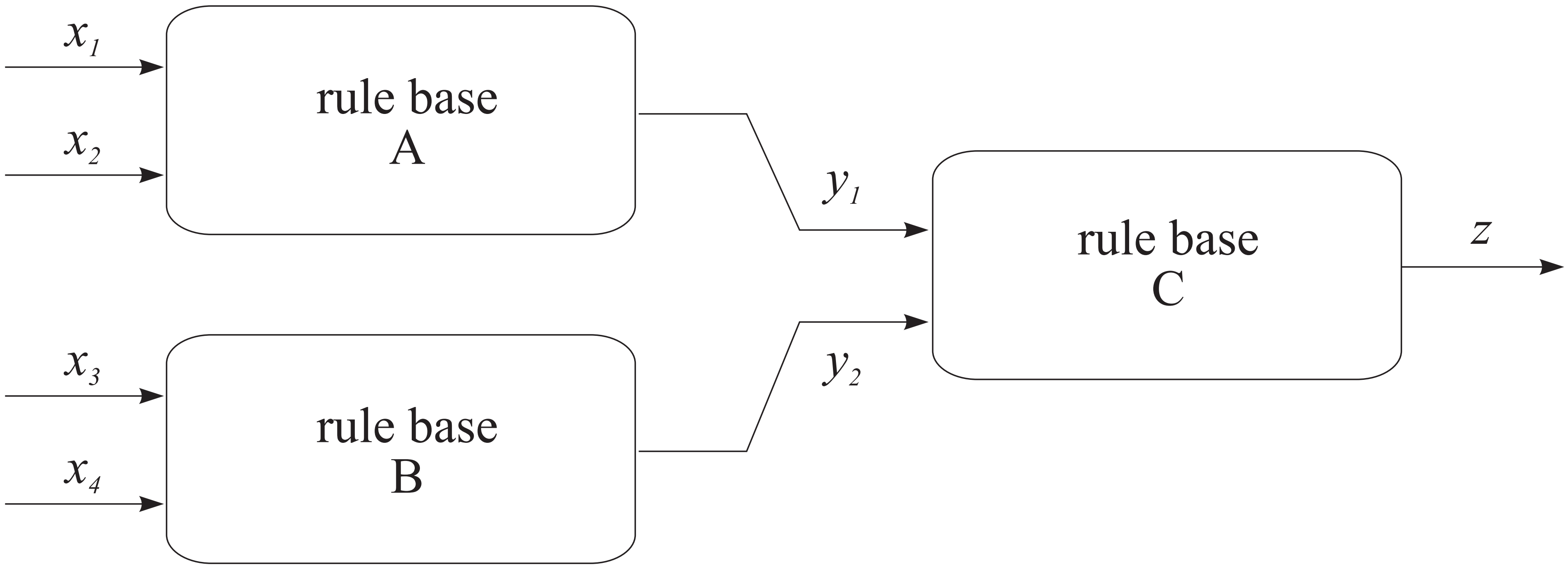
Why Knowledge-Based Modeling?

- Interaction between tool and environment is complex, dynamic and highly nonlinear, rigorous mathematical models are not available.
- Little data (15 data points) to develop statistical regression models.
- Input data are a mixture of numerical measurements (rock strength, joint spacing, trench dimensions) and qualitative information (joint orientation).
- Precise numerical output not needed, qualitative assessment is sufficient.

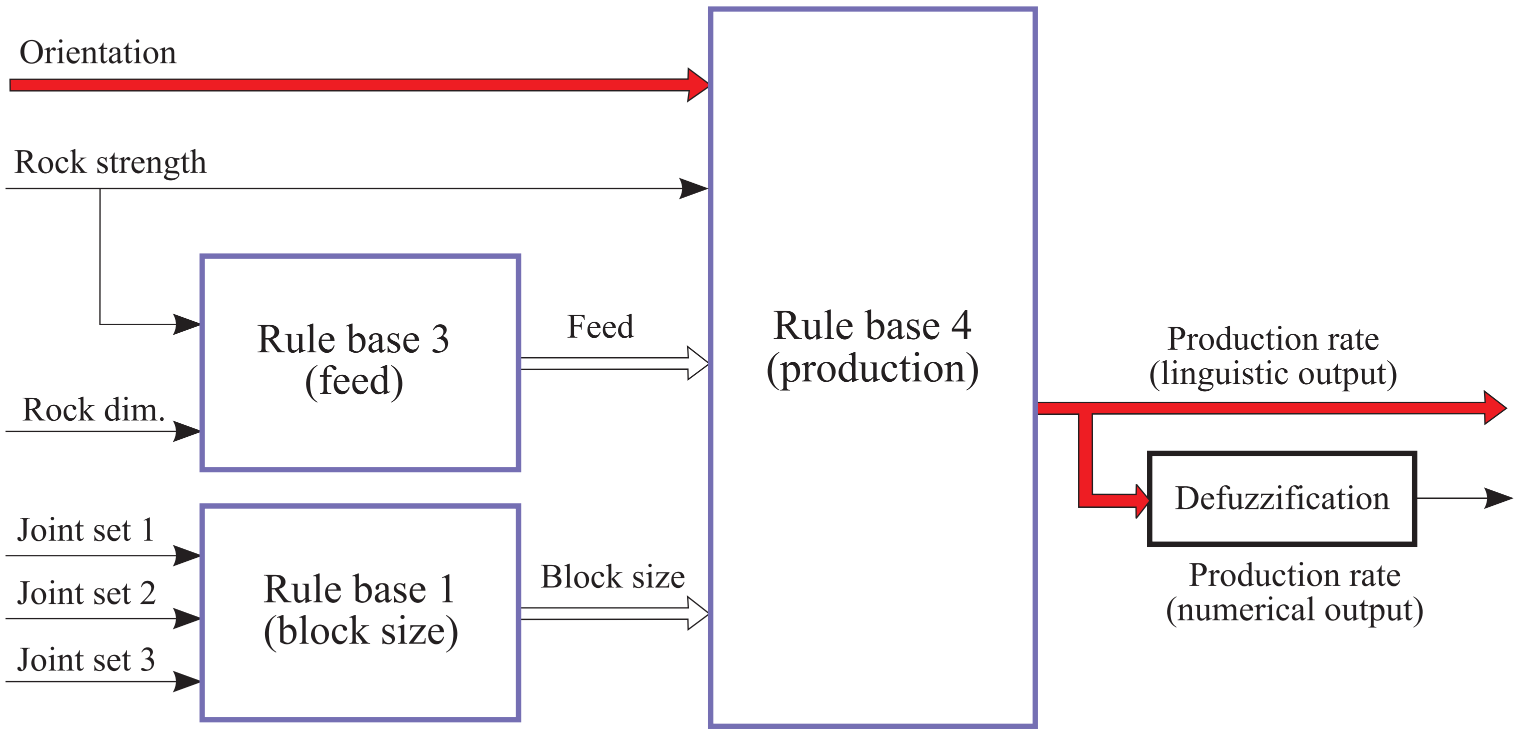
Dimensionality Problem: Hierarchical Structure

