Solution Concepts and Well-posedness of Hybrid Systems

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DISC Course on Modeling and Control of Hybrid Systems 2017

Key issues:

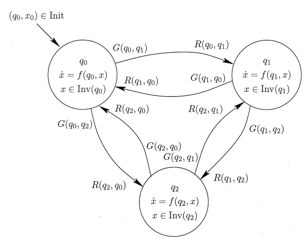
- Dynamics and Funny Behaviour: Solution concepts
- Well-posedness: existence & uniqueness of solutions given initial condition
- Second hour: (Start with) Stability and stabilisation (control/observer design)

Outline lecture

- How does this work for continuous systems (differential equations)?
- How does this work for hybrid automata?
- What can happen?
  - Zeno behaviour: infinite number of discrete events/actions in a finite length interval ...
- Various examples
- Switched systems/Discontinuous dynamical systems
  - Sliding modes

Solution concept

Description format / syntax / model
↓
solutions / trajectories / executions/ semantics/ behavior



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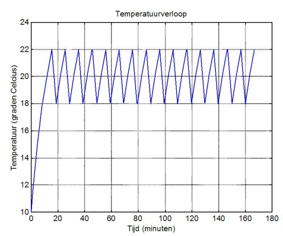


Well-posedness: given initial condition does there **exist** a solution and is it **unique**?

Let's start simple ...

Hybrid systems

Hybrid systems are combinations of

- discrete models such as finite state machines / automata
- continuous models such as differential equations

How does this work for continuous systems, i.e. differential eqs?

Continuous systems: differential equations

Example $\dot{x} = f(t,x) \quad x(0) = x_0$.

A solution trajectory is a function $x : [0,T] \to \mathbb{R}^n$ that is continuous, differentiable and satisfies $x(0) = x_0$ and

$$\dot{x}(t) = f(t,x(t)) \text{ for all } t \in (0,T)$$

Well-posedness: given initial condition does there **exists** a solution and is it **unique**?

Question: Who can say something about this that makes sense???

Well-posedness

Example $\dot{x} = 2\sqrt{|x|}, x(0) = 0$. Solutions: $x(t) = 0$ and $x(t) = t^2$.

Local existence and uniqueness of solutions given an initial condition:

Theorem 1 Let $f(t,x)$ be piecewise continuous in t and satisfy the following Lipschitz condition: there exist an $L > 0$ and $r > 0$ such that

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|$$

and all x and y in a neighborhood $B := \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$ of x_0 and for all $t \in [0,T]$.

$$\Downarrow$$

There is a $\delta > 0$ s.t. a unique solution exists on $[0,\delta]$ starting in x_0 at time 0.

- what if f is continuously differentiable?

Global well-posedness

Example $\dot{x} = x^2 + 1, x(0) = 0$. Solution: $x(t) = \tan t$. **Local** on $[0, \pi/2)$.

• Note that we have $\lim_{t\uparrow\pi/2} x(t) = \infty$. Finite escape time!

Theorem 2 (Global Lipschitz condition) Suppose $f(t,x)$ is piecewise continuous in t and satisfies

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|$$

for all x, y in $\mathbb{R}^n$ and for all $t \in [0,T]$. Then, a unique solution exists on $[0,T]$ for any initial state x_0 at 0.

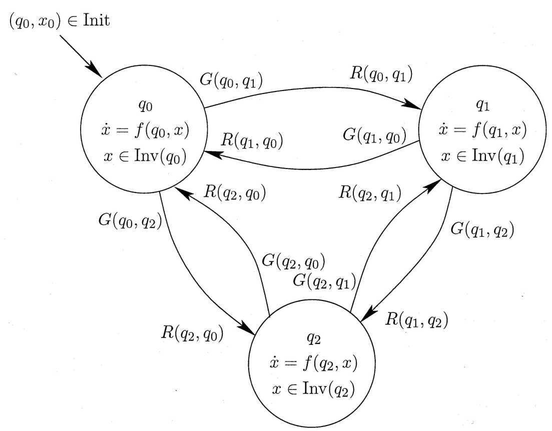
• Not necessary: $\dot{x} = -x^3$ not glob. Lipsch., but unique global solutions.

