Abstract: Deconvolution filtering where the system and noise dynamics are obtained by parametric system identification is considered. Consistent with standard identification methods, ellipsoidal uncertainty in the estimated parameters is considered. Three problems are considered: 1) Computation of the worst case $H_2$ performance of a given deconvolution filter in this uncertainty set. 2) Design of a filter which minimizes the worst case $H_2$ performance in this uncertainty set. 3) Input design for the identification experiment, subject to a limited input power budget, such that the filter in 2) gives the smallest possible worst-case $H_2$ performance. It is shown that there are convex relaxations of the optimization problems corresponding to 1) and 2) while the third problem can be treated via iterating between two convex optimization problems.

1. INTRODUCTION

In many applications it is of interest to estimate the input to a dynamical system given measurements of the output, examples include seismic signal processing [Mendel, 1983], restoration of old recordings [Stockham et al., 1975], astronomy [Windhorst et al., 1994] and telecommunication [Godard, 1980]. Compounding factors are that in general measurements are noise corrupted and, furthermore, that the system through which the signal passes is not known exactly but uncertain. In deconvolution filtering the original signal is estimated by filtering the measured signal(s) through a filter. The measured signal is $\{y\}$ which is given by

$$y(t) = G_0(z)u(t) + H_0(z)e(t)$$

where $G_0$ is the dynamics of the system through which $\{u\}$ passes and where term $H_0(z)e(t)$ represents an aggregation of disturbances, modelled as zero mean white noise $\{e\}$ filtered through some dynamics $H_0$. The measured signal $y$ is filtered through the causal deconvolution filter $F$, resulting in the estimate

$$\hat{u}(t - n_k) = F(z)y(t)$$

The delay $n_k$ corresponds to the degree of non-causality (or smoothing) that will be allowed in the filtering.

In the sequel, we will suppose that all signals are wide-sense stationary and that the power spectrum $\Phi_u(\omega)$ of the signal $u(t)$ is given by $\Phi_u(\omega) = |W_u(e^{j\omega})|^2$ with $W_u(z)$ being a known stable transfer function. The variance of $e(t)$ is denoted by $\sigma_e^2$. We assume that $u$ and $e$ are independent. We will also assume that $G_0$ and $H_0$ are stable linear time invariant systems.

The filter $F(z)$ should be designed in order to give an accurate estimate of the input signal. By this we mean an estimate such that the mean-square error $J_0(F) = \mathbb{E}[(\hat{u}(t) - u(t))^2]$ is small. When the system $G_0$ and noise dynamics $H_0$ are perfectly known, the Wiener filter is the optimal filter [Wiener, 1949].

When $G_0$ and $H_0$ are uncertain, the filter $F$ should be robust, i.e. it should give good (but maybe not optimal) performance for any system in the class of systems to which $G_0$ and $H_0$ may belong. There is a rich literature on robust filtering in general, see, e.g., Neveux et al. [2007], Li and Fu [1997], Xie et al. [1994], and also on robust deconvolution filtering, see Chen and Chen [1994], Chen and Peng [1994], Eldar [2005a,b].

In this contribution we will suppose that an identification experiment has been performed and that we have identified a parametric uncertainty region which contains the true system:

$$G_0(z) \in \mathcal{D}_G = \{G(z, \theta) \mid \theta \in U\}$$
$$H_0(z) \in \mathcal{D}_H = \{H(z, \theta) \mid \theta \in U\}$$

where $U$ is an ellipsoid in the parameter space centered at the identified parameter vector $\hat{\theta}_N$

$$U = \{\theta \mid (\theta - \hat{\theta}_N)^T R (\theta - \hat{\theta}_N) < 1\}$$

where $R$ is a positive definite matrix. This ellipsoid $U$ thus characterizes the uncertainty of both $G$ and $H$. To the best of the authors’ knowledge, the use of parametric ellipsoidal uncertainty in robust $H_2$-deconvolution filtering is new. Furthermore, the size and shape of this ellipsoid $U$ is dependent on the spectrum $\Phi_{u_{id}}(\omega)$ of the input signal used during the identification. To show this dependence, $U$ will be also sometimes denoted by $U(\Phi_{u_{id}})$. The subscript id in $\Phi_{u_{id}}(\omega)$ is there to distinguish the input signal during the identification (i.e. $u_{id}$) and the one that has to be reconstructed (i.e. $u(t)$). The main contribution of this paper is to explore this dependence and to develop a computationally feasible method to determine the optimal $\Phi_{u_{id}}$. 