Assume 5 membership functions for each input

625 rules in a flat rule base vs. 75 rules in a hierarchical one

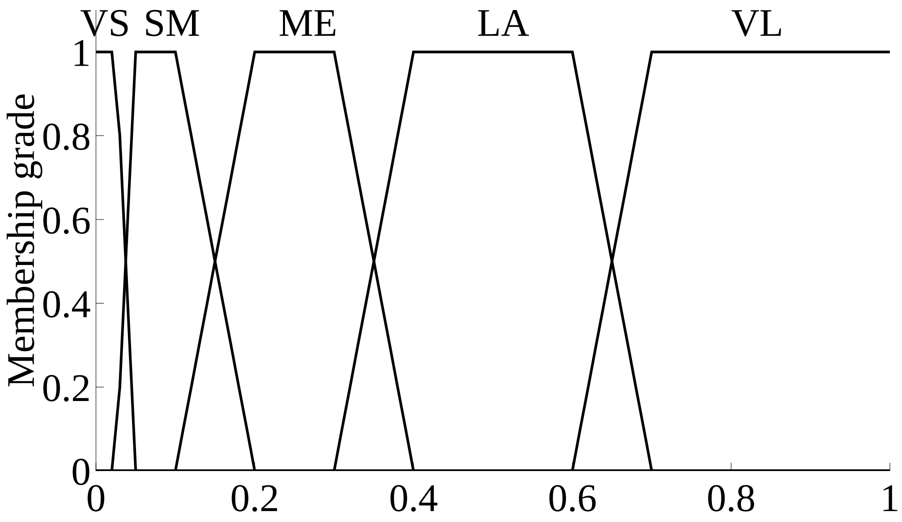


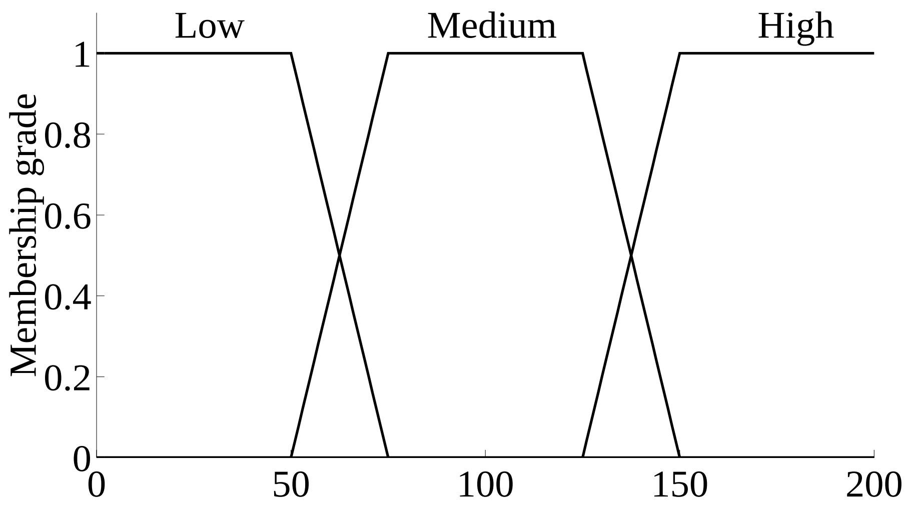
Trencher: Fuzzy Rule Bases

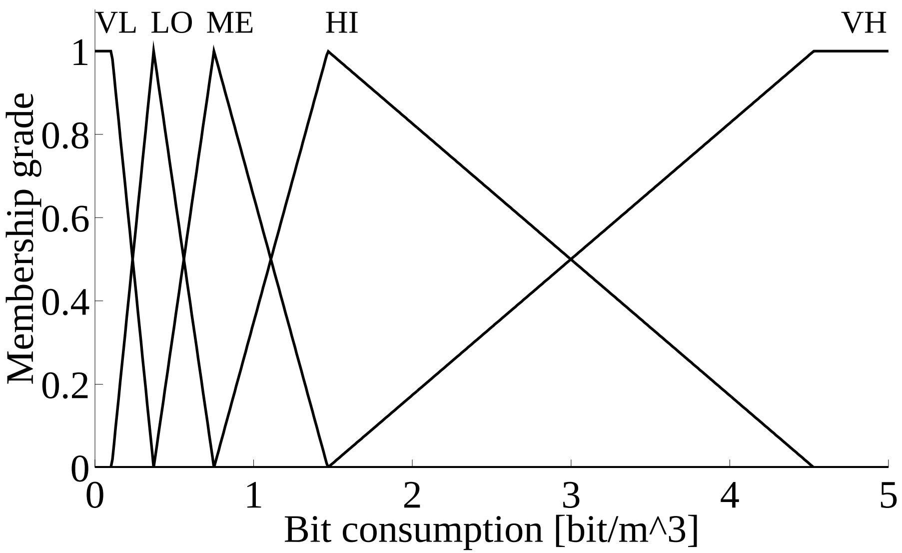



```
If TRENCH-DIM is SMALL  and STRENGTH is LOW    Then FEED is VERY-HIGH;
If TRENCH-DIM is SMALL  and STRENGTH is MEDIUM Then FEED is HIGH;
....
If JOINT-SP is EXTRA-LARGE and FEED is VERY-HIGH Then PROD is VERY-HIGH
```

Example of Membership Functions







Output: Prediction of Production Rate

data no.	measured value	predicted linguistic value(s)			
1	2.07	VERY-LOW	1.00		
2	5.56	HIGH	1.00		
3	23.60	VERY-HIGH	0.50		
4	11.90	HIGH	0.40	VERY-HIGH	0.60
5	7.71	MEDIUM	1.00		
6	7.17	LOW	0.72		
7	8.05	MEDIUM	0.80		
8	7.39	LOW	1.00		
9	4.58	LOW	0.50		
10	8.74	MEDIUM	1.00		
11	134.84	EXTREMELY-HIGH	1.00		

Data-Driven Construction

Structure and Parameters

Structure:

- Input and output variables. For dynamic systems also the representation of the dynamics.
- Number of membership functions per variable, type of membership functions, number of rules.

Parameters:

- Consequent parameters (least squares).
- Antecedent membership functions (various methods).

Least-Squares Estimation of Singletons

$$R_i\colon\ \ \textbf{If } \mathbf{x} \textbf{ is } A_i \textbf{ then } y = b_i$$

- **Given A_i and a set of input–output data:**

$$\{(\mathbf{x}_k, y_k) \mid k = 1, 2, \ldots, N\}$$

- **Estimate optimal consequent parameters b_i.**

Least-Squares Estimation of Singletons

1. Compute the membership degrees $\mu_{A_i}(\mathbf{x}_k)$

2. Normalize

$$\gamma_{ki} = \mu_{A_i}(\mathbf{x}_k) / \sum_{j=1}^{K} \mu_{A_j}(\mathbf{x}_k)$$

(Output: $y_k = \sum_{i=1}^{K} \gamma_{ki} b_i$**, in a matrix form:** $\mathbf{y} = \mathbf{\Gamma b}$**)**

3. Least-squares estimate: $\mathbf{b} = \left[\mathbf{\Gamma}^T \mathbf{\Gamma}\right]^{-1} \mathbf{\Gamma}^T \mathbf{y}$

Least-Square Estimation of TS Consequents

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_N^T \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}, \quad \mathbf{W}_i = \begin{bmatrix} \gamma_{i1} & 0 & \cdots & 0 \\ 0 & \gamma_{i2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_{iN} \end{bmatrix}$$

$$\boldsymbol{\theta}_i = \left[\mathbf{a}_i^T \quad b_i\right]^T, \quad \mathbf{X}_e = [\mathbf{X} \quad \mathbf{1}]$$

Least-Square Estimation of TS Consequents

- **Global LS:** $\boldsymbol{\theta}' = \left[(\mathbf{X}')^T \mathbf{X}' \right]^{-1} (\mathbf{X}')^T \mathbf{y}$

  with $\mathbf{X}' = [\mathbf{W}_1 \mathbf{X}_e \quad \mathbf{W}_2 \mathbf{X}_e \quad \ldots \quad \mathbf{W}_c \mathbf{X}_e]$

  and $\boldsymbol{\theta}' = \left[\boldsymbol{\theta}_1^T \quad \boldsymbol{\theta}_2^T \quad \ldots \quad \boldsymbol{\theta}_c^T \right]^T$

Least-Square Estimation of TS Consequents

- **Global LS:** $\boldsymbol{\theta}' = \left[(\mathbf{X}')^T \mathbf{X}'\right]^{-1} (\mathbf{X}')^T \mathbf{y}$

  with $\mathbf{X}' = [\mathbf{W}_1 \mathbf{X}_e \quad \mathbf{W}_2 \mathbf{X}_e \quad \dots \quad \mathbf{W}_c \mathbf{X}_e]$

  and $\boldsymbol{\theta}' = \left[\boldsymbol{\theta}_1^T \quad \boldsymbol{\theta}_2^T \quad \dots \quad \boldsymbol{\theta}_c^T\right]^T$

- **Local LS:** $\boldsymbol{\theta}_i = \left[\mathbf{X}_e^T \mathbf{W}_i \mathbf{X}_e\right]^{-1} \mathbf{X}_e^T \mathbf{W}_i \mathbf{y}$

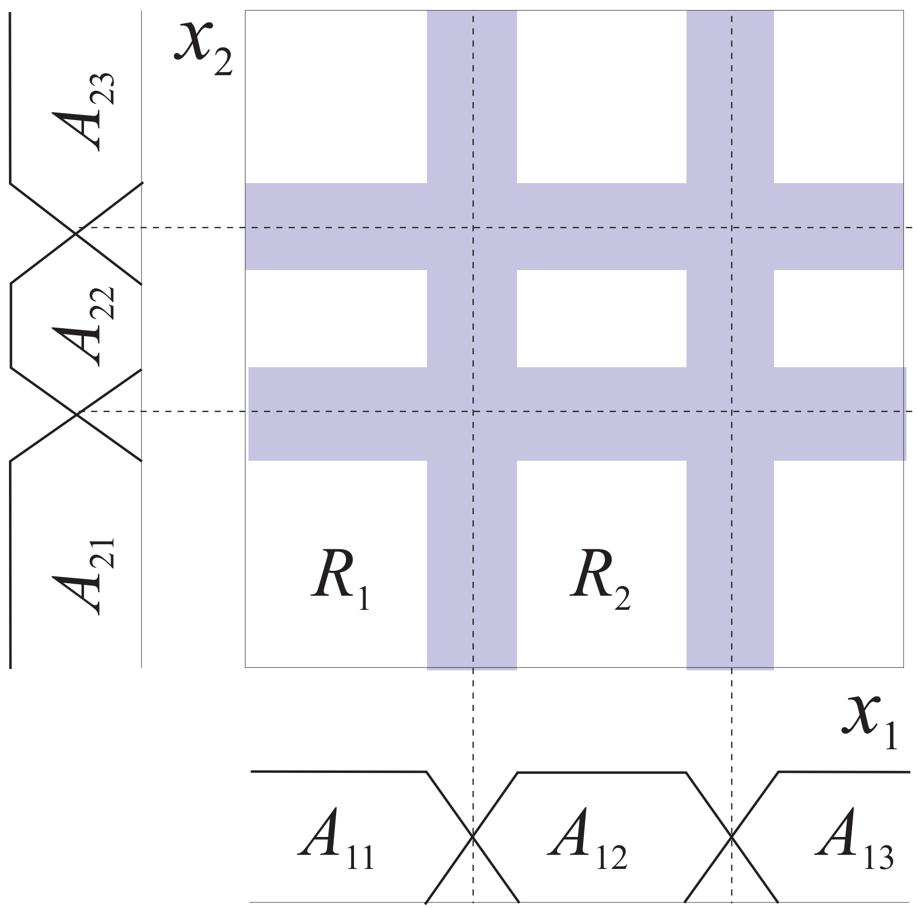
Antecedent Membership Functions

- templates (grid partitioning),
- nonlinear optimization (neuro-fuzzy methods),
- tree-construction
- product space fuzzy clustering

Template-Based Modeling

Determine membership functions a priori (shape, number).