• As hybrid system = discrete system $\times$ continuous system, the above can happen, but even more awkward stuff (Zeno)

Hybrid automaton

Hybrid automaton H is collection $H = (Q, X, f, \mathsf{Init}, \mathsf{Inv}, E, G, R)$ with

- $Q = \{q_1, \ldots, q_N\}$ is finite set of discrete states or *modes*
- $X = \mathbb{R}^n$ is set of continuous states
- $f : Q \times X \to X$ is vector field
- $\mathsf{Init} \subseteq Q \times X$ is set of initial states
- $\mathsf{Inv} : Q \to P(X)$ describes the *invariants*
- $E \subseteq Q \times Q$ is set of edges or *transitions*
- $G : E \to P(X)$ is *guard condition*
- $R : E \to P(X \times X)$ is *reset map*



Evolution of hybrid automaton

- Initial hybrid state $(q_0, x_0) \in \mathsf{Init}$
- Continuous state x evolves according to

$$\dot{x} = f(q_0, x) \quad \text{with } x(0) = x_0$$

discrete state q remains constant: $q(t) = q_0$

- Continuous evolution can go on as long as $x \in \mathsf{Inv}(q_0)$
- If at some point state x reaches guard $G(q_0, q_1)$, then
  - transition $q_0 \to q_1$ is enabled
  - discrete state *may* change to q_1, continuous state then jumps from current value x^- to new value x^+ with $(x^-, x^+) \in R(q_0, q_1)$
- Next, continuous evolution resumes and whole process is repeated

Formalization: Hybrid time trajectory

Definition 3 A hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$ is a finite ($N < \infty$) or infinite ($N = \infty$) sequence of intervals of the real line, such that

- $I_i = [\tau_i, \tau_i']$ with $\tau_i \leq \tau_i' = \tau_{i+1}$ for $0 \leq i < N$;
- if $N < \infty$, either $I_N = [\tau_N, \tau_N']$ or $I_N = [\tau_N, \tau_N')$ with $\tau_N \leq \tau_N' \leq \infty$.

- For instance,

$$\tau = \{[0,2],[2,3],\{3\},\{3\},[3,4.5],\{4.5\},[4.5,6]\}$$

$$\tau = \{[0,2],[2,3],[3,4.5],\{4.5\},[4.5,6],[6,\infty)\}$$

$$I_0 = [0,\frac{1}{2}], I_1 = [\frac{1}{2},\frac{3}{4}], I_2 = [\frac{3}{4},\frac{7}{8}], \ldots, I_i = [1-2^i, 1-2^{i+1}]$$

- $\lim \tau_i' = 1$ for latter case!

Execution of hybrid automaton

Definition 4 An execution χ of a HA consists of $\chi = (\tau, q, x)$

- τ a hybrid time trajectory;
- $q = \{q_i\}_{i=0}^N$ with $q_i : I_i \to Q$; and
- $x = \{x_i\}_{i=0}^N$ with $x_i : I_i \to X$

Initial condition $(q(\tau_0), x(\tau_0)) \in \mathsf{Init}$;

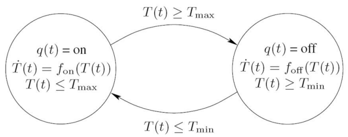
Continuous evolution for all i

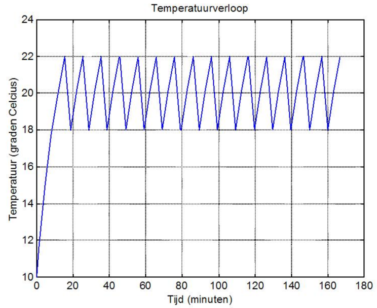
- q_i is constant, i.e. $q_i(t) = q_i(\tau_i)$ for all $t \in I_i$;
- x_i is solution to $\dot{x}(t) = f(q_i(t), x(t))$ on I_i with initial condition $x_i(\tau_i)$ at τ_i;
- for all $t \in [\tau_i, \tau_i')$ it holds that $x_i(t) \in \mathsf{Inv}(q_i(t))$.

Discrete evolution for all i,

- $e = (q_i(\tau_i'), q_{i+1}(\tau_{i+1})) \in E$,
- $x(\tau_i') \in G(e)$;
- $(x_i(\tau_i'), x_{i+1}(\tau_{i+1})) \in R(e)$.

