OPTIMAL CONTROL OF HYBRID SYSTEMS: *STOCHASTIC MODELS AND APPLICATIONS*

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OUTLINE

DYNAMIC OPTIMIZATION

- *Receding Horizon* control
- Robustness Properties
- **Single Stage Manufacturing Systems**: Receding Horizon vs Optimal Control

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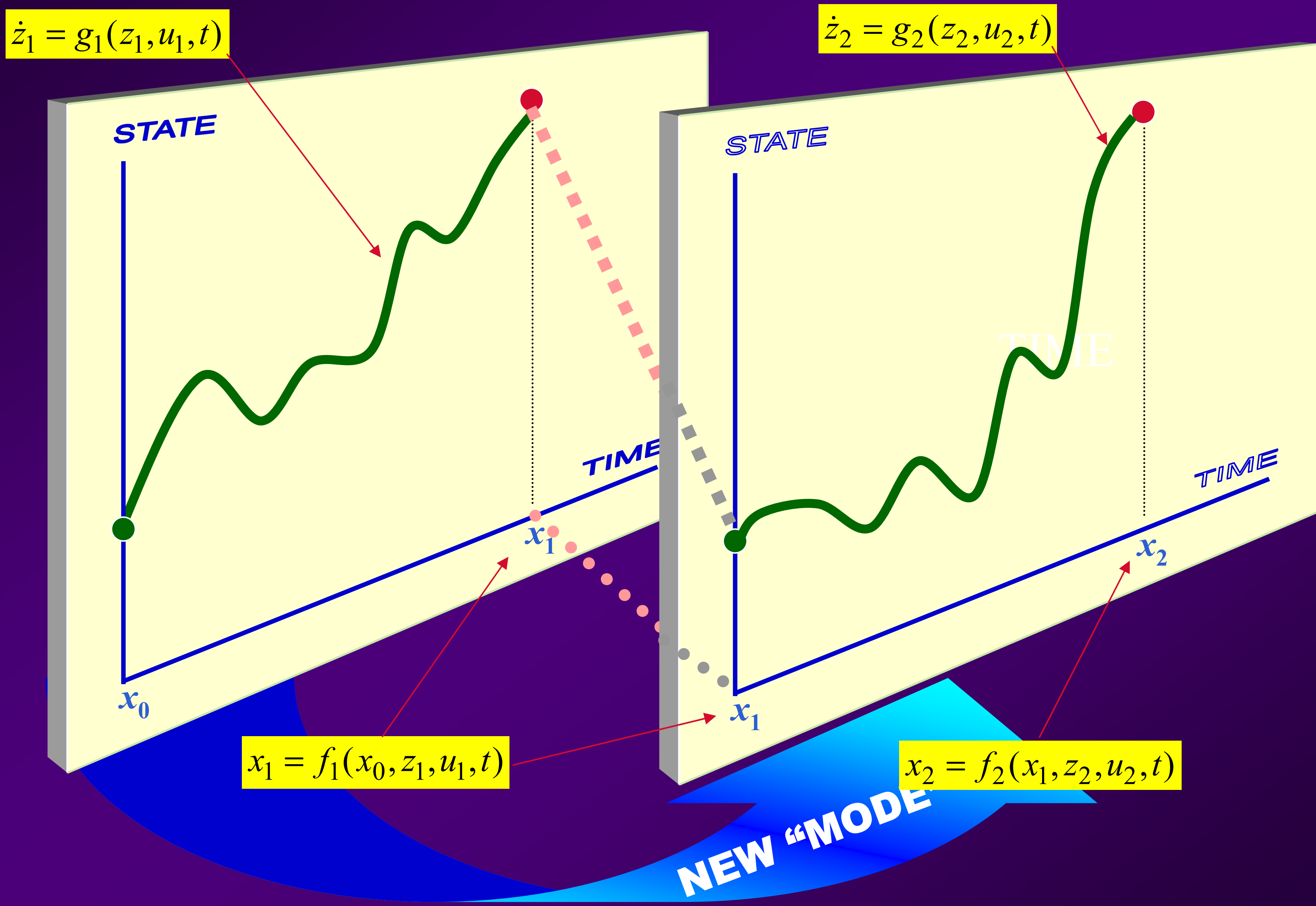
PARAMETRIC OPTIMIZATION

- *Stochastic Flow Models (SFM)* as Hybrid Systems
- Perturbation Analysis of SFMs
- **Communication Networks**: Threshold-based on-line control

WHAT'S A HYBRID SYSTEM?