- **Only for small problems (1 to 3 inputs).**

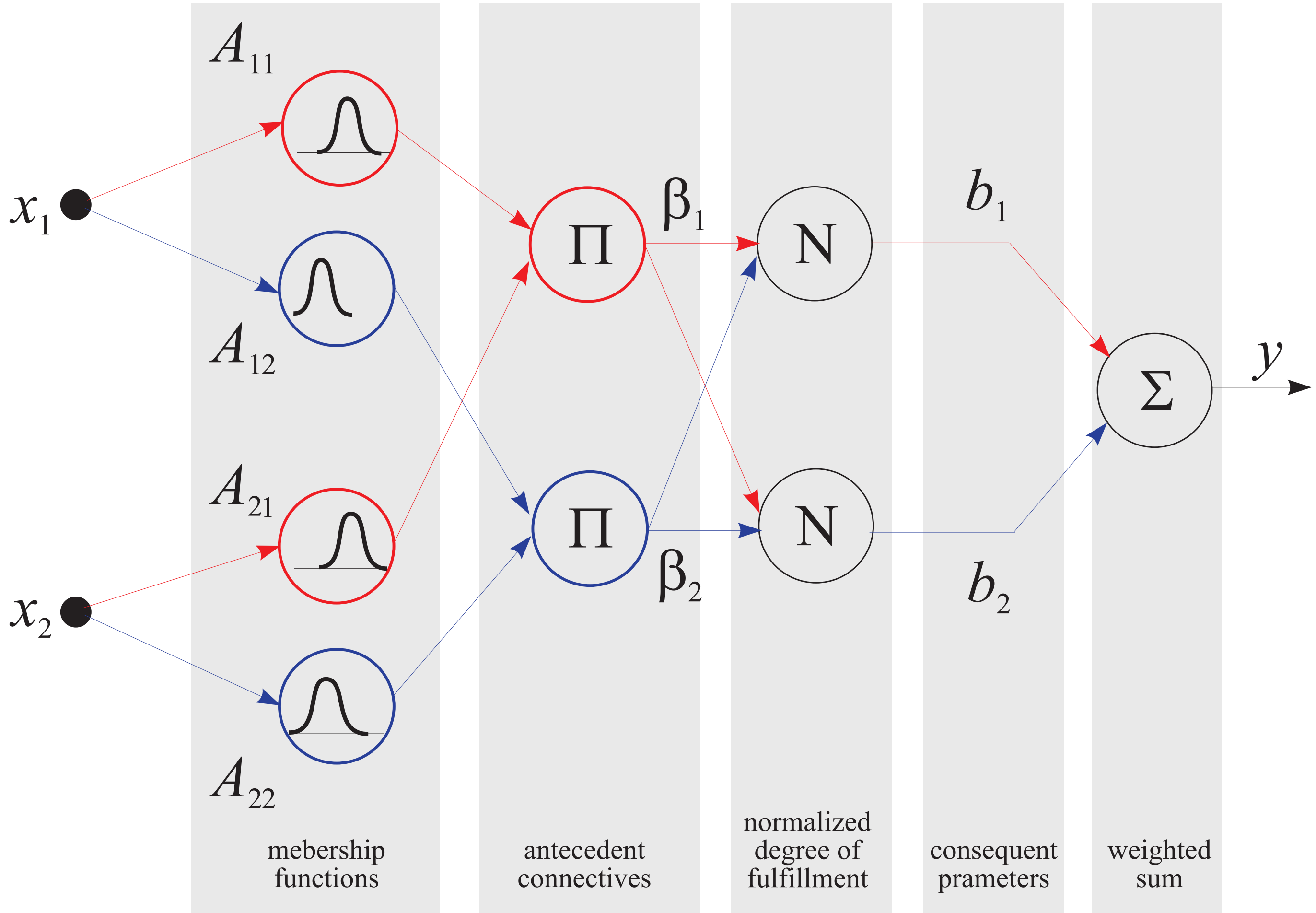


$$K = \prod_{j=1}^{p} N_j$$

Nonlinear Optimization (Neuro–Fuzzy Learning)

If x_1 **is** A_{11} **and** x_2 **is** A_{21} **then** $y = b_1$

If x_1 **is** A_{12} **and** x_2 **is** A_{22} **then** $y = b_2$



Smooth Membership Functions

$$\mu(x; c, \sigma) = \exp\left(-(\frac{x-c}{2\sigma})^2\right)$$

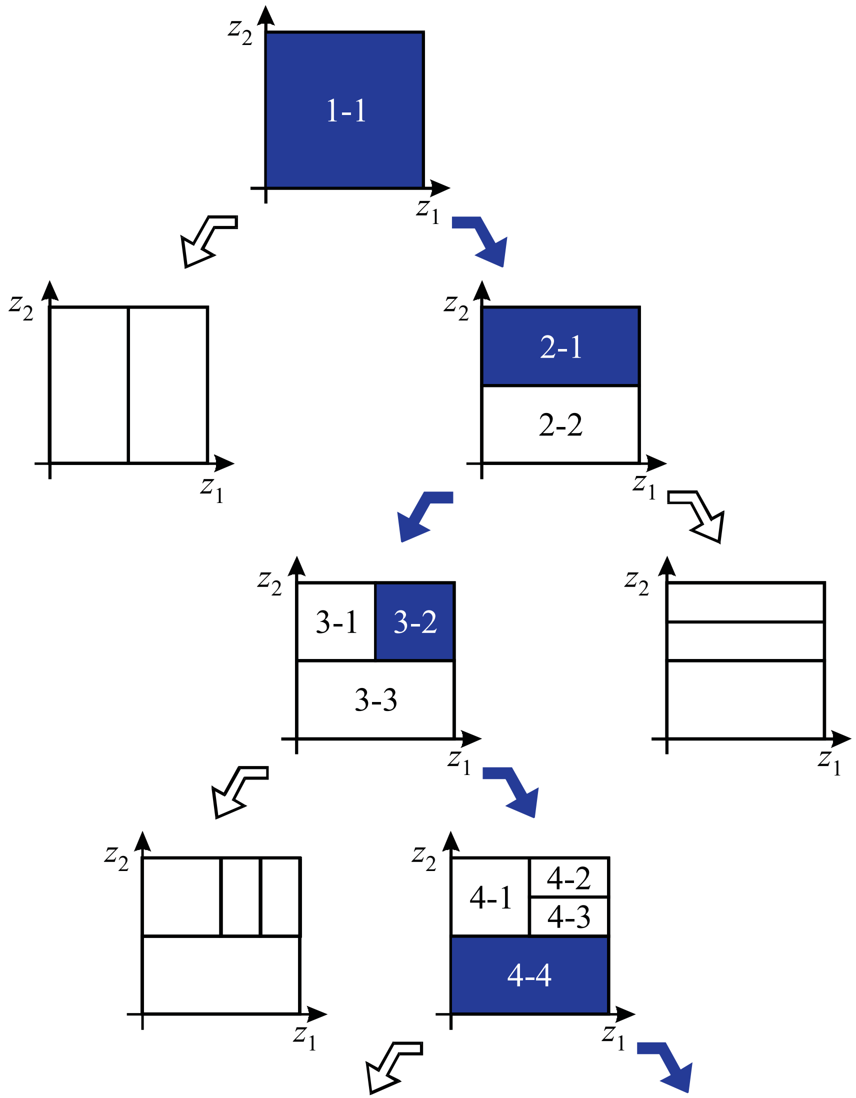
$$y = \frac{\sum_{i=1}^{K} \exp\left(-(\frac{x-c_i}{2\sigma_i})^2\right) b_i}{\sum_{i=1}^{K} \exp\left(-(\frac{x-c_i}{2\sigma_i})^2\right)}$$

Adjust parameters c_i and σ_i (nonlinear optimization):