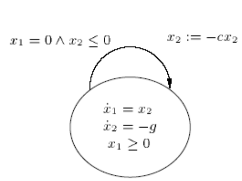
Executions of hybrid automata: the thermostat example

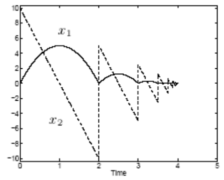




This is an infinite execution as it can be defined for all times $t \in [0, \infty)$

Executions of hybrid automata: the bouncing ball example

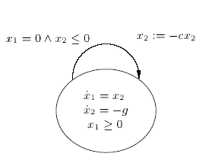


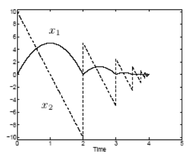


- Dynamics: $\ddot{x}_1 = -g$ subject to $x_1 \geq 0$ ($x_1(t)$: height)
- $x_2(t)$ is velocity
- Newton's restitution rule ($0 < c < 1$):

$$x_2(\tau+) = -cx_2(\tau-) \quad \text{when } x_1(\tau-) = 0, x_2(\tau-) \leq 0$$

Bouncing ball





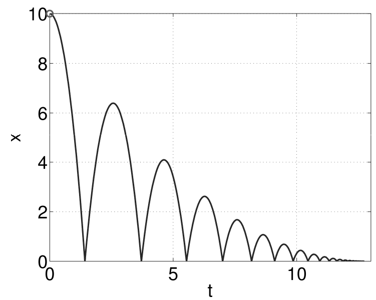
- Assuming $x_1(0) = 0$, $x_2(0) > 0$, event times are related through

$$\tau_{i+1} = \tau_i + \frac{2c^i x_2(0)}{g}$$

- Sequence has finite limit $\tau^* = \frac{2x_2(0)}{g - gc} < \infty$ (geometric series)
- Physical interpretation: ball is at rest within finite time span, but after infinitely many bounces $\rightarrow$ Zeno behavior

  In this case: infinite number of state re-initializations, set of event times contains *right-accumulation point*

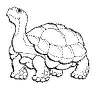
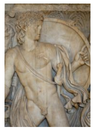
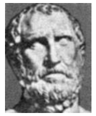
Bouncing ball



This is also called an infinite execution as it has an infinite number of transitions ...

Zeno behavior: infinitely many mode switches in finite length time interval

Zeno of Elea and one of his paradoxes

Distance Travelled (m) by Achilles	Event times of A reaching previous T position
1	1
0.5	1.5
0.25	1.75
0.125	1.875
0.0625	1.9375
0.03125	1.96875
0.015625	1.984375
0.0078125	1.9921875
0.00390625	1.99609375
0.001953125	1.998046875

Executions of hybrid automata: the M system

$Q = \{q_1, q_2\}$, $X = \mathbb{R}$, Init = $\{(q_1, 0)\}$

Mode q_1:

- $\dot{x} = f(q_1, x) = 1$
- $\mathsf{Inv}(q_1) = \{x \in X \mid x \leq 1\}$

Mode q_2:

- $\dot{x} = f(q_2, x) = -1$
- $\mathsf{Inv}(q_2) = \{x \in X \mid x \geq 0\}$

Transitions: $E = \{(q_1, q_2)\}$ with Guard $G((q_1, q_2)) = \{x \in X \mid x \geq \frac{1}{2}\}$

Reset relation $R(q_1, q_2) = \{(x, 0) \mid x \in X\}$.

System is not deterministic

No infinite solutions, but - as any HA - it does have so-called maximal ones .... at end of execution, the system is in deadlock

Well-posedness for hybrid automata

- $\mathscr{H}^{\infty}_{(q_0,x_0)}$: infinite executions: either defined on $[0, \infty)$ or infinite number of transitions.
  Formally: τ is an infinite sequence or if $\lim_{i \to N} \tau'_i = \sum_i (\tau'_i - \tau_i) = \infty$
- $\mathscr{H}^{M}_{(q_0,x_0)}$: maximal executions: solution cannot be continued any further (at end of the execution system in deadlock or it is an infinite execution).
  Formally: τ is not a strict prefix of another one!

- A hybrid automaton is called *non-blocking*, if $\mathscr{H}^{\infty}_{(q_0,x_0)}$ is non-empty for all $(q_0, x_0) \in \mathsf{Init}$.
- It is called *deterministic*, if $\mathscr{H}^{M}_{(q_0,x_0)}$ contains at most one element for all $(q_0, x_0) \in \mathsf{Init}$.

Well-posedness for hybrid automata - continued

Assumption

- The vector field $f(q, \cdot)$ is globally Lipschitz continuous for all $q \in Q$.
- The edge $e = (q, q')$ is contained in E if and only if $G(e) \neq \emptyset$ and $x \in G(e)$ if and only if there is an $x' \in X$ such that $(x, x') \in R(e)$.

A state $(\hat{q}, \hat{x}) \in \mathsf{Reach}$, if there exists a finite execution (τ, q, x) with $\tau = \{[\tau_i, \tau'_i]\}_{i=0}^{N}$ and $(q(\tau'_N), x(\tau'_N)) = (\hat{q}, \hat{x})$.