More specifically, let us consider Figure 1 (left-hand side) where the true system has been replaced by \( G(z, \theta) \) and \( H(z, \theta) \) for one arbitrary \( \theta \in U \) and let us define \( J(\theta, F) = \mathbb{E}[\epsilon^2(t, \theta)] \) where \( \epsilon(t, \theta) \triangleq \tilde{u}(t-n_k, \theta) - u(t-n_k) \). We have thus that \( J_0(\theta) < \gamma \) if

\[
J(\theta, F) < \gamma \quad \forall \theta \in U
\]

Given this set-up, we will in this contribution study the following three problems:

1. **Performance validation.** Given a filter \( F(z) \) and an uncertainty region \( U \), verify that \( J(\theta, F) < \gamma \) \( \forall \theta \in U \) for some given \( \gamma \). We are in fact interested by the smallest \( \gamma \) for which the latter holds (i.e. the worst case performance achieved by the filter over the plants described by \( \theta \) in \( U \)). This worst case performance \( J_{WC} \) is thus the solution \( \gamma_{opt} \) of the following optimization problem:

\[
\min_{\gamma} \gamma \\
\text{s.t.} \quad J(\theta, F) < \gamma, \quad \forall \theta \in U
\]  

(2)

2. **Robust \( H_2 \) filter design.** Given an uncertainty region \( U \), determine a filter \( F \) such that \( J(\theta, F) < \gamma \) \( \forall \theta \in U \) for the smallest possible \( \gamma \). The to-be-designed filter \( F \) is thus the solution \( F_{opt} \) of the following optimization problem:

\[
\min_{\gamma} \gamma \\
\text{s.t.} \quad J(\theta, F) < \gamma, \quad \forall \theta \in U
\]  

(3)

For simplicity, we will restrict attention to filters where the poles are pre-specified. This covers, e.g., FIR filters, which are commonly used in Telecommunications applications, but also Laguerre [Wahlberg, 1991] and Kautz filters.

3. **Optimal input design for robust \( H_2 \) filtering.** Suppose that there is a constraint on the maximal power for the excitation signal \( u_{id} \) for the identification experiment, which is supposed to be conducted in open loop. The optimal experiment design problem consists in this case to determining the input spectrum \( \Phi_{u_{id}} \) of \( u_{id} \) that delivers an uncertainty region \( U(\Phi_{u_{id}}) \) for which the corresponding robust filter \( F \) leads to the smallest worst case performance \( J_{WC} \). The optimal \( \Phi_{u_{id}} \) is the solution \( \Phi_{opt} \) of the following optimization problem:

\[
\min_{\gamma,F} \gamma \\
\text{s.t.} \quad J(\theta, F) < \gamma, \quad \forall \theta \in U
\]  

where \( \alpha \) represents the maximum allowed power. The corresponding robust filter is then given by the optimal filter \( F_{opt} \) in the same optimization problem.

These three problems will be covered in the ensuing three sections. Section 5 contains a numerical illustration and the paper is concluded in Section 6.