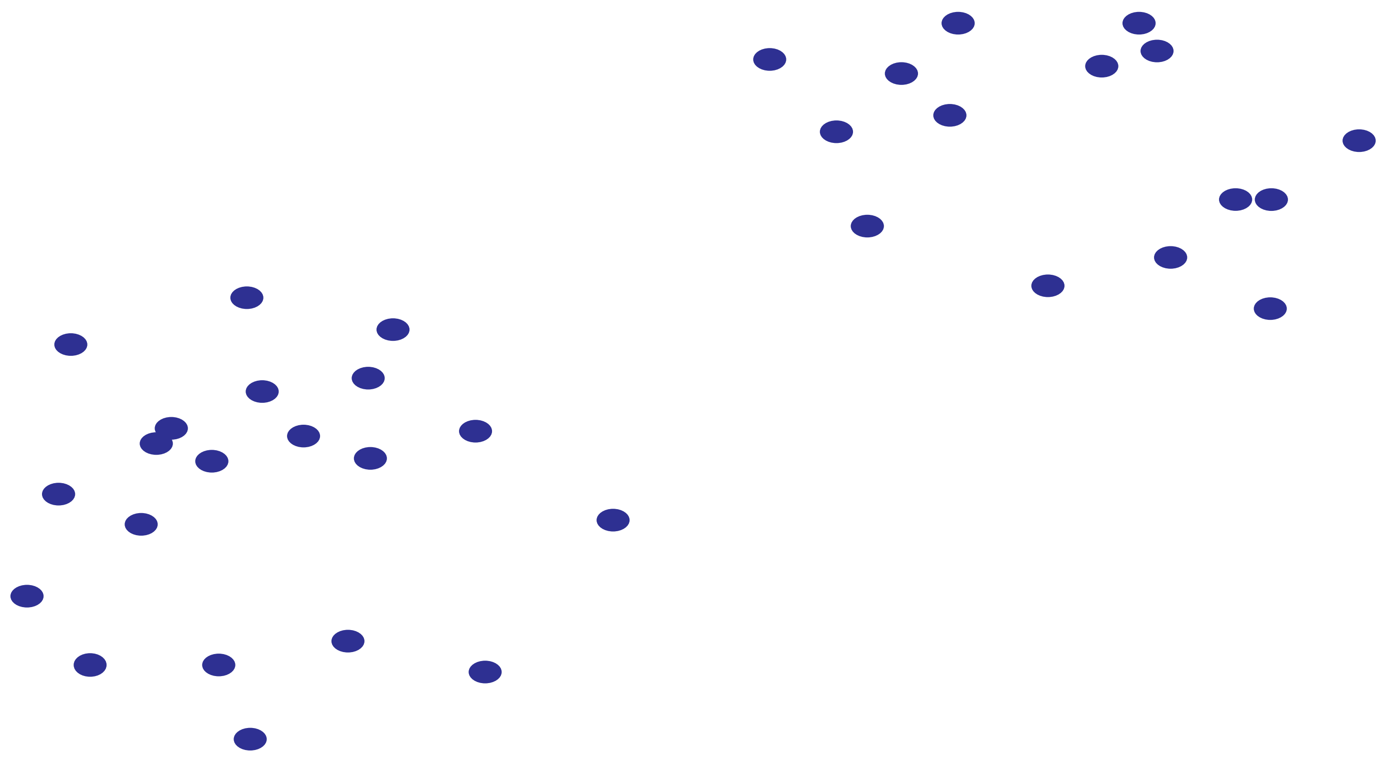
- **Gradient-based: (back-propagation, Levenberg-Marquardt).**
- **Gradient-free: (Nelder-Mead, GA, simulated annealing).**

Tree-Construction Methods

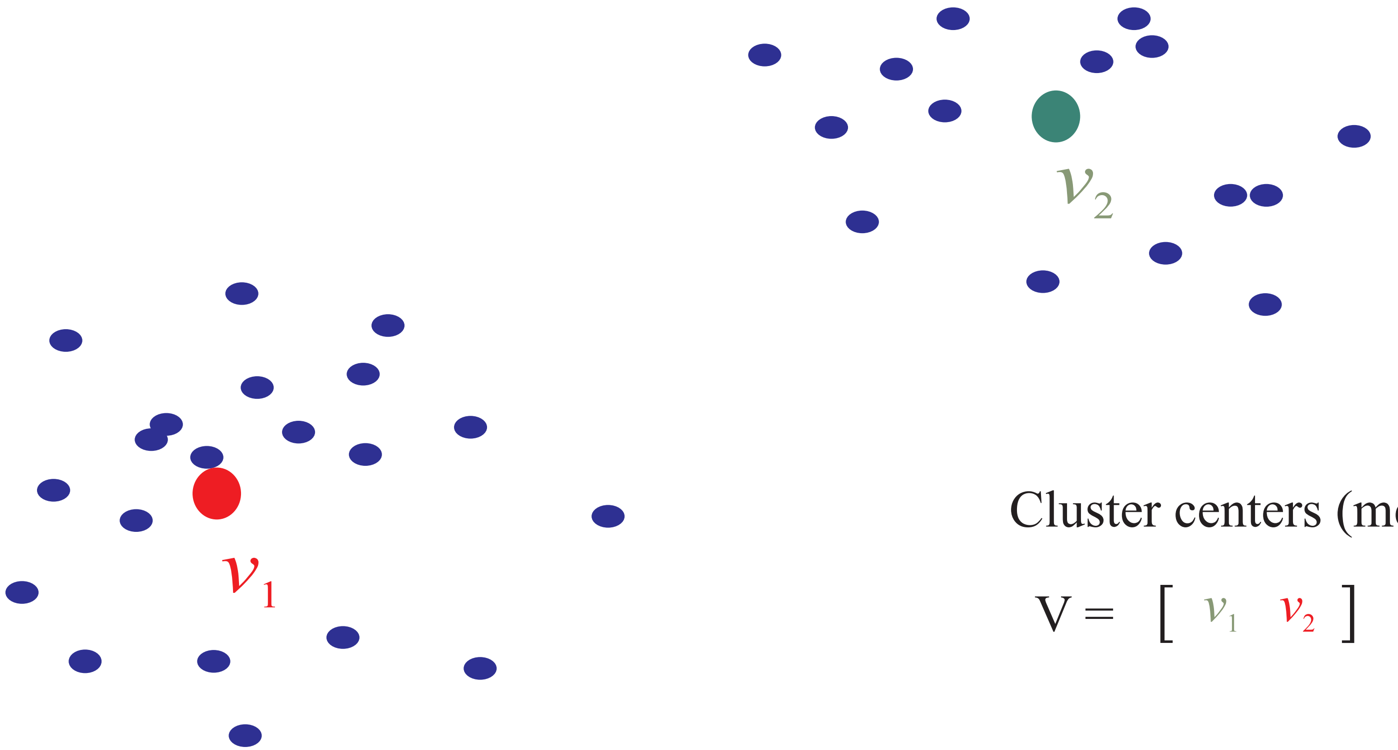
- Growing: Adds one LLM/rule in each iteration
- Axis-orthogonal partition of the input space
- Placement of the membership functions
  1. Division of the worst performing LLM
  2. Test division in each input dimension
  3. Best performing division is realized
- Separate estimation of the newly generated LLMs (weighted least squares)