SmoothContinuation and Out

- The set of states from which continuous evolution is possible:

$$\texttt{SmoothContinuation} = \{(q_0,x_0) \in Q \times X \mid \exists \varepsilon > 0 \forall t \in [0,\varepsilon)\ x_{q_0,x_0}(t) \in \mathsf{Inv}(q_0)\}$$

- The set of states from which continuous evolution is impossible :

$$\mathsf{Out} = \{(q_0,x_0) \in Q \times X \mid \forall \varepsilon > 0 \exists t \in [0,\varepsilon)\ x_{q_0,x_0}(t) \notin \mathsf{Inv}(q_0)\}$$

in which $x_{q_0,x_0}(\cdot)$ denotes the unique solution to $\dot{x} = f(q_0,x)$ with $x(0) = x_0$.

Well-posedness theorems

Theorem A hybrid automaton is non-blocking, if for all $(q,x) \in \mathsf{Reach} \cap \mathsf{Out}$, there exists $e = (q,q') \in E$ with $x \in G(e)$. In case the automaton is deterministic, this condition is also necessary.

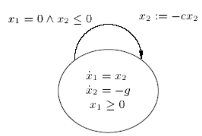
Theorem A hybrid automaton is deterministic, if and only if for all $(q,x) \in \mathsf{Reach}$

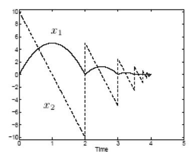
- if $x \in G((q,q'))$ for some $(q,q') \in E$, then $(q,x) \in \mathsf{Out}$;
- if $(q,q') \in E$ and $(q,q'') \in E$ with $q' \neq q''$, then $x \notin G((q,q')) \cap G((q,q''))$; and
- if $(q,q') \in E$ and $x \in G((q,q'))$, then there is at most one $x' \in X$ with $(x,x') \in R((q,q'))$.

Examples with Zeno behavior

- Zeno behavior : infinitely many mode switches in finite time interval
- Prevents that solutions are globally defined $[0,\infty)$ ("simulators get stuck")
- Examples
  1. bouncing ball
  2. two-tank system

Bouncing ball





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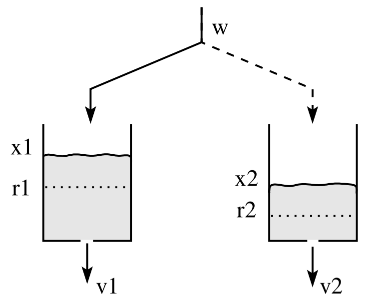
- Physical interpretation: ball is at rest within finite time span, but after infinitely many bounces $\rightarrow$ Zeno behavior

  In this case: infinite number of state re-initializations, set of event times contains *right-accumulation point*

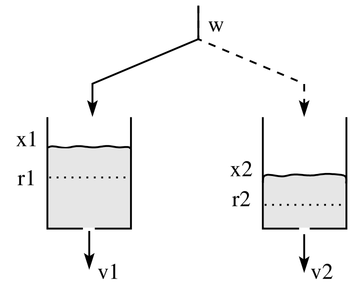
Non-blocking and deterministic HA, but no solutions on $[0,\infty)$

Two tank system



- Two tanks (x_i: volume of water in tank)
- Tanks are leaking at constant rate $v_i > 0$
- Water is added at constant rate w through hose, which at any point in time is dedicated to either one tank or the other
- Objective: keep water volumes above r_1 and r_2
- Controller that switches inflow to tank 1 whenever $x_1 \leq r_1$ and to tank 2 whenever $x_2 \leq r_2$

Description of two-tank system as hybrid automaton

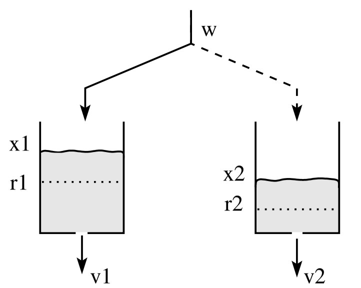


- Two modes: filling tank 1 (mode q_1) or tank 2 (mode q_2)
- Evolution of continuous state:

$$\begin{cases} \dot{x}_1 = w - v_1 \\ \dot{x}_2 = -v_2 \end{cases} \text{in mode } q_1 \qquad \begin{cases} \dot{x}_1 = -v_1 \\ \dot{x}_2 = w - v_2 \end{cases} \text{in mode } q_2$$

- Init $= \{q_1, q_2\} \times \{(x_1, x_2) \mid x_1 \geq r_1 \text{ and } x_2 \geq r_2\}$