HIGHER LEVEL PROBLEM

$$\min_{\mathbf{s}} \sum_{i=1}^{N} \left[\phi_i^*(s_i) + \psi_i(x_i) \right] \quad s.t. \quad x_i = \max\{x_{i-1}, a_i\} + s_i(u_i)$$

Cost of optimal process control over interval [0, s_i]

Cost related to event timing

Given arrival sequence (INPUT)

Processing time (CONTOLLABLE)

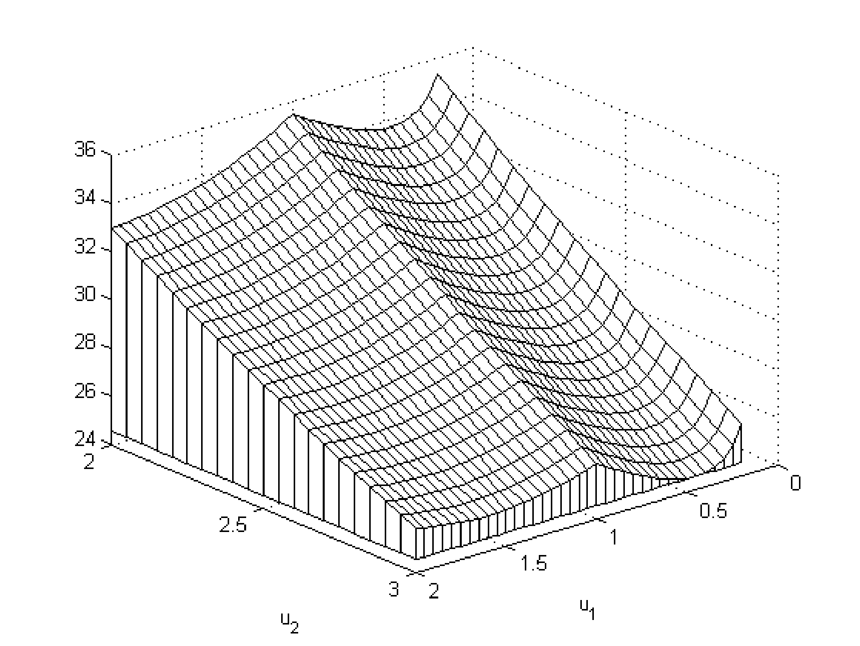
$$\text{EXAMPLE:} \quad \psi_i(x_i) = (x_i - \tau_i)^2$$

HOW DO WE SOLVE THE HIGHER LEVEL PROBLEM?

$$\min_{\mathbf{s}} \sum_{i=1}^{N} \left[\phi_i^*(s_i) + \psi_i(x_i) \right] \quad s.t. \quad x_i = \max\{x_{i-1}, a_i\} + s_i(u_i)$$

Even if these are convex, problem is still NOT convex in s!