2. VALIDATION PROBLEM

2.1 Reformulation in the LFT framework

We will start by reformulating the problem (2) in the LFT framework. Let us take a look at Figure 1. Since \( e \) and \( u \) are independent and since \( \Phi_u(\omega) = |W_u(e^{j\omega})|^2 \), we can say that \( u(t) = W_u(z)\tilde{u}(t) \) and \( e(t) = \tilde{e}(t) \) with \( \tilde{u}(t) \) and \( \tilde{e}(t) \) independent white noises with unit variance. Consequently, the cost function \( J(F, \theta) \) is equal to the squared \( H_2 \)-gain of the vector of transfer functions:

\[
\left[ \begin{array}{c}
\sigma_u(F(z)H(z, \theta)) \\
((F(z)G(z, \theta)) - z^{-n_k})W_u(z)
\end{array} \right]^T
\]

which is the transfer function between \( [\tilde{u}(t)] \) and \( \epsilon(t) \). We have thus:

\[
J(F, \theta) = \left\| \left( \begin{array}{c}
\sigma_u(F(z)H(z, \theta)) \\
((F(z)G(z, \theta)) - z^{-n_k})W_u(z)
\end{array} \right) \right\|_2
\]

where \( \|G\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Trace}(G^*(e^{j\omega})G(e^{j\omega})) \, d\omega} \).

Since both the parametrization of \( G(z, \theta) \) and \( H(z, \theta) \) are generally chosen as LFT in \( \theta \), the transfer function between \( v(t) \) and \( e(t) \) can be easily rewritten both as an LFT in \( \Theta_{bias} = \text{Diag}(\theta, \theta) \) (\( \theta \in \mathbb{R}^k \)) and as an LFT in \( \Theta = \text{Diag}(\Delta \Theta, \Delta \Theta) \) (\( \Delta \Theta := \theta - \bar{\theta}_N \)). Thus, we can write:

\[
\begin{align*}
\begin{bmatrix}
= M(z) \\
p \\
q
\end{bmatrix} &=
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\begin{bmatrix}
p \\
q
\end{bmatrix},
q &=
\begin{bmatrix}
\Delta \Theta \\
0 \\
\Delta \Theta
\end{bmatrix}
p
\end{align*}
\]

(7)

(see the right-hand side of Figure 1), which is an LFT transformation. We will use the notation

\[
\epsilon(t) = F(M(z), \Theta)v(t)
\]

(8)

To be able to develop some of the coming results, it will be necessary to constrain \( M_{12}(z) \) to have no direct term i.e. that \( M_{12} \) contains at least one delay. However, this can always be achieved.

**Lemma 2.1.** Let

\[
G(z, \theta) = \frac{Z_1 \theta}{1 + Z_2 \theta} \quad H(z, \theta) = \frac{1 + Z_3 \theta}{1 + Z_4 \theta}
\]

with \( Z_i(\cdot) \) row vectors of dimension \( k \) containing only delays and zeros and \( \theta \in \mathbb{R}^{k \times 1} \). Then the LFT relation (7) can be chosen such that the matrix \( M_{12}(z) \) has no direct term.
Due to space limitations the proof of Lemma 2.1 is omitted.

Remark 2.1. The result in Lemma 2.1 is equivalent with the fact that $p(t = 0) = 0$ and $q(t = 0) = 0$ for all signals $v(t)$ such that $v(t < 0) = 0$.

Based on what has been developed in this section, Problem (2) can be reformulated as follows: With $\Theta$ as in (7),
\[
\min_{\gamma} \gamma \quad \text{s.t.} \quad \|F(M(z), \Theta)\|_2 < \gamma \quad \forall \theta \in U
\]
(9)

2.2 Multiplier describing the uncertainty

To solve (9), a very important step will be to describe the uncertainty matrix $\Theta$ by a set of multipliers $\Pi(\omega)$:
\[
\theta \in U \implies \left[ \begin{array}{c} \theta \\ \ell \end{array} \right]^T \Pi(\omega) \left[ \begin{array}{c} \theta \\ \ell \end{array} \right] > 0
\]
(10)