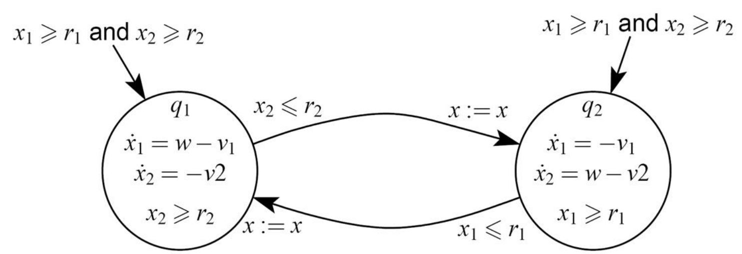
Description of two-tank system as hybrid automaton (cont.)



- Invariants: $\text{Inv}(q_1) = \{x \in \mathbb{R}^2 \mid x_2 \geq r_2\}$
  $\text{Inv}(q_2) = \{x \in \mathbb{R}^2 \mid x_1 \geq r_1\}$
- Guards: $G(q_1, q_2) = \{x \in \mathbb{R}^2 \mid x_2 \leq r_2\}$
  $G(q_2, q_1) = \{x \in \mathbb{R}^2 \mid x_1 \leq r_1\}$
- No resets:

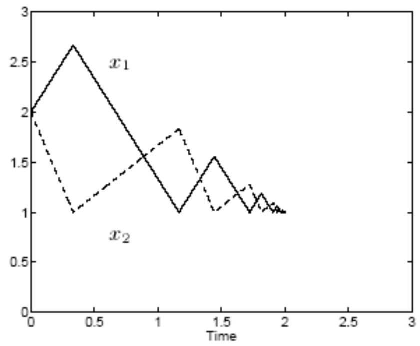
$$R(q_1, q_2) = R(q_2, q_1) = \{(x^-, x^+) \mid x^-, x^+ \in \mathbb{R}^2 \text{ and } x^- = x^+\}$$

Description of two-tank system as hybrid automaton (cont.)

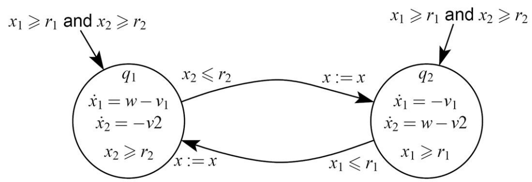


A simulation

$r_1 = r_2 = 1$, $v_1 = 2$, $v_2 = 3$, $w = 4$, $x_1(0) = x_2(0) = 2$, $q(0) = q_1$



Two-tank system and Zeno behavior



- Assume total outflow $v_1 + v_2 > w$
- Control objective cannot be met and tanks will empty in finite time
- Infinitely many switchings in finite time $\rightarrow$ Zeno behavior

Be careful with your conclusions!

Summary

- Smooth differential equations
  - Lipschitz continuity sufficient for well-posedness
  - absence Lipschitz: possibly non-uniqueness $\dot{x} = 2\sqrt{x}$
  - absence global Lipschitz: possibly finite escape times and no global existence $\dot{x} = x^2 + 1$
- Hybrid automata:
  - Non-blocking and deterministic can be checked via Out and Reach
  - Complications due to Zeno: non-blocking HA might have no solutions on $[0, \infty)$
  - Zenoness might also lead to erroneous conclusions ... tanks do not stay full ...

Dynamics of switched systems? Funny phenomena?

Switched systems

$$\dot{x} = f_\sigma(x)$$

$\{f_1(x), f_2(x), \ldots, f_N(x)\}$ family of smooth vector fields from $\mathbb{R}^n$ to $\mathbb{R}^n$

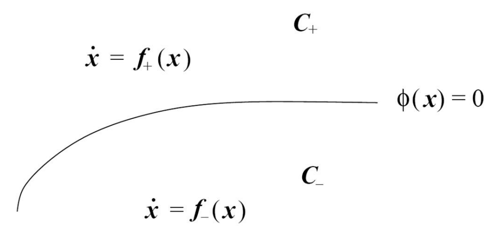
Switching signal $\sigma : [0,\infty) \mapsto \{1,2,\ldots,N\}$ piecewise constant function of time

- Function of time t: $\sigma(t)$
- Function of state $x(t)$: $\sigma(x)$
  Piecewise smooth systems or Discontinuous dynamical systems
- Combinations: $\sigma(t,x)$

No resets, continuous state variable x evolves continuously, only its derivative may be discontinuous ...