Fuzzy Clustering: Data



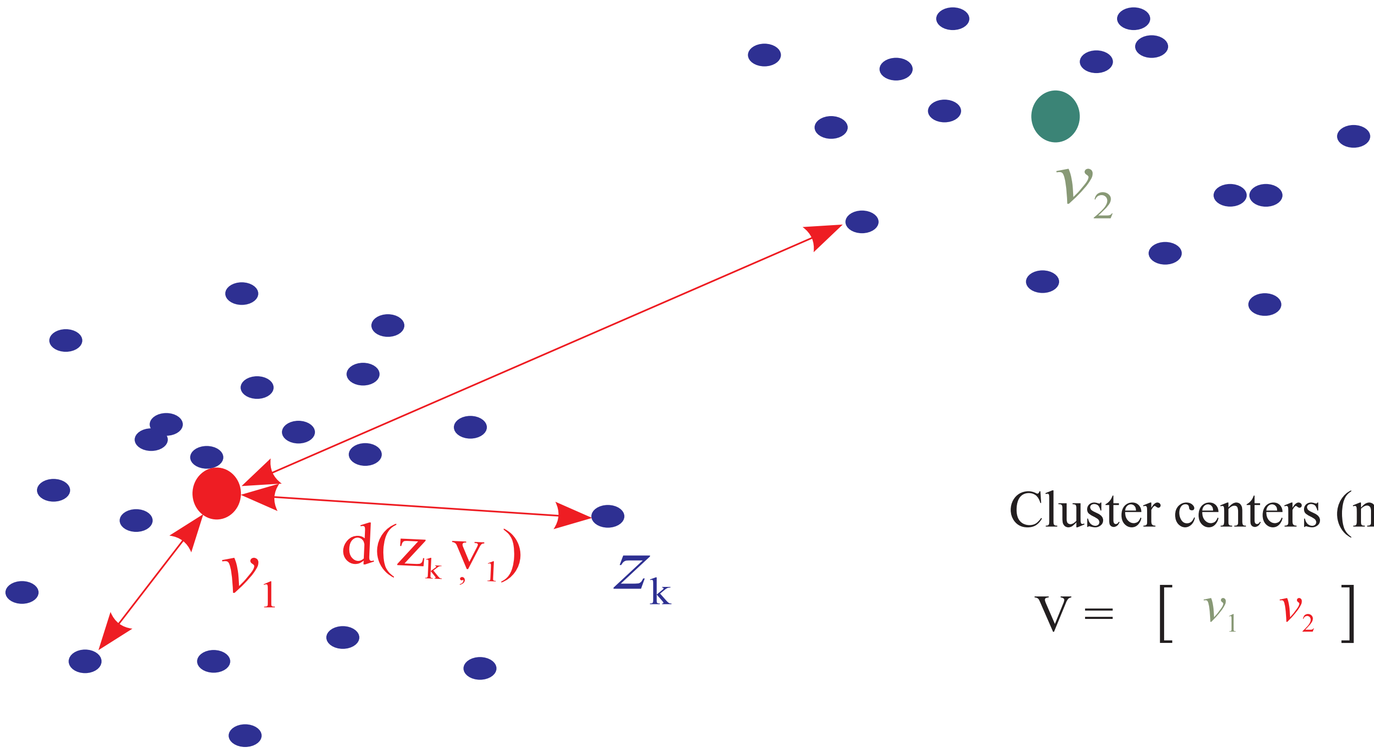
Fuzzy Clustering: Prototypes



Cluster centers (means):

$$V = [\ v_1 \quad v_2\ ]$$

Fuzzy Clustering: Distance



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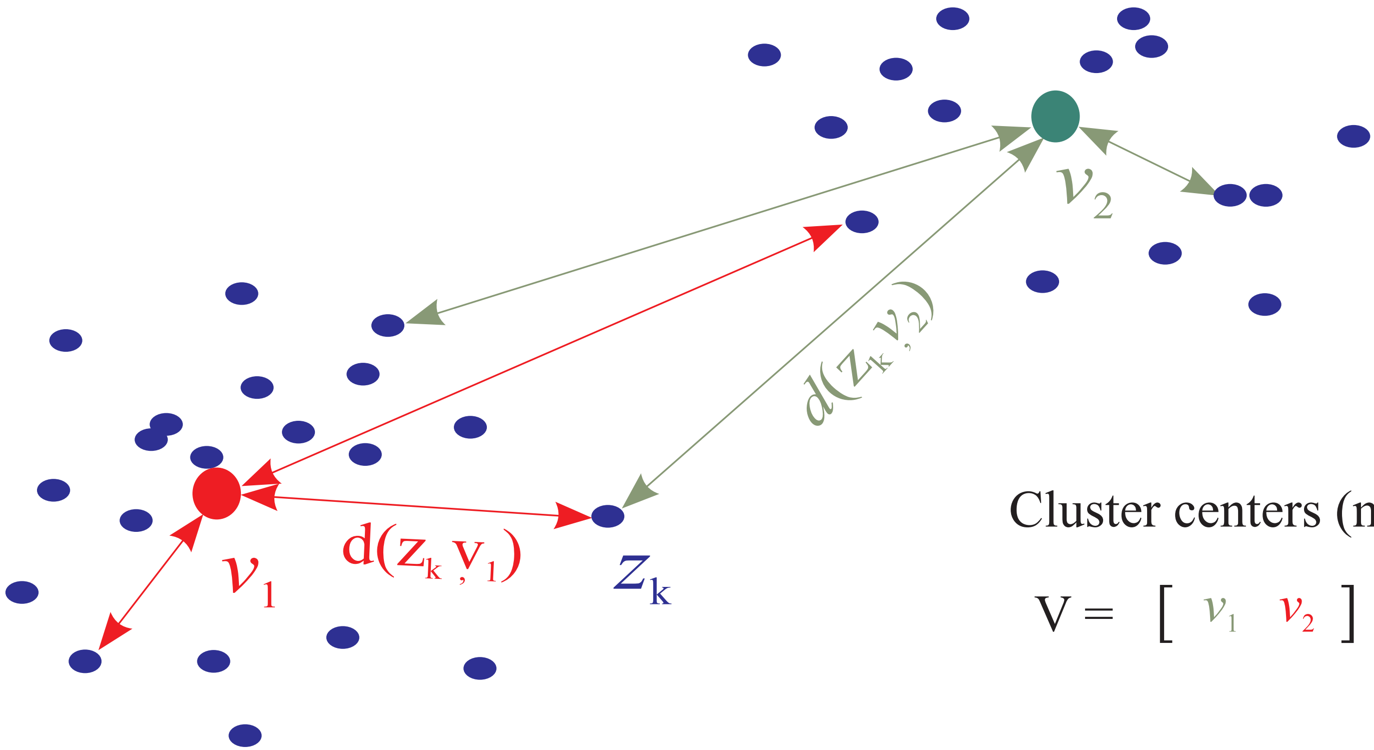
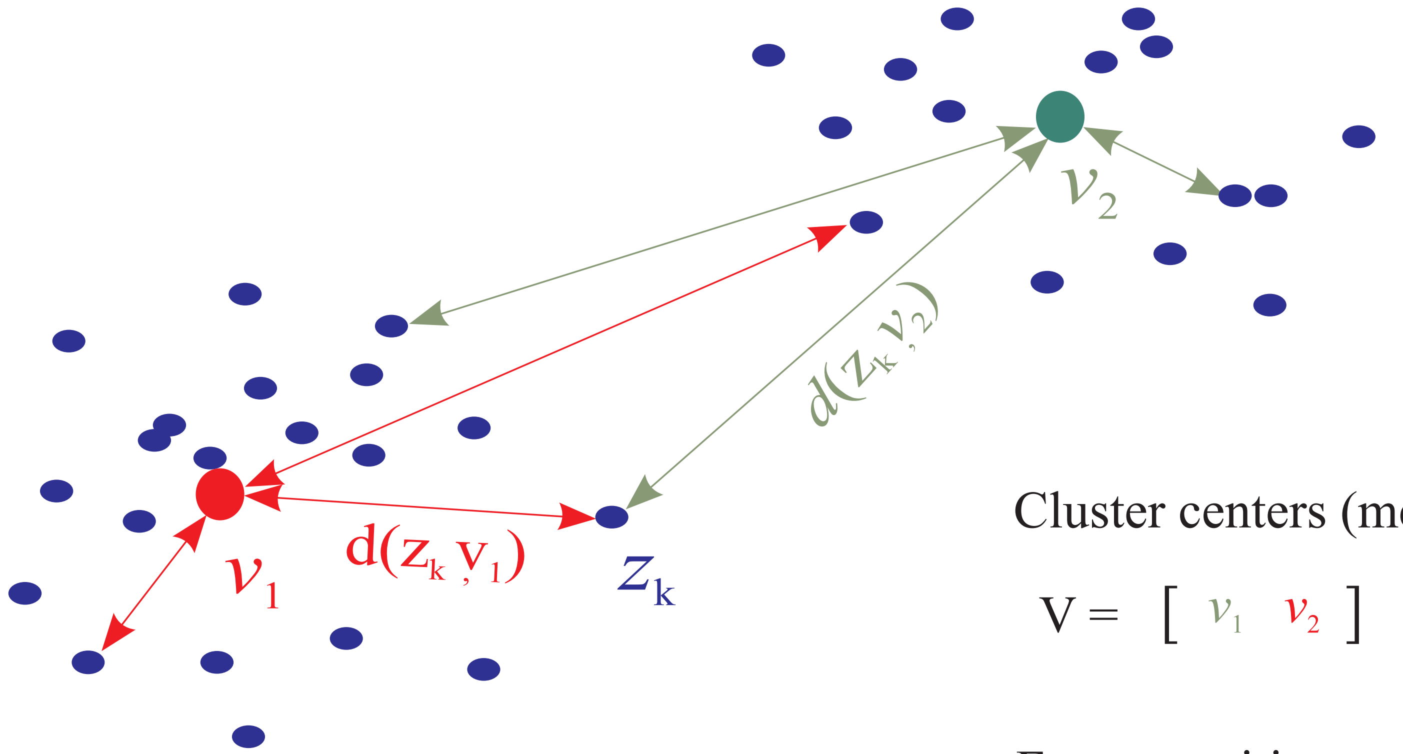


Cluster centers (means):