This can be done as in Barethin et al. [2006] which yields the multiplier (where the frequency argument $\omega$ has been omitted)
\[
\Pi = \begin{bmatrix}
-Z_0 \otimes R & \begin{bmatrix} j A_{11} & B_{12} + j A_{12} \\ j A_{12} & j A_{22} \end{bmatrix} & Z_0 \\
-B_{12} + j A_{12} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

with $p_{ij}(\omega)$, $\bar{p}_{ij}(\omega) \in \mathbb{R}^{2 \times k}$, $Z_0(\omega)$ a positive definite Hermitian matrix of dimension 2 and $A_{ij}(\omega)$ $B_{ij}(\omega) \in \mathbb{R}^{1 \times k}$ with $A_{ij} = -A_{ij}^T$ and $B_{ij} = -B_{ij}^T$. Finally, $R$ comes from (1). In the sequel, we will need a factorized version of $\Pi$, i.e. we will need to rewrite $\Pi(\omega)$ as
\[
\Pi(\omega) = \Psi^*(e^{j\omega}) P \Psi(e^{j\omega})
\]
(11)

where $P$ is a frequency-independent matrix and $\Psi(z)$ is a matrix of transfer functions. This can be done using a very similar reasoning as in Appendix B of Bombois et al. [2006] and involves the parametrization of the frequency dependence using the basis functions $B(z) = (1 \ z^1 \ldots z^{(b-1)})^T$, so that, e.g.,
\[
Z_0(\omega) = \Lambda_0 + \sum_{k=1}^{b-1} \left( \Lambda_k e^{j\omega k} + \Lambda_k^T e^{-j\omega k} \right)
\]
(12)

where $\Lambda_k \in \mathbb{R}^{2 \times 2}$ and $\Lambda_0 = \Lambda_0^T$. The positivity of $Z_0(\omega)$ can be forced by an extra LMI conditions on $[\Lambda_k]_{k=1}^{b-1}$ using the the KYP-lemma [Yakubovich, 1962]. Notice that
\[
Z_0(\omega) \otimes R = \Lambda_0 \otimes R + \sum_{k=1}^{b-1} \left( (\Lambda_k \otimes R) e^{j\omega k} + (\Lambda_k^T \otimes R) e^{-j\omega k} \right)
\]
(13)

Omitting details, the matrix $P$ in (11) will have a certain structure, linear in variables $S$ that parametrize $p_{ij}$, $\bar{p}_{ij}$, $A_{ij}$, $i, j = 1, 2$ and $B_{12}$ and bilinear in $R$ and $\Lambda := [\Lambda_0, \ldots, \Lambda_{b-1}]$, according to (13). We therefore write $P = P(R, \Lambda, S)$.

2.3 LMI formulation for the validation problem

Such as e.g. Scherer and Weiland [1999], we define
\[
\begin{bmatrix}
z_1(e^{j\omega}) \\
z_2(e^{j\omega})
\end{bmatrix} = \begin{bmatrix}
\Psi(e^{j\omega}) & \begin{bmatrix} I_{2k} \\ M_{11}(e^{j\omega}) & M_{12}(e^{j\omega}) \\ M_{21}(e^{j\omega}) & M_{22}(e^{j\omega}) \end{bmatrix} \\
\end{bmatrix} \begin{bmatrix}
q(e^{j\omega}) \\
w_1 \\
w_2
\end{bmatrix}
\]
(14)

where $w_1, w_2 \in \mathbb{R}$, with state-space representation
\[
x(t + 1) = A x(t) + B_1 q(t) + B_2 \begin{bmatrix} w_1 \delta(t) \\
w_2 \delta(t) \end{bmatrix}
\]
(21)

\[
\begin{bmatrix}
z_1(t) \\
z_2(t)
\end{bmatrix} = \begin{bmatrix} C_1 & D_{11} \\ C_2 & D_{21} \end{bmatrix} x(t) + \begin{bmatrix} D_{12} & D_{22} \end{bmatrix} \begin{bmatrix} w_1 \delta(t) \\
w_2 \delta(t) \end{bmatrix}
\]
(22)