Example: thermostat

Discontinuous dynamical systems / Piecewise smooth systems

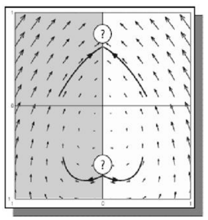


State-dependent switching

Example 1: Piecewise linear system

.

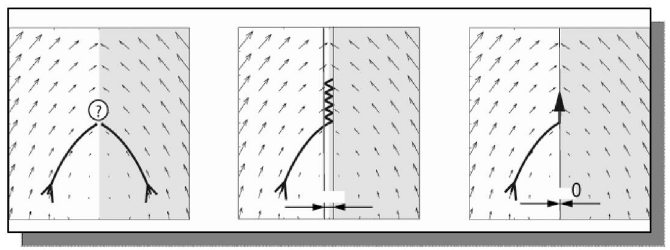
$$\begin{cases} \dot{x}_1 = -2x_1 - 2x_2 \operatorname{sgn}(x_1) \\ \dot{x}_2 = x_2 + 4x_1 \operatorname{sgn}(x_1) \end{cases}$$



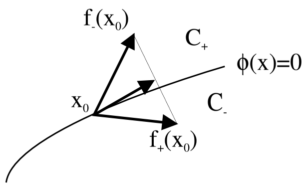
$$\dot{x} = \begin{cases} A_1 x & \text{, when } x_1 < 0 \\ A_2 x & \text{, when } x_1 > 0 \end{cases}$$

where $A_1 = \begin{pmatrix} -2 & 2 \\ -4 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} -2 & -2 \\ 4 & 1 \end{pmatrix}$.

Example 1: Piecewise linear system



Sliding modes



$f_+(x)$ points towards C_- and $f_-(x)$ points towards C_+.

No classical solution

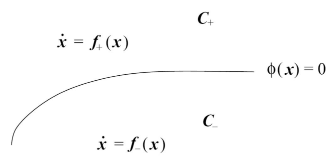
- Relaxation: spatial (hysteresis) Δ, time delay τ, smoothing ε
- Chattering / infinitely fast switching (limit case $\Delta \downarrow 0$, $\varepsilon \downarrow 0$, and $\tau \downarrow 0$)

Filippov's convex definition: convex combination of both dynamics

$$\dot{x} = \lambda f_+(x) + (1-\lambda) f_-(x) \text{ with } 0 \le \lambda \le 1$$

such that x moves ("slides") along $\phi(x) = 0$. **"Third mode ... "**

Discontinuous differential equations: a class of switched systems



$$\dot{x} = \begin{cases} f_+(x) & , \text{ if } x \in C_+ := \{x \in \mathbb{R}^n \mid \phi(x) > 0\} \\ f_-(x) & , \text{ if } x \in C_- := \{x \in \mathbb{R}^n \mid \phi(x) < 0\} \end{cases}$$

- x in interior of C_- or C_+: just follow!
- $f_-(x)$ and $f_+(x)$ point in same direction: just follow!
- $f_+(x)$ points towards C_+ and $f_-(x)$ points towards C_-: At least two trajectories
- $f_+(x)$ points towards C_- and $f_-(x)$ points towards C_+: sliding mode! Filippov

Differential inclusions

$$\dot{x} = \begin{cases} f_+(x), & \text{if } \phi(x) > 0 \\ \lambda f_+(x) + (1-\lambda) f_-(x), & \text{if } \phi(x) = 0,\ 0 \le \lambda \le 1 \\ f_-(x), & \text{if } \phi(x) < 0, \end{cases}$$

Differential inclusion $\dot{x} \in F(x)$ with set-valued

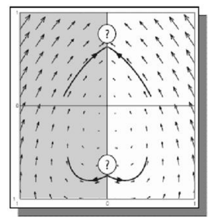
$$F(x) = \begin{cases} \{f_+(x)\}, & \phi(x) > 0 \\ \{\lambda f_+(x) + (1-\lambda) f_-(x) \mid \lambda \in [0,1]\}, & \phi(x) = 0 \\ \{f_-(x)\}, & \phi(x) < 0 \end{cases}$$

Definition 5 A function $x : [a,b] \mapsto \mathbb{R}^n$ is a *solution* of $\dot{x} \in F(x)$, if x is absolutely continuous and satisfies $\dot{x}(t) \in F(x(t))$ for almost all $t \in [a,b]$.

