Reinforcement Learning

Part I: The Classical Setting

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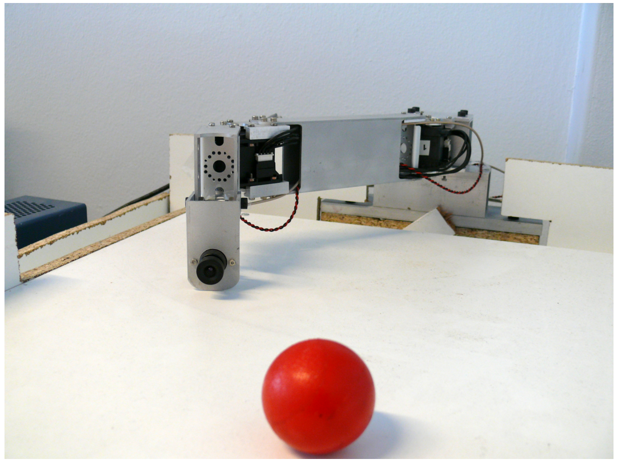
Knowledge-Based Control Systems

Demo: RL for a robot goalkeeper

Learn how to catch ball, using video camera image



Outline

1. Learning paradigms
2. Elements of RL
3. Algorithms
4. Summary and outlook

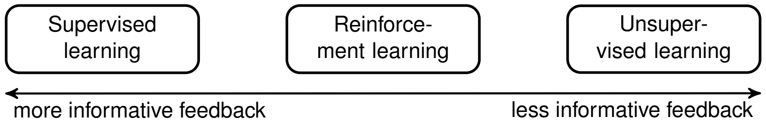
Why learning?

Learning can find solutions that:

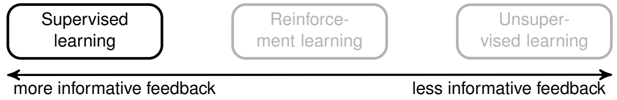
1. cannot be found in advance
   - problem too complex
     (e.g., controlling highly nonlinear systems)
   - problem not fully known beforehand
     (e.g., robotic exploration of extraterrestrial planets)
2. steadily improve
3. adapt to time-varying environments

Essential for any **intelligent** system

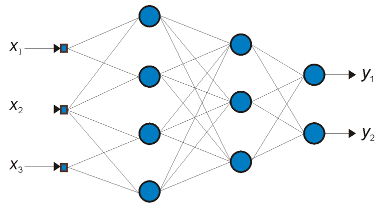
RL on the Machine Learning spectrum

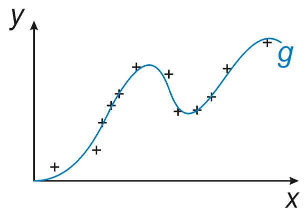


Spectrum: Supervised learning

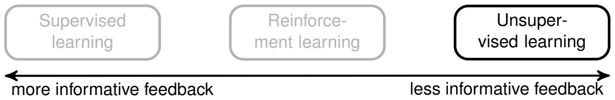


- For each input sample x, **correct output** y is known
- Infer input-output relationship $y \approx g(x)$
- Example: neural networks

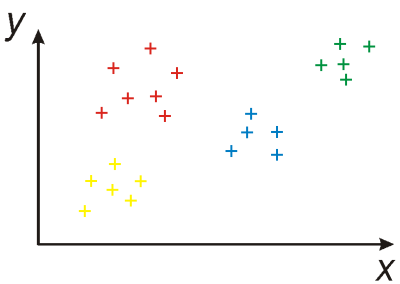




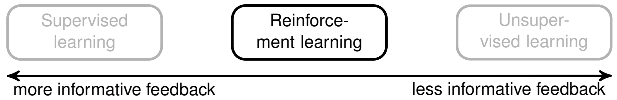
Spectrum: Unsupervised learning



- Only input samples available – **no outputs**
- Find patterns in the data
- Example: clustering



Spectrum: Reinforcement learning



- Correct outputs not available, **only rewards**
- Find optimal control behavior