with $\delta(t)$ the discrete-time pulse signal. We are now in position to present a method to compute an upper bound for the worst case performance $J_{WC}$. This upper bound is given by the solution $\gamma_{\text{opt}}$ of the following LMI optimization problem:
\[
\begin{aligned}
\min_{\gamma, Q, K = K^T, \Lambda_0 = \Lambda_0^T, \Lambda_1, \ldots, \Lambda_{b-1}, S} & \gamma \\
\text{s.t.} \quad \text{Trace}(Q) < \gamma, \\
B_2^T K B_2 + \begin{bmatrix} D_{12} & D_{22} \end{bmatrix} \begin{bmatrix} P(R, \Lambda, S) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} D_{12} & D_{22} \end{bmatrix} < Q,
\end{aligned}
\]
(15)

\[
F_1(K) + F_2(P(R, \Lambda, S)) < 0,
\]
(16)

\[
X(\Lambda) < 0
\]
(17)

where (18) is an LMI in $\Lambda$ that ensures positivity of (12), where
\[
\begin{bmatrix}
F_1(K) := & \begin{bmatrix} I & 0 \\ A & B_1 \end{bmatrix} \begin{bmatrix} -K & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} I & 0 \end{bmatrix}, \\
F_2(P) := & \begin{bmatrix} C_1 & D_{11} \\ C_2 & D_{21} \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 & D_{11} \\ C_2 & D_{21} \end{bmatrix}
\end{bmatrix}
\]
(19)

The proof that the above LMI problem results in an upper bound for the worst case performance has been omitted due to space limitation.

3. ROBUST $H_2$ FILTER DESIGN

In this section, we consider Problem (3) i.e. the robust filter design problem. The only difference from the performance validation problem is that, instead of considering $F$ as a given transfer function, this filter is now also a decision variable.

3.1 Adding $F$ as extra decision variable

We restrict attention to filters with pre-specified poles, i.e.
\[
F(z) = f_0 + \frac{f_1 z^{-1} + \ldots + f_n z^{-n_f}}{1 + a_1 z^{-1} + \ldots + a_{n_f} z^{-n_f}}.
\]
(23)

Note that a FIR-filter corresponds to the case where $a_i = 0$ ($i = 1..n_f$). The filter $F(z)$ has the following minimal state-space realization
\[
F = \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix}, \quad A_F = \begin{bmatrix} -a & -a_{n_f} \\ I_{n_f+1} & 0 \end{bmatrix}, \quad B_F = \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}, \quad C_F = [f_1 \ldots f_{n_f}], \quad D_F = f_0, a = [-a_1 \ldots -a_{n_f}]
\]
(24)

The matrices of interest are $C_F$ and $D_F$ since those matrices contain the to-be-determined coefficients. We
will now investigate how these matrices appear in the LMI’s (16), (17).

In order to be able to rewrite the LMI’s (16), (17) as a function of $C_F$ and $D_F$, we need to derive a (minimal) state-space representation for the system (14). Indeed, the state-space matrices of this system appear in the considered LMI’s. The system (14) can be represented as in Figure 2 where $q$ and $p$ are fictive signals such that it holds that $q = \Theta p$ and where the dependence of (14) on $F$ is represented. The minimal state-space representation for the system (14) can be easily deduced from a minimal state-space representation for the transfer matrix $L$ defined in Figure 2. Let us therefore suppose that we have deduced a minimal realization for the transfer matrix $L$:

$$
\begin{align*}
\dot{x}(t + 1) &= Ax(t) + B_1 q(t) + B_2 v(t) \\
\begin{bmatrix}
q(t) \\
p(t) \\
u(t-n_k)
\end{bmatrix}
&= \begin{bmatrix}
\bar{C}_1 \\
\bar{C}_2 \\
\bar{C}_3
\end{bmatrix} x(t) + \begin{bmatrix}
\bar{D}_{13} \\
0 \\
0
\end{bmatrix} v(t) \\
y(t) &= \begin{bmatrix}
\bar{B}_3 \\
\bar{B}_2 \\
0
\end{bmatrix} v(t)
\end{align*}
$$