$V = [\ v_1\ \ v_2\ ]$

Fuzzy Clustering: Partition Matrix



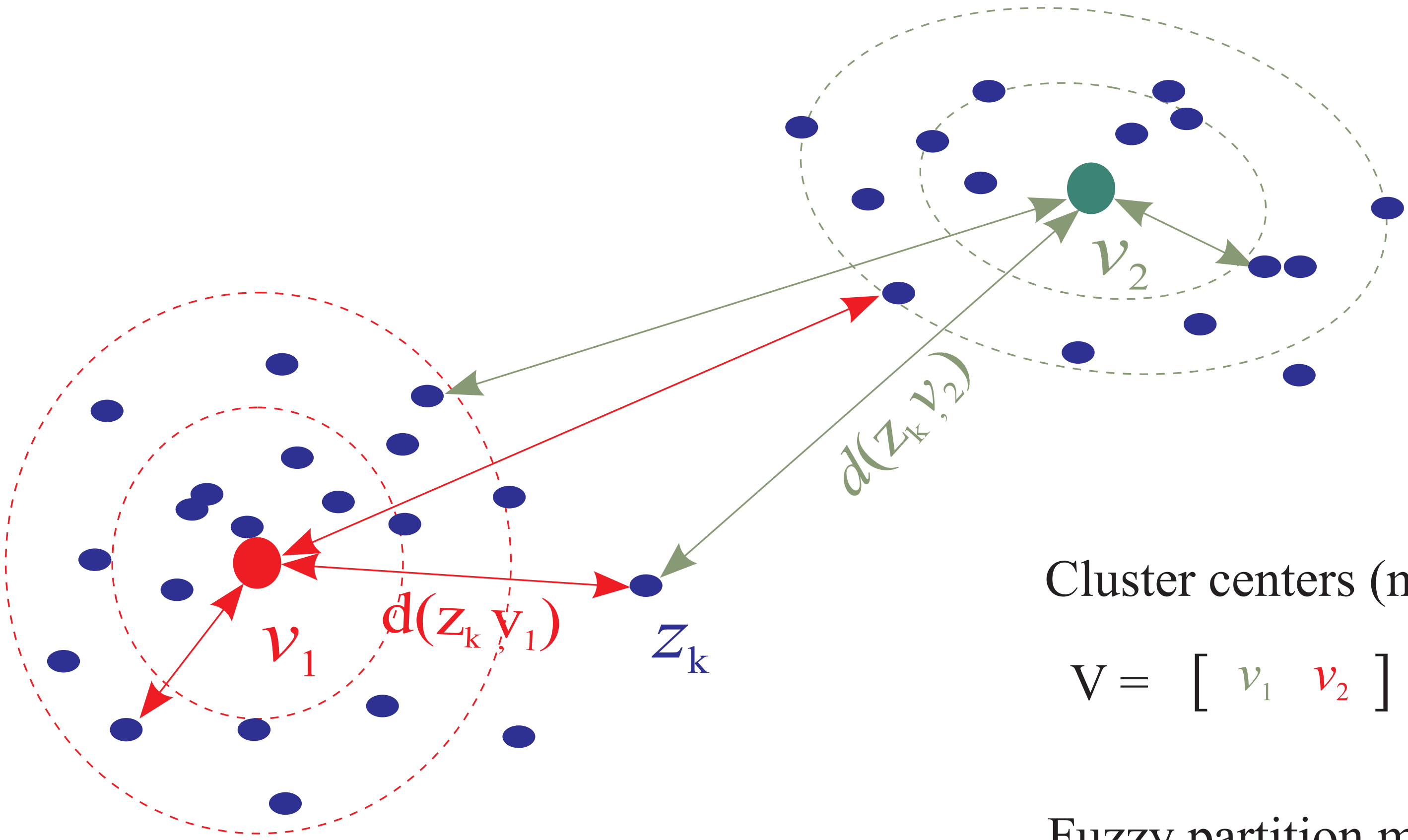
Cluster centers (means):

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

Fuzzy partition matrix:

$$U = \begin{bmatrix} \mu_{11} & \mu_{12} & \dots & \mu_{1N} \\ \mu_{21} & \mu_{22} & \dots & \mu_{2N} \end{bmatrix}$$

Fuzzy Clustering: Shapes



Cluster centers (means):

$$V = [\; v_1 \quad v_2 \;]$$

Fuzzy partition matrix:

$$U = \begin{bmatrix} \mu_{11} & \mu_{12} & \dots & \mu_{1N} \\ \mu_{21} & \mu_{22} & \dots & \mu_{2N} \end{bmatrix}$$

Fuzzy Clustering Problem

Given the data:

$$\mathbf{z}_k = [z_{1k}, z_{2k}, \ldots, z_{nk}]^T \in \mathbb{R}^n, \quad k = 1, \ldots, N$$

Find:

the fuzzy partition matrix:

$$\mathbf{U} = \begin{bmatrix} \mu_{11} & \ldots & \mu_{1k} & \ldots & \mu_{1N} \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ \mu_{c1} & \ldots & \mu_{ck} & \ldots & \mu_{cN} \end{bmatrix}$$

and the cluster centers:

$$\mathbf{V} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_c\}, \quad \mathbf{v}_i \in \mathbb{R}^n$$

Fuzzy Clustering: an Optimization Approach

Objective function (least-squares criterion):

$$J(Z; V, U, A) = \sum_{i=1}^{c} \sum_{j=1}^{N} \mu_{i,j}^{m} d_{A_i}^2(\mathbf{z}_j, \mathbf{v}_i)$$

subject to constraints:

$$0 \le \mu_{i,j} \le 1, \qquad i = 1, \ldots, c, \; j = 1, \ldots, N \qquad \textbf{membership degree}$$

$$0 < \sum_{j=1}^{N} \mu_{i,j} < 1, \qquad i = 1, \ldots, c \qquad \textbf{no cluster empty}$$

$$\sum_{i=1}^{c} \mu_{i,j} = 1, \qquad j = 1, \ldots, N \qquad \textbf{total membership}$$

Fuzzy c-Means Algorithm

Repeat:

1. Compute cluster prototypes (means): $v_i = \dfrac{\sum_{k=1}^{N} \mu_{i,k}^m \mathbf{z}_k}{\sum_{k=1}^{N} \mu_{i,k}^m}$

Fuzzy c-Means Algorithm

Repeat:

1. Compute cluster prototypes (means): $v_i = \frac{\sum_{k=1}^{N} \mu_{i,k}^m \mathbf{z}_k}{\sum_{k=1}^{N} \mu_{i,k}^m}$

2. Calculate distances: $d_{ik} = (\mathbf{z}_k - \mathbf{v}_i)^T (\mathbf{z}_k - \mathbf{v}_i)$

Fuzzy c-Means Algorithm

Repeat:

1. Compute cluster prototypes (means): $v_i = \frac{\sum_{k=1}^{N} \mu_{i,k}^m \mathbf{z}_k}{\sum_{k=1}^{N} \mu_{i,k}^m}$

2. Calculate distances: $d_{ik} = (\mathbf{z}_k - \mathbf{v}_i)^T (\mathbf{z}_k - \mathbf{v}_i)$

3. Update partition matrix: $\mu_{ik} = \frac{1}{\sum_{j=1}^{c} (d_{ik}/d_{jk})^{1/(m-1)}}$

until $\|\Delta \mathbf{U}\| < \epsilon$

Distance Measures

- **Euclidean norm:**

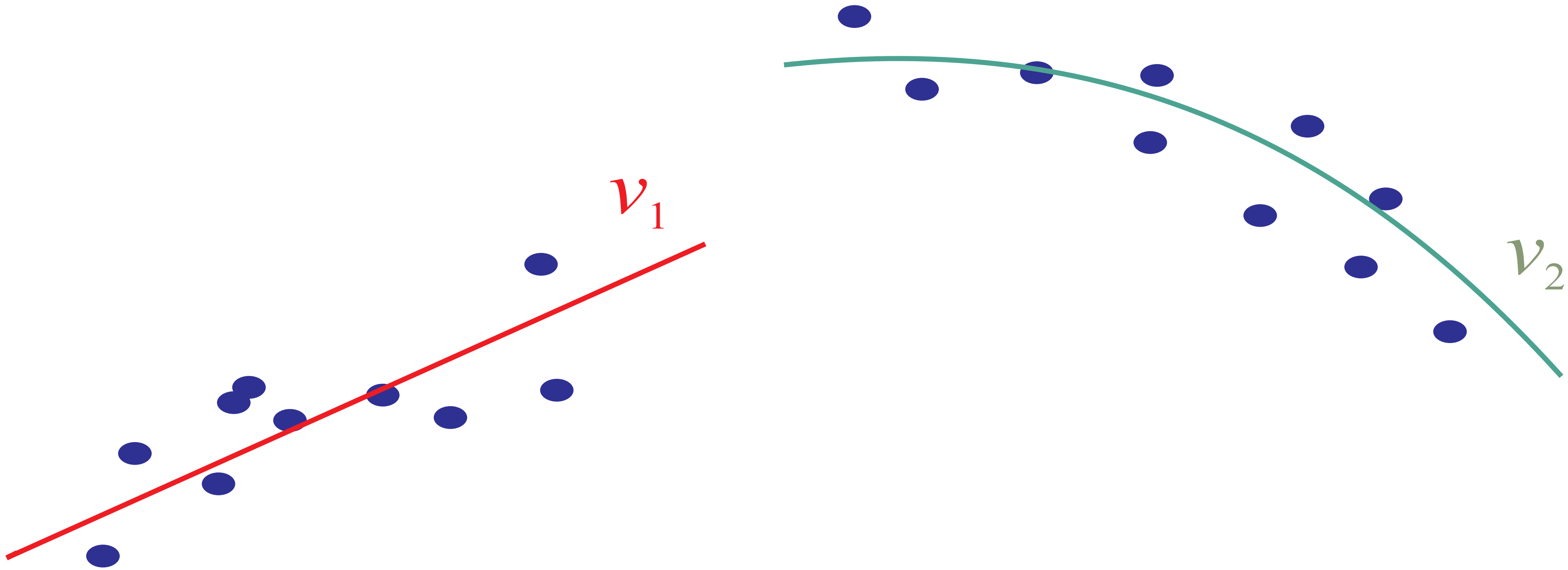
$$d^2(\mathbf{z}_j, \mathbf{v}_i) = (\mathbf{z}_j - \mathbf{v}_i)^T(\mathbf{z}_j - \mathbf{v}_i)$$