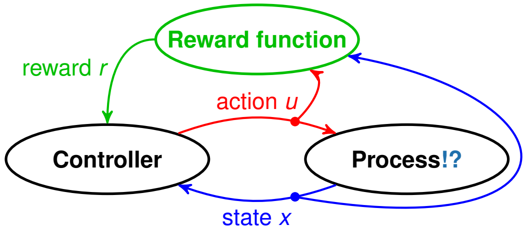
1 Learning paradigms

2 Elements of RL
Markov decision process, learning goal, policy
Bellman equation, optimality, solutions

3 Algorithms

4 Summary and outlook

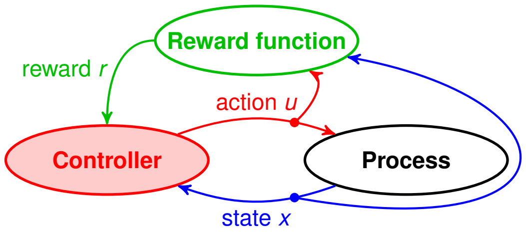
Principle of RL



- Interact with a system through states and actions
- Inspired by human and animal learning
- Receive rewards as performance feedback

Reinforcement learning = Control

Reinforcement learning is about **control**:
optimal, adaptive, and model-free



This lecture: **classical RL** – discrete states and actions

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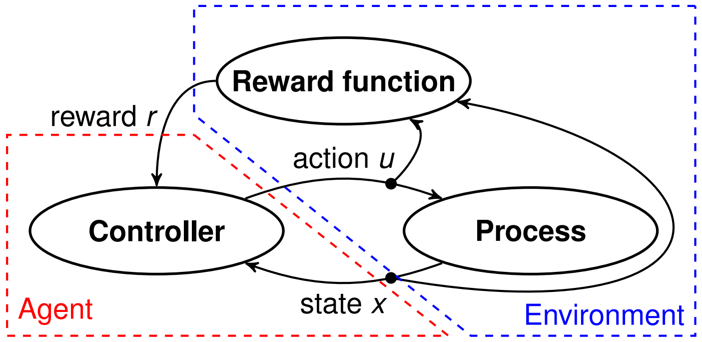
Environment and agent



The environment

The environment is modeled by an MDP:

Markov Decision Process (MDP)

An MDP is a tuple $\langle X, U, f, \rho \rangle$ where:

- X is the finite state space
- U is the finite action space
- $f : X \times U \to X$ is the state transition function
- $\rho : X \times U \to \mathbb{R}$ is the reward function

$x_{k+1} = f(x_k, u_k)$, with k the discrete time
$r_k = \rho(x_k, u_k)$

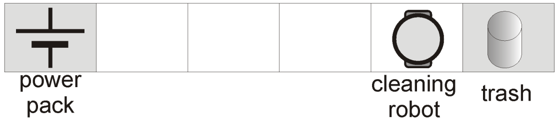
Note: stochastic formulation is possible

The agent

The agent is a state feedback controller:

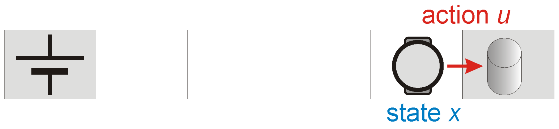
- Learns optimal mapping from states to actions
- Policy $\pi : X \mapsto U$ is the control law

A simple cleaning robot example

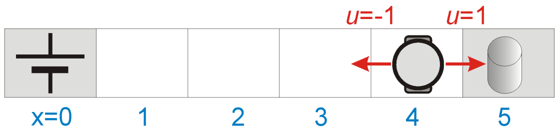


- Cleaning robot in a 1-D world
- Goal: pick up trash (reward $+5$) or power pack (reward $+1$)
- After picking up item, episode terminates

Cleaning robot: State & action

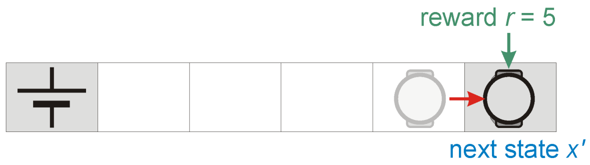


- Robot in given state x (cell)
- and takes action u (e.g., move right)



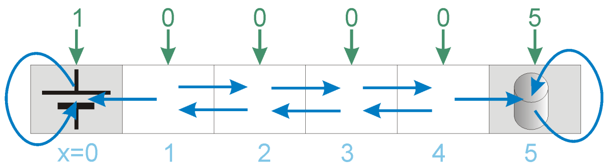
- State space $X = \{0, 1, 2, 3, 4, 5\}$
- Action space $U = \{-1, 1\} = \{\text{left, right}\}$

Cleaning robot: Transition & reward



- Robot reaches next state x'
- and receives reward r = quality of transition (here, $+5$ for collecting trash)

Cleaning robot: Transition & reward functions



- Transition function (process behavior):

$$x' = f(x, u) = \begin{cases} x & \text{if } x \text{ is terminal (0 or 5)} \\ x + u & \text{otherwise} \end{cases}$$