Note that $\bar{D}_{3v} \neq 0$ only when $n_k = 0$. Using now the formulae on page 34 of Zhou and Doyle [1998], we can easily deduce a minimal realization of the system (14):

$$
\begin{align*}
\dot{x}(t + 1) &= A x(t) + B_1 q(t) + B_2 v(t) \\
\begin{bmatrix}
q(t) \\
p(t) \\
u(t-n_k)
\end{bmatrix}
&= \begin{bmatrix}
\bar{C}_1 \\
\bar{C}_2 \\
\bar{C}_3
\end{bmatrix} x(t) + \begin{bmatrix}
\bar{D}_{13} \\
0 \\
0
\end{bmatrix} v(t) \\
y(t) &= \begin{bmatrix}
\bar{B}_3 \\
\bar{B}_2 \\
0
\end{bmatrix} v(t)
\end{align*}
$$

$$
\begin{align*}
z_1(t) &= [D_F \bar{C}_1 \bar{C}_2 \bar{C}_3] x(t) + D_{13} q(t) + D_{13}^T v(t) \\
z_2(t) &= [D_F \bar{C}_2 - \bar{C}_1 \bar{C}_3 \bar{C}_2] x(t) + D_{22} q(t) + D_{22}^T v(t)
\end{align*}
$$

where $(A_F, B_F, C_F, D_F)$ is a minimal state-space representation of $\Psi(z)$. What is important to see is that $C_2$, $D_{21}$, and $D_{22}$ are the only matrices which are function of $C_F$ and $D_F$ and that they are furthermore linear function of $C_F$ and $D_F$.

Let us now consider the dependence on $C_F$ and $D_F$ of the LMI (16):

$$
B_2^T K B_2 + D_1^T P D_{12} + D_{22}^T D_{22} < Q
$$

Because of the product $D_{22}^T D_{22}$, such an expression is not an LMI in $D_F$. However, using the Schur complement, we see that the above expression is equivalent to:

$$
\begin{align*}
0 > \begin{bmatrix}
Q - B_2^T K B_2 - D_1^T P D_{12} + D_{22}^T D_{22} & 1 \\
1 & 0
\end{bmatrix}
\end{align*}
$$

The latter expression is an LMI in $Q, P, K$ but now also in $D_F$. Similarly (17) is equivalent to an LMI in $Q, P, K, C_F$ and $D_F$.

$$
\begin{align*}
0 > \begin{bmatrix}
-F_1(K) - [C_1 D_{11}]^T P [C_1 D_{11}] + [C_2 D_{21}]^T & 1 \\
1 & 0
\end{bmatrix}
\end{align*}
$$

### 3.2 LMI formulation of the robust filter design problem

The robust filter design problem (3) now can be written

$$
\begin{align*}
\min_{\gamma, Q=Q^T, K=K^T, \Lambda_0=\Lambda_0^T, \Lambda_1, \ldots, \Lambda_{n-1}, S, C_F, D_F} & \gamma \\
\text{subject to} & \text{Trace}(Q) < \gamma, \\
& [Q - B_2^T K B_2 - D_1^T P (R, \Lambda) D_{12} + D_{22}^T D_{22}] > 0, \\
& [-F_1(K) - [C_1 D_{11}]^T P (R, \Lambda) [C_1 D_{11}] + [C_2 D_{21}]^T > 0, \\
& \begin{bmatrix}
[C_2 D_{21}] \\
1
\end{bmatrix} \leq X(A) < 0
\end{align*}
$$

We denote this problem $ROBFILT$ for future reference.

The robust filter has then the state-space realization $(A_F, B_F, C_F^{opt}, D_F^{opt})$ with $C_F^{opt}$ and $D_F^{opt}$ solutions of the above LMI problem and $A_F$ and $B_F$ as given in (21).

The results in Sections 2 and 3 are in fact an extension of continuous-time robust filtering results (see e.g. Scherer and Kose [2006], Scherer and Weiland [1999]) to the discrete-time case and to the case of an ellipsoidal parametric uncertainty region $U$ (coming e.g., from an identified model).