Example 1: Piecewise linear

.

$$\begin{cases} \dot{x}_1 = -2x_1 - 2x_2 \text{sgn}(x_1) \\ \dot{x}_2 = x_2 + 4x_1 \text{sgn}(x_1) \end{cases}$$



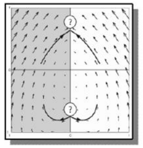
Example 1: Piecewise linear system

Equivalent dynamics on sliding modes

Example: Piecewise linear system

$$\begin{cases} \dot{x}_1 = -2x_1 - 2x_2\,\mathrm{sgn}(x_1) \\ \dot{x}_2 = x_2 + 4x_1\,\mathrm{sgn}(x_1) \end{cases}$$

on $\mathcal{S}_1^+ = \{x \mid x_1 = 0 \wedge x_2 \geq 0\}$



Filippov solutions satisfy $\dot{x}(t) \in \alpha A_1 x(t) + (1-\alpha) A_2 x(t)$ for some $\alpha \in [0,1]$

If x(t) should stay on $\mathcal{S}_1^+$, we must have $\dot{x}_1(t) = 0$, i.e.,

$$\alpha \cdot 2x_2 + (1-\alpha)\cdot(-2x_2) = x_2(4\alpha - 2) = 0$$

The only solution is given by α=1/2, resulting in the unique sliding dynamics

$$\dot{x}_1 = 0, \qquad \dot{x}_2 = x_2$$

- Attractive and repulsive sliding mode
- Sliding modes might be non-unique

Example 2:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - u \\ \dot{x}_2 &= 2x_2(u^2 - u - 1) \\ u &= \begin{cases} 1, & \text{if } x_1 > 0 \\ -1, & \text{if } x_1 < 0. \end{cases} \end{aligned}$$

Two "original" dynamics:

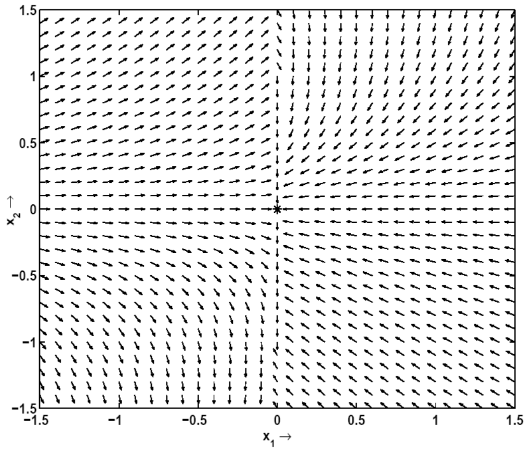
- C_+ the region $x_1 > 0$: $\dot{x} = f_+(x)$

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - 1 \\ \dot{x}_2 &= -2x_2 \end{aligned}$$

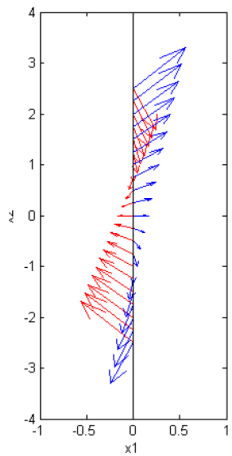
- C_- the region $x_1 < 0$ $\dot{x} = f_-(x)$

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 + 1 \\ \dot{x}_2 &= 2x_2 \end{aligned}$$

Vector fields



Vector fields: zoom



Sliding modes?

Two “original” dynamics:

- C_+ the region $x_1 > 0$: $\dot{x} = f_+(x)$

$$\dot{x}_1 = -x_1 + x_2 - 1$$
$$\dot{x}_2 = -2x_2$$

- C_- the region $x_1 < 0$: $\dot{x} = f_-(x)$

$$\dot{x}_1 = -x_1 + x_2 + 1$$
$$\dot{x}_2 = 2x_2$$

- $n(x)^T f_+(x) = x_2 - 1 < 0 \quad \longrightarrow \quad x_2 < 1$
- $n(x)^T f_-(x) = x_2 + 1 > 0 \quad \longrightarrow \quad x_2 > -1$
- Sliding possible in $x_1 = 0$ and $x_2 \in [-1, 1]$.