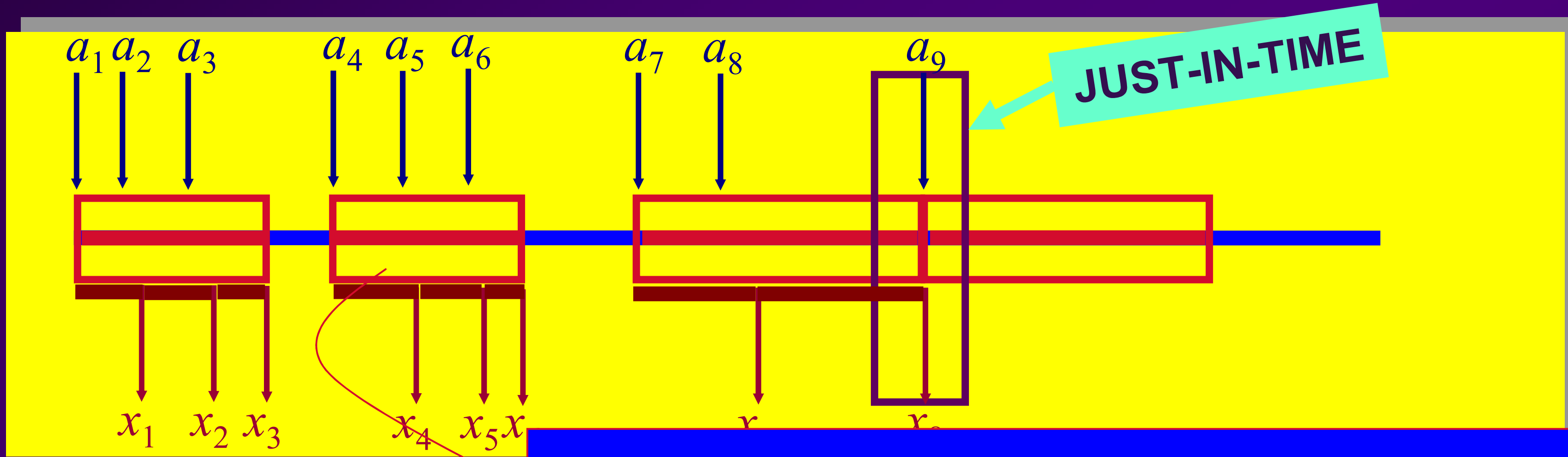
Causes nondifferentiabilities!



Even though problem is **NONDIFFERENTIABLE** and **NONCONVEX**, optimal solution shown to be ***unique***.

[Cassandras, Pepyne, Wardi, IEEE TAC 2001]

SOLVING THE HIGHER LEVEL PROBLEM *CONTINUED*



Each "block" corresponds to a ***Constrained Convex Optimization*** problem

$\Rightarrow$ search over 2^{N-1} possible *Constrained Convex Optimization* problems

BUT algorithms that only need *N Constrained Convex Optimization* problems have been developed $\Rightarrow$ SCALEABILITY

[*Cho, Cassandras, Intl. J. Rob. and Nonlin.Control, 2001*]

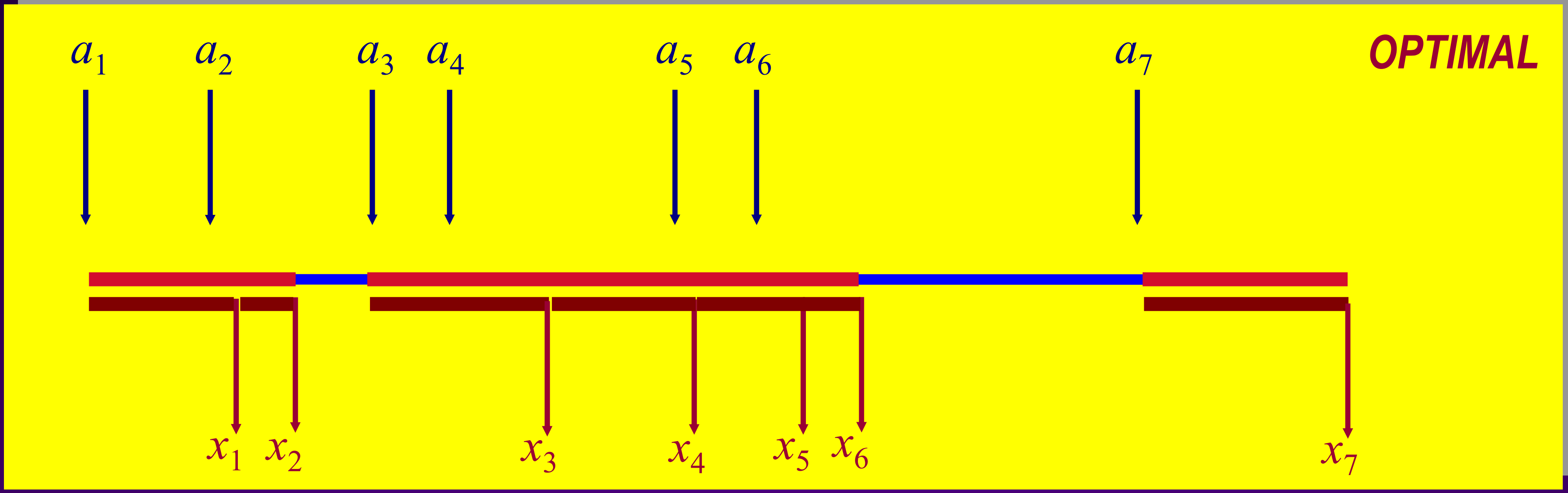
WHAT IF EVENT TIMES ARE UNKNOWN?