### 4. OPTIMAL EXPERIMENT DESIGN FOR ROBUST $H_2$ FILTERING

Now let us turn to our third problem i.e the optimal experiment design problem (4). We then add the constraint (6) on the maximal power of $u_{id}$ to $ROBFILT$. Furthermore, we observe that the input spectrum used during the identification experiment, $\Phi_{u_{id}}$, determines the matrix $R$ which defines the ellipsoidal uncertainty set $U$, cf. (1), and furthermore, $R$ is affine in $\Phi_{u_{id}}$, Ljung [1999].

Now, the optimization problem $ROBFILT$ only depends on $R$ through $P(R, \Lambda, S)$ where $P$ is bilinear in $R$ and $\Lambda$. Thus if we modify $ROBFILT$ by keeping $A$ fixed and adding $\Phi_{u_{id}}$ to the decision variables (and adding (6)), the modified problem is still an LMI. We can thus proceed as in Barenthin et al. [2006] using iterations inspired by the DK-approach in $\mu$-synthesis whereby one alternates between solving $ROBFILT$ and the modified problem.
We consider the following ARMAX true system:

\[ y(t) = G_0(z)u(t) + H_0(z)e(t) \]

Both \( G_0 \) and \( H_0 \) are represented in a Bode plot in Figure 3. We would like to reconstruct the signal \( u(t) \) of this system based on a measurement of the output \( y(t) \). We will suppose that the input of the system has a power spectrum \( \Phi_u(\omega) = |W_u(e^{j\omega})|^2 \) located mostly in the frequency range \([0, 0.04]\) (see Figure 6). The transfer function \( W_u(z) \) is indeed given by \( W_u(z) = \frac{1}{1 - 0.9z^{-1}} \). The variance \( \sigma^2_e \) of the noise \( e(t) \) is here chosen equal to 1.

Using the technique presented in Section 4, we will determine the optimal input spectrum \( \Phi_{uid}(\omega) \) for the identification of a model of the true system when this model has to be used for robust deconvolution. We suppose that \( \alpha = 1 \) in (6) i.e. the power of the excitation signal for the identification is constrained to be smaller than one. We need also to fix some other variables such as \( b \) and \( n_f \). The scalar \( b \) is the size of the basis function vector \( \mathbf{B}(z) \) which allows to factorize the multiplier \( \Pi(\omega) \). It is to be noted that the larger \( b \), the closer the optimal \( \gamma \) deduced with the optimization problem of Section 4 will be from the actual value of the worst case performance \( J_{WC} \). On the other hand, the larger \( b \), the larger the complexity of the optimization problem. As a trade-off choice, the value of \( b \) is here chosen equal to 3. The scalar \( n_f \) is the order of the parametrization of the filter \( F(z) \) that will be here chosen as a FIR. This filter \( F(z) \) is also a decision variable of the optimization problem of Section 4. A similar trade-off as for the value of \( b \) must be done for \( n_f \): we have here chosen \( n_f = 3 \).

We first suppose that \( \Phi_{uid}(\omega) = 1 \) for all \( \omega \) (white noise). Having chosen a spectrum \( \Phi_{uid}(\omega) \), the matrix \( R \) in the decision variable \( P \) is entirely determined and we can therefore use the LMI optimization problem of Section 3.2 to determine the optimal multipliers \( \Lambda_i \) \( (i = 0, \ldots, (b-1)) \) for this \( \Phi_{uid}(\omega) \). For this spectrum \( \Phi_{uid}(\omega) \), the upper bound \( \gamma_{opt} \) for the worst case reconstruction error \( J_{WC} \) is equal to 0.0498.

By fixing the multipliers \( \Lambda_i \) to their values found when \( \Phi_{uid}(\omega) = 1 \ \forall \omega \), it is now possible to consider \( \Phi_{uid}(\omega) \) as a decision variable. Indeed, the optimization problem of Section 4 is then a LMI optimization problem. To do this optimization on \( \Phi_{uid}(\omega) \), we in fact determine the parameters \( \alpha_r = \alpha_{r_{id}} \) in the following classical parametrization for \( \Phi_{uid}(\omega) \):

\[ \Phi_{uid}(\omega) = \sum_{r=-M}^{M} \alpha_r e^{j\omega r} \geq 0 \quad (23) \]

where \( M \) is here chosen equal to 3. The spectrum \( \Phi_{uid}(\omega) \) obtained after this first iteration is represented in blue dashdot in Figure 4 where it is compared to the initial spectrum \( \Phi_{uid}(\omega) = 1 \ \forall \omega \) (red dashed).