- **Inner-product norm:**

$$d^2_{A_i}(\mathbf{z}_j, \mathbf{v}_i) = (\mathbf{z}_j - \mathbf{v}_i)^T A_i(\mathbf{z}_j - \mathbf{v}_i)$$

- **Many other possibilities . . .**

Generalized Prototypes



lines, circles, ellipses, functions, etc.

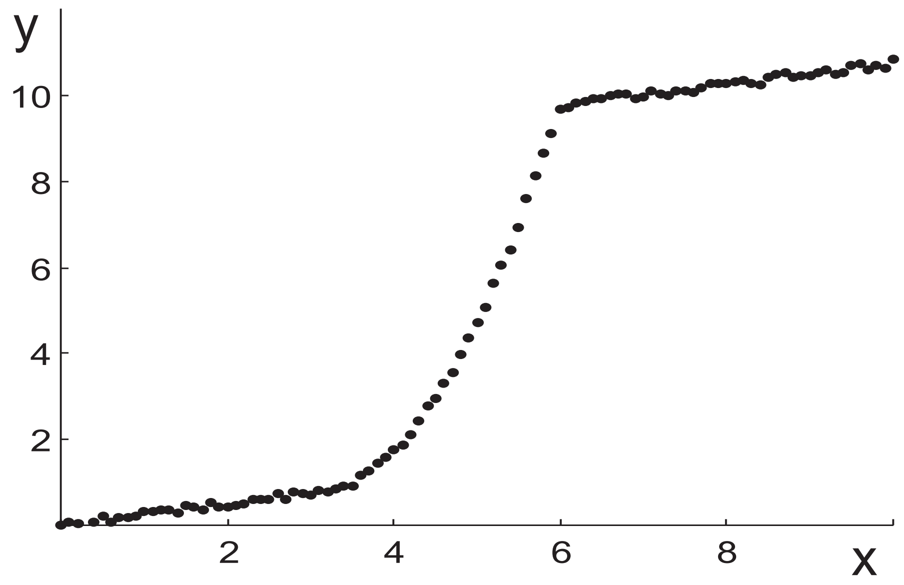
Fuzzy Clustering – Demo

1. **Fuzzy c-means**

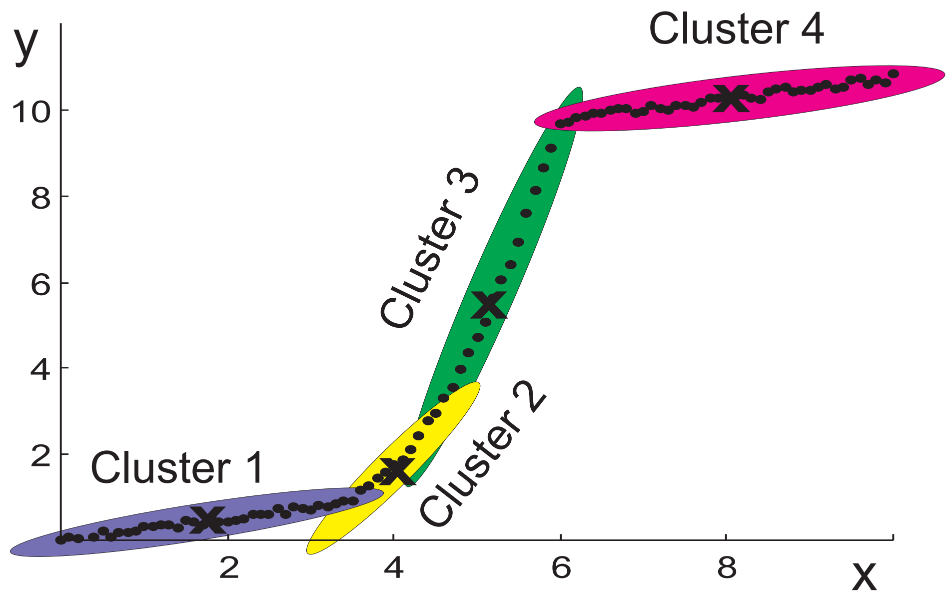
2. **Clustering with adaptive distance measure**