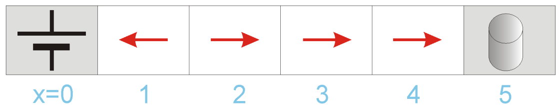
- Reward function (immediate performance):

$$r = \rho(x, u) = \begin{cases} 1 & \text{if } x = 1 \text{ and } u = -1 \text{ (powerpack)} \\ 5 & \text{if } x = 4 \text{ and } u = 1 \text{ (trash)} \\ 0 & \text{otherwise} \end{cases}$$

Cleaning robot: Policy

- Policy π: mapping from x to u (state feedback)
- Determines controller behavior

Example:



$\pi(0) = *$ $\quad \pi(1) = -1$ $\quad \pi(2) = 1$

$\pi(3) = 1$ $\quad \pi(4) = 1$ $\quad \pi(5) = *$

$*$ action irrelevant in terminal state

Learning goal

Find π that maximizes discounted return:

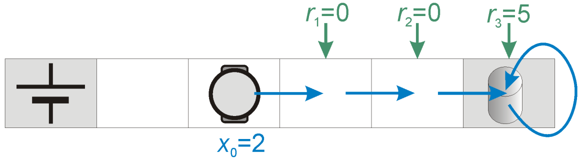
$$R^\pi(x_0) = \sum_{k=0}^{\infty} \gamma^k r_{k+1} = \sum_{k=0}^{\infty} \gamma^k \rho(x_k, \pi(x_k))$$

from any x_0

Discount factor $\gamma \in [0, 1)$:

- induces a “pseudo-horizon” for optimization
- bounds infinite sum
- encodes increasing uncertainty about the future
- helps convergence of algorithms

Cleaning robot: Return



Assume π always goes right

$$R^\pi(2) = \gamma^0 r_1 + \gamma^1 r_2 + \gamma^2 r_3 + \gamma^3 0 + \gamma^4 0 + \dots$$
$$= \gamma^2 \cdot 5$$

Because x_3 is terminal, all remaining rewards are 0

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Value function

One of these two is used:

- **V-function** (state value) of policy π:

$$V^\pi(x_0) = R^\pi(x_0)$$

- **Q-function** (state-action value) of policy π:

$$Q^\pi(x_0, u_0) = \rho(x_0, u_0) + \gamma R^\pi(x_1)$$

(return after taking u_0 in x_0 and then following π)

Q-function

$$\begin{aligned} R^\pi(x_0) &= \sum_{k=0}^{\infty} \gamma^k r_{k+1} = \sum_{k=0}^{\infty} \gamma^k \rho(x_k, \pi(x_k)) \\ &= \rho(x_0, \pi(x_0)) + \sum_{k=1}^{\infty} \gamma^k \rho(x_k, \pi(x_k)) \\ &= \rho(x_0, \pi(x_0)) + \gamma \sum_{k=0}^{\infty} \gamma^k \rho(x_{k+1}, \pi(x_{k+1})) \\ &= \rho(x_0, \pi(x_0)) + \gamma R^\pi(x_1) \end{aligned}$$

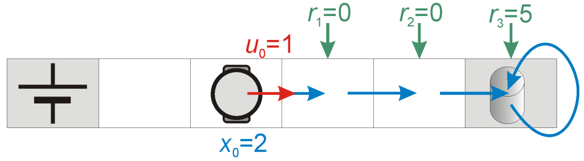
Q-function makes first action a free variable u_0:

$$Q^\pi(x_0, u_0) = \rho(x_0, u_0) + \gamma R^\pi(x_1)$$

Q-function (cont'd)

$$Q^\pi(x_0, u_0) = \rho(x_0, u_0) + \gamma R^\pi(x_1)$$

- First action in the sequence independent of policy
- Rest of the sequence follows the policy



- Q-function allows direct derivation of policy

Bellman equation

- Develop Q-function one step ahead:

$$\begin{aligned} Q^\pi(x_0, u_0) &= \rho(x_0, u_0) + \gamma R^\pi(x_1) \\ &= \rho(x_0, u_0) + \gamma[\rho(x_1, \pi(x_1)) + \gamma R^\pi(x_2)] \\ &= \rho(x_0, u_0) + \gamma Q^\pi(x_1, \pi(x_1)) \end{aligned}$$