By now iterating between optimizing the multipliers \( \Lambda_i \) and the parameters \( \alpha_i \) determining the input spectrum yields after four iterations a decrease of the value of the upper bound \( \gamma_{opt} \) for the worst case reconstruction error \( J_{WC} \) from 0.0498 (when \( \Phi_{uid}(\omega) = 1 \ \forall \omega \)) to 0.0374. This is a reduction \(^1\) of 25%.

To improve the optimal spectrum obtained at the fourth iteration even more, we have increased the value of \( M \) in (23) to \( M = 6 \). The spectrum obtained with \( M = 6 \) is represented in black solid in Figure 4. This new spectrum does not change so much the upper bound on the worst case performance i.e. \( \gamma_{opt} = 0.0373 \). The corresponding optimal filter \( F(\omega) \) is represented in Figure 3 where we can observe that \( |F(e^{j\omega})| \approx |G_0(e^{j\omega})|^{-1} \). In Figure 5, we represent the performance of this filter on a realization of length 1000 of the to be reconstructed input signal \( u(t) \). By looking at the optimal spectrum represented in black solid in Figure 4, we observe that this spectrum \( \Phi_{uid}(\omega) \) is basically a spectrum located in the frequency range \([0 0.2]\). The contribution outside this band is nevertheless nonzero, but this could perhaps be due to the restricted parametrization of \( \Phi_{uid}(\omega) \). Consequently, we test whether a RBS signal of power \( \alpha = 1 \) and with a clock period \( \nu = 6 \) delivers a better worst-case reconstruction error \( J_{WC} \). Indeed, this signal has a spectrum (see Figure 6) with much less contribution outside the band \([0 0.2]\). We observe that this RBS signal indeed delivers a better \( J_{WC} = 0.0346 \).

It is important to note that \([0 0.2]\) is in fact the bandwidth of the system \( G_0(z) \) (see Figure 3) and thus not the bandwidth of the signal \( u(t) \) that must be reconstructed as can be seen in Figure 6 where we also represent the spectrum \( \Phi_u(\omega) \) of the to-be-reconstructed signal (blue dashdot).

In this contribution we have addressed the problem of robust deconvolution for ellipsoidal parametric uncertainty. In addition, we have taken the further step of considering the problem of how to design the input in an identification experiment such that the resulting robust filter gives the smallest possible worst-case \( H_2 \) performance in the set of uncertain systems. We believe these methods to have wide applicability.

\(^1\) This decrease of the upper bound on \( J_{WC} \) is confirmed by a decrease of the lower bound on \( J_{WC} \). This lower bound is reduced from a value of 0.0336 (when \( \Phi_{uid}(\omega) = 1 \)) to a value of 0.0295 after four iterations. The lower bound can be computed using a gridding of the uncertainty set \( U \) corresponding to the chosen spectrum.
Fig. 4. Some of the spectra $\Phi_{u\omega}(\omega)$ found during the performed iterations: initial spectrum (red dashed), first iteration spectrum (blue dashdot), spectrum obtained after four iterations and $M = 6$ (black solid).

Fig. 5. A realization of the to-be-reconstructed signal $u(t)$ (blue) and the corresponding reconstruction error $\epsilon(t)$ (red) when using the filter $F(z)$ represented in Figure 3.

Fig. 6. Final spectrum $\Phi_{u\omega}(\omega)$ obtained after four iterations and with $M = 6$ (black solid), $\Phi_{u\omega}(\omega)$ of the optimal RBS signal ($\nu = 6$) (red dashed) and spectrum $\Phi_\nu(\omega)$ of the to-be-reconstructed signal (blue dashdot).

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