3. **Line detection by clustering**

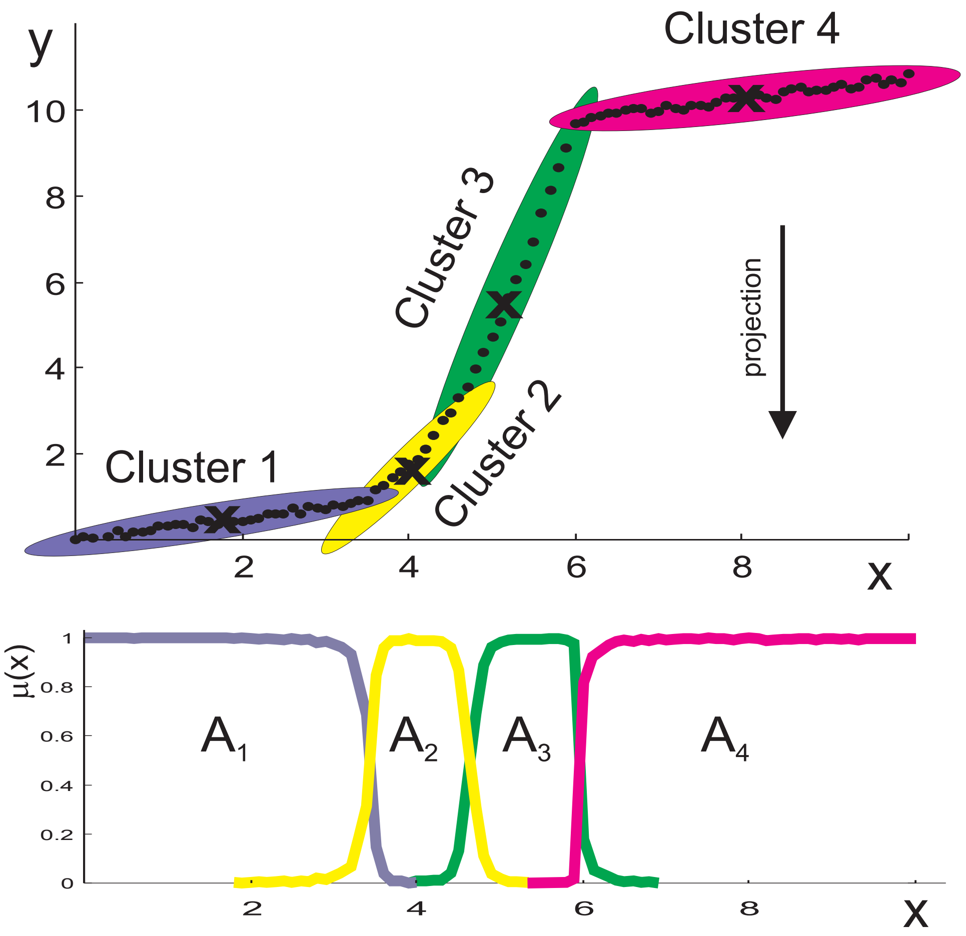
Extraction of Rules by Fuzzy Clustering



Extraction of Rules by Fuzzy Clustering



Extraction of Rules by Fuzzy Clustering



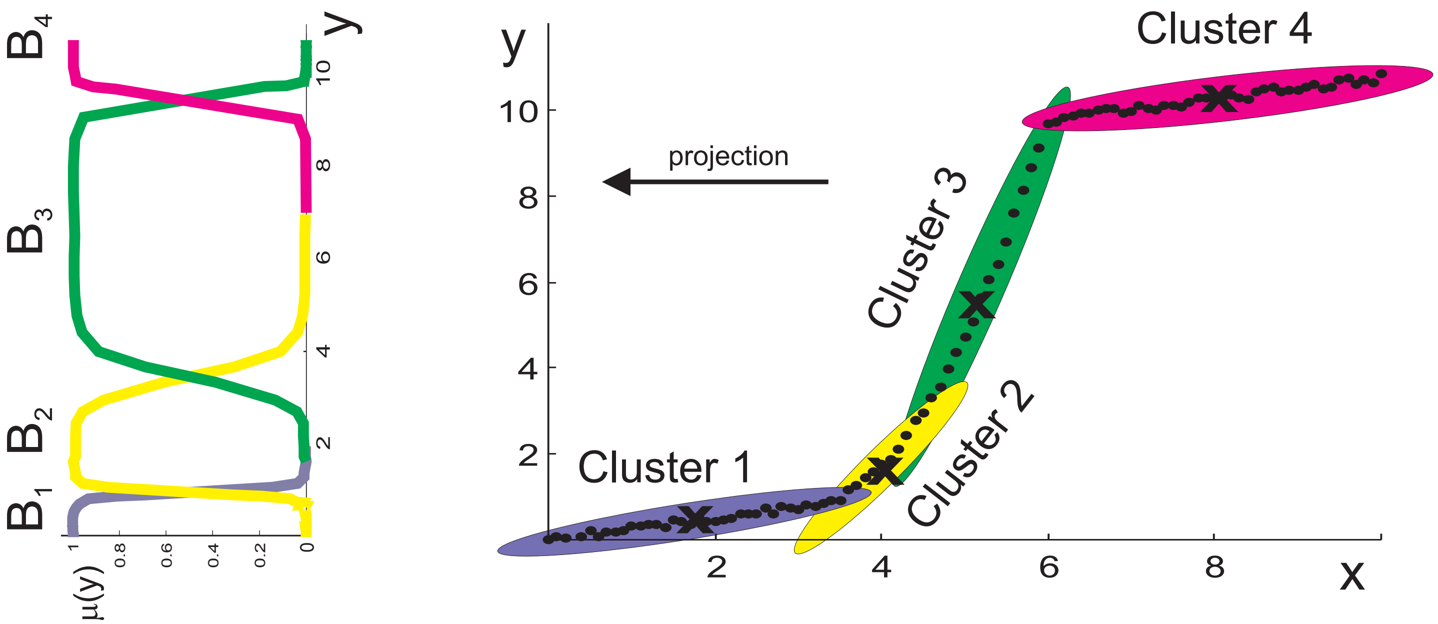
Takagi-Sugeno model

Rule-based description:

If x is A_1 then $y = a_1x + b_1$
If x is A_2 then $y = a_2x + b_2$

etc...

Extraction of Rules by Fuzzy Clustering



Rule-based description:

If y is B_1 then $x = a_1 y + b_1$
If y is B_2 then $x = a_2 y + b_2$

etc...

Inverse Takagi-Sugeno model

Rule Extraction – Demo

- **Extraction of Takagi–Sugeno rules**

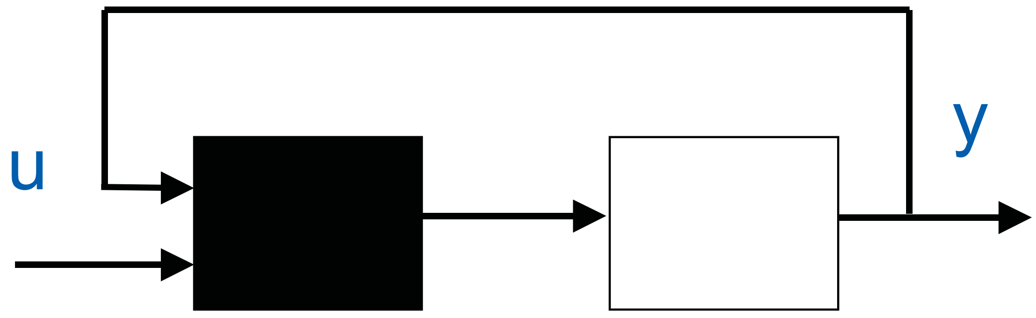
Semi-Mechanistic Modeling

- White-box model developed for well-known parts.
- Black-box or knowledge-based model for unknown relationships.

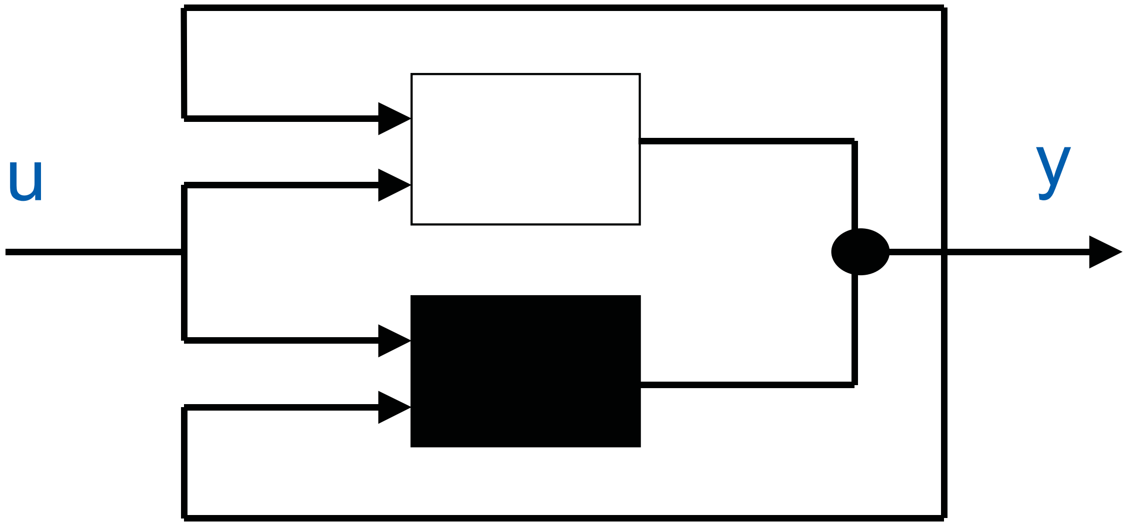
+ effective use of all available information
+ extrapolation, scalability
+ short development time

Semi-Mechanistic Modeling – Structures

Serial

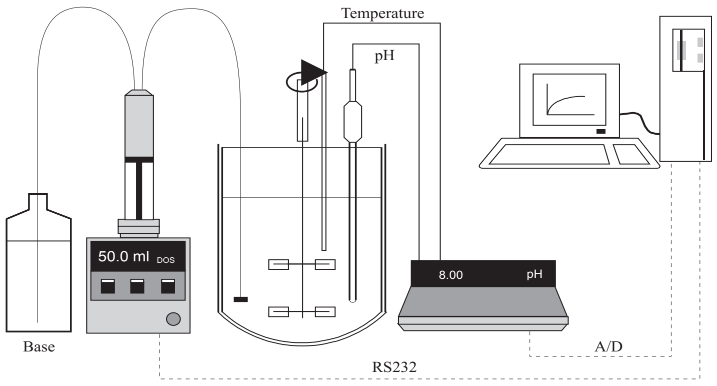


Parallel



Semi-Mechanistic Modeling: Example

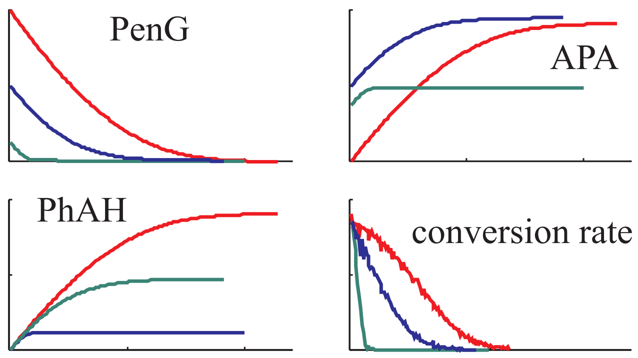
$$PenG + H_20 \xrightarrow{\text{enzyme}} 6\text{-APA} + PhAH$$



Mechanistic model (balance equations)

$$
\begin{aligned}
B_{k+1} &= B_k + \Delta T \frac{[E]_k \cdot V_k \cdot r_e}{M_B} \\
V_{k+1} &= V_k + \Delta T \frac{[E]_k \cdot V_k \cdot r_e}{M_B} \\
[\mathrm{PenG}]_{k+1} &= \frac{V_k}{V_{k+1}}([\mathrm{PenG}]_k \quad \Delta T \cdot R_{\mathrm{G}} \cdot [E]_k \cdot r_e) \\
[6-\mathrm{APA}]_{k+1} &= \frac{V_k}{V_{k+1}}([6-\mathrm{APA}]_k + \Delta T \cdot R_{\mathrm{A}} \cdot [E]_k \cdot r_e) \\
[\mathrm{PhAH}]_{k+1} &= \frac{V_k}{V_{k+1}}([\mathrm{PhAH}]_k + \Delta T \cdot R_{\mathrm{P}} \cdot [E]_k \cdot r_e) \\
[E]_{k+1} &= \frac{V_k}{V_{k+1}}[E]_k
\end{aligned}
$$

Data from batch experiments



Semi-mechanistic model