Remember: $x_1 = f(x_0, u_0)$

Bellman equation for Q^π

$$Q^\pi(x, u) = \rho(x, u) + \gamma Q^\pi(f(x, u), \pi(f(x, u)))$$

Optimal solution

- **Optimal Q-function**:

$$Q^* = \max_\pi Q^\pi$$

$\Rightarrow$ Greedy policy in Q^*:

$$\pi^*(x) = \arg\max_u Q^*(x, u)$$

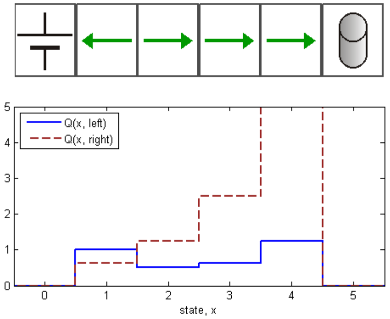
is **optimal** (achieves maximal returns)

Bellman optimality equation (for Q^*)

$$Q^*(x, u) = \rho(x, u) + \gamma \max_{u'} Q^*(f(x, u), u')$$

Cleaning robot: Optimal solution

Discount factor $\gamma = 0.5$



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   - Taxonomy
   - Q-learning
   - SARSA
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Types of algorithms

By model knowledge

1. Model-based – dynamic programming
   f, ρ known
2. Model-free – proper reinforcement learning
   f, ρ unknown, only transition data (x, u, x', r) available
3. Model-learning RL
   estimate f and ρ from transition data

Types of algorithms (cont'd)

By level of interaction

1. Offline
   data collected in advance
2. Online
   controller learns by interacting with the process

By path to optimal solution

1. Off-policy
   find Q^*, use it to compute π^*
2. On-policy
   find Q^π, improve π, repeat

Algorithms in this lecture

Online model-free reinforcement learning:

Off-policy	On-policy
Q-learning	SARSA

Both methods are **temporal difference (TD)** methods

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Off-policy online RL: Q-learning

Off-policy: **find** Q^*, use it to compute π^*

1. Take Bellman optimality equation at some (x, u):
$$Q^*(x, u) = \rho(x, u) + \gamma \max_{u'} Q^*(f(x, u), u')$$
2. Turn into **iterative update**:
$$Q(x, u) \leftarrow \rho(x, u) + \gamma \max_{u'} Q(f(x, u), u')$$
3. Instead of model f, ρ, use **transition sample** $(x_k, u_k, x_{k+1}, r_{k+1})$ at each step k:
$$Q(x_k, u_k) \leftarrow r_{k+1} + \gamma \max_{u'} Q(x_{k+1}, u')$$
Note: $x_{k+1} = f(x_k, u_k), r_{k+1} = \rho(x_k, u_k)$

Q-learning (cont'd)

4. Finally, make update **incremental**:
$$Q(x_k, u_k) \leftarrow Q(x_k, u_k) + \alpha_k \cdot \left[r_{k+1} + \gamma \max_{u'} Q(x_{k+1}, u') - Q(x_k, u_k)\right]$$
with learning rate $\alpha_k \in (0, 1]$.

The expression
$$r_{k+1} + \gamma \max_{u'} Q(x_{k+1}, u') - Q(x_k, u_k)$$
is called the **temporal difference**.

Complete Q-learning algorithm

Q-learning

for every trial **do**
  initialize x_0
  repeat for each step k
    take action u_k
    measure x_{k+1}, receive r_{k+1}
    $Q(x_k, u_k) \leftarrow Q(x_k, u_k) + \alpha_k \cdot$
      $[r_{k+1} + \gamma \max_{u'} Q(x_{k+1}, u') - Q(x_k, u_k)]$
  until terminal state
end for

Exploration-exploitation tradeoff

- Essential condition for convergence to Q^*:
  all (x, u) pairs must be visited infinitely often

$\Rightarrow$ **Exploration** necessary:
sometimes, choose actions randomly

- **Exploitation** of current knowledge is also necessary:
  sometimes, choose actions greedily:

$$u_k = \arg\max_{\bar{u}} Q(x_k, \bar{u})$$

Exploration-exploitation tradeoff crucial
for performance of online RL

Exploration-exploitation: ε-greedy strategy

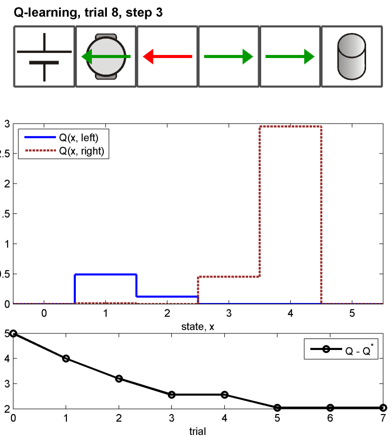
- Simple solution: **ε-greedy**

$$u_k = \begin{cases} \arg\max_{\bar{u}} Q(x_k, \bar{u}) & \text{with probability } (1 - \varepsilon_k) \\ \text{a random action} & \text{with probability } \varepsilon_k \end{cases}$$