Filippov’s solution concept

Two “original” dynamics:

- C_+ the region $x_1 > 0$: $\dot{x} = f_+(x)$

$$\dot{x}_1 = -x_1 + x_2 - 1$$
$$\dot{x}_2 = -2x_2$$

- C_- the region $x_1 < 0$: $\dot{x} = f_-(x)$

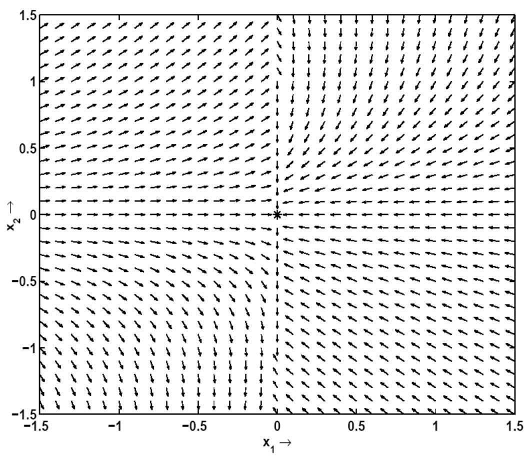
$$\dot{x}_1 = -x_1 + x_2 + 1$$
$$\dot{x}_2 = 2x_2$$

- Filippov: Take convex combination of dynamics

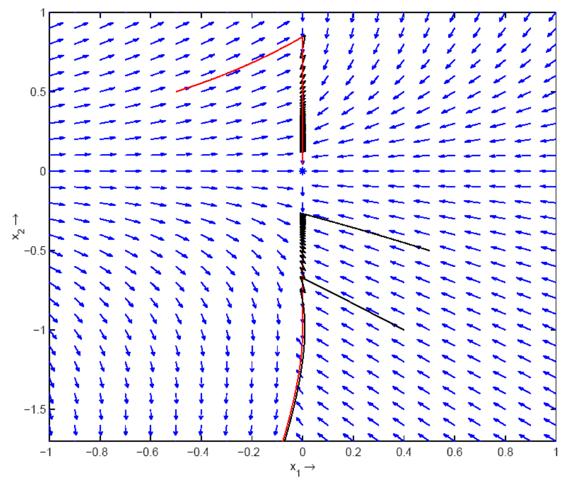
$$\dot{x} = \lambda f_+(x) + (1-\lambda) f_-(x)$$

  such that state slides on $x_1 = 0$: Hence, $x_1 = \dot{x}_1 = 0$.
- $\lambda(x_2 - 1) + (1-\lambda)(x_2 + 1) = 0$ implies $\lambda = \frac{1}{2}(x_2 + 1)$
- Hence, $\dot{x}_2 = \lambda(-2x_2) + (1-\lambda)(2x_2) = -2x_2^2$
- 0 is unstable equilibrium.

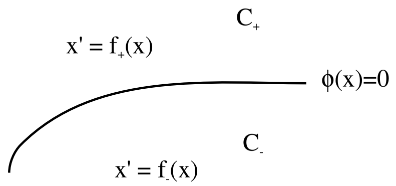
Vector fields: Filippov’s case



Solution trajectories: Filippov’s case + hysteresis



A well-posedness result



- f_- and f_+ are continuously differentiable (C^1)
- ϕ is C^2
- the discontinuity vector $h(x) := f_+(x) - f_-(x)$ is C^1

If for each point x with $\phi(x) = 0$ at least one of the two condition holds:

- $f_+(x)$ points strictly towards C_- ($n(x)^T f_+(x) < 0$ where $n(x) = \frac{\nabla\phi(x)}{\|\nabla\phi(x)\|}$ is normal to switching surface at the point x)
- $f_-(x)$ points strictly towards C_+ ($n(x)^T f_-(x) > 0$)

(for different points a different inequality may hold), then the Filippov solutions are unique.

Local existence is always guaranteed under continuity of f_+ and f_-.

Summary discontinuous dynamical systems

- Discontinuous dynamical systems and piecewise smooth systems
- Dynamics: sliding modes
- Formalized this
  - Filippov's convex definition (limit case / idealization of hysteresis, spatial delay)
- (Local) existence of solutions guaranteed.
- Well-posedness: directions of vector field at switching plane

Summary

- Continuous differential equations
  - Solution concept straightforward
  - Continuity sufficient for local existence, not for uniqueness
  - Local Lipschitz continuity sufficient for local well-posedness
  - Global Lipschitz continuity sufficient for global well-posedness
  - absence global Lipschitz finite escape times and no global existence
- Hybrid automata: non-blocking and deterministic
- Characterizations of well-posedness using `Reach` and `Out`!
- Conditions for hybrid automata: implicit!
- Be careful with conclusions due to Zeno!

Summary - continued

- Discontinuous differential equations
  - Interpret idealized simple models such that they match underlying real plant (hysteresis).
  - Filippov's convex definition
  - Solution concept from differential inclusions
  - Sliding motions
  - Local existence of solutions always guaranteed
  - For uniqueness conditions on directions of vector field at switching plane