- Exploration probability $\varepsilon_k \in (0, 1)$
  is usually decreased over time

Cleaning robot: Q-learning demo

Parameters: $\alpha = 0.2$, $\varepsilon = 0.3$ (constant)
$x_0 = 2$ or 3 (randomly)



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On-policy online RL: SARSA

On-policy: **find Q^π**, improve π, repeat

Similar to Q-learning:

1. Take Bellman equation for Q^π, at some (x, u):
$$Q^\pi(x,u) = \rho(x,u) + \gamma Q^\pi(f(x,u), \pi(f(x,u)))$$
2. Turn into iterative update:
$$Q(x,u) \leftarrow \rho(x,u) + \gamma Q(f(x,u), \pi(f(x,u)))$$
3. Use sample $(x_k, u_k, r_{k+1}, x_{k+1}, u_{k+1})$ at each step k:
$$Q(x_k,u_k) \leftarrow r_{k+1} + \gamma Q(x_{k+1}, u_{k+1})$$
Note: $u_{k+1} = \pi(f(x_k,u_k))$, π **= policy being followed**

SARSA (cont'd)

4. Make update incremental:

$$Q(x_k,u_k) \leftarrow Q(x_k,u_k) + \alpha_k \cdot [r_{k+1} + \gamma Q(x_{k+1}, u_{k+1}) - Q(x_k,u_k)]$$

Note that
$$r_{k+1} + \gamma Q(x_{k+1}, u_{k+1}) - Q(x_k,u_k)$$
is the **temporal difference** here

$(x_k, u_k, r_{k+1}, x_{k+1}, u_{k+1})$ =
(**S**tate, **A**ction, **R**eward, **S**tate, **A**ction) = **SARSA**

Complete SARSA algorithm

SARSA

for every trial **do**
  initialize x_0, choose initial action u_0
  repeat for each step k
    apply u_k, measure x_{k+1}, receive r_{k+1}
    choose **next** action u_{k+1}
    $Q(x_k,u_k) \leftarrow Q(x_k,u_k) + \alpha_k \cdot [r_{k+1} + \gamma Q(x_{k+1}, u_{k+1}) - Q(x_k,u_k)]$
  until terminal state
end for

Exploration-exploitation in SARSA

- For convergence—besides infinite exploration—
  SARSA requires **policy to eventually become greedy**
- E.g., ε-greedy

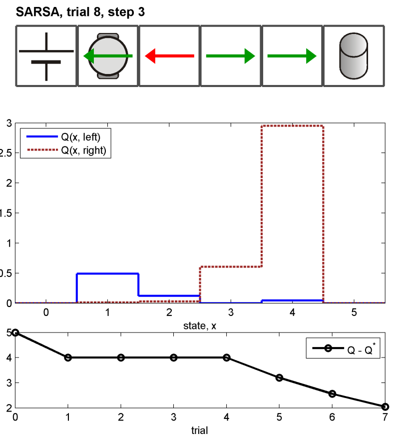
$$u_k = \begin{cases} \arg\max_{\bar{u}} Q(x_k, \bar{u}) & \text{with probability } (1 - \varepsilon_k) \\ \text{a random action} & \text{with probability } \varepsilon_k \end{cases}$$

  with $\lim_{k\to\infty} \varepsilon_k = 0$

- Greedy actions $\Rightarrow$ policy implicitly improved!
  (Recall on-policy: find Q^π, **improve** π, repeat)

Cleaning robot: SARSA demo

Parameters like Q-learning: $\alpha = 0.2$, $\varepsilon = 0.3$ (constant)
$x_0 = 2$ or 3 (randomly)



Summary

- **Reinforcement learning** =
  optimal, adaptive, model-free control
- Principle: reward signal as performance feedback
- Inspired from human and animal learning,
  but solid mathematical foundation
- Classical RL: small, discrete X and U (this lecture)

A final look at the algorithms

Off-policy: Q-learning
On-policy: SARSA

Typical parameter values:

γ 0.9 or larger
α_k under 0.5 or diminishing schedule
ε_k around 0.1 or diminishing schedule

Next lecture

Still to address:

- Continuous state and action spaces X, U
- More algorithms: actor-critic, model-learning, etc.

Part II – RL using function approximation