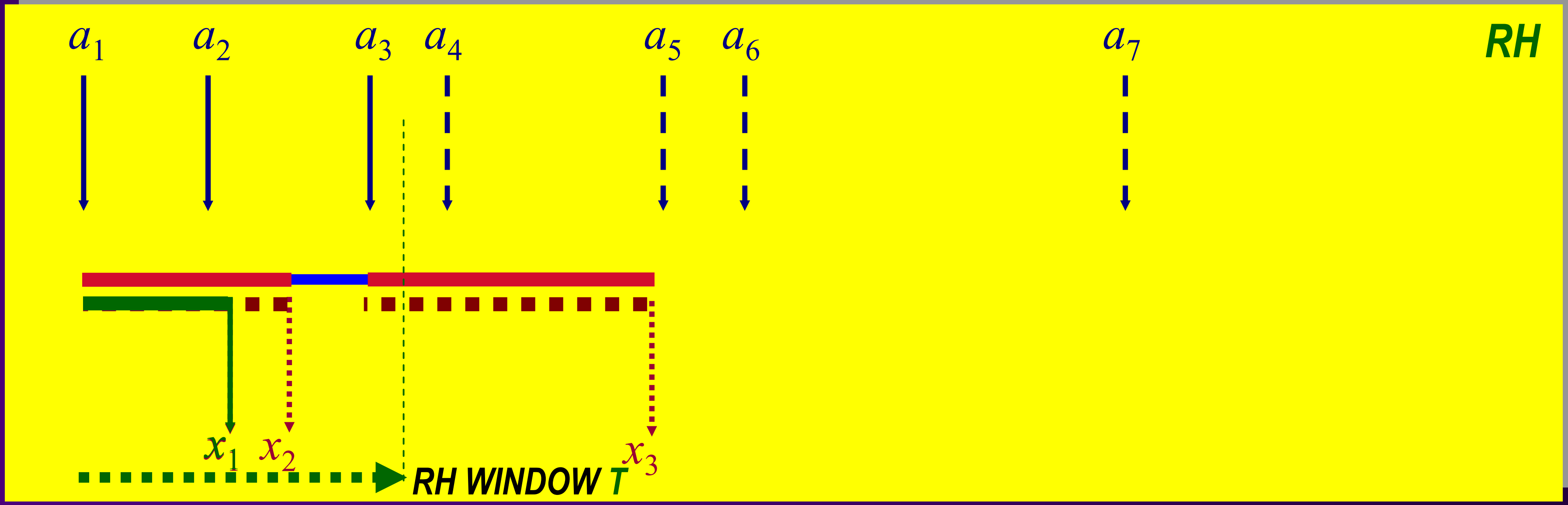
- **Event times known only over a "look-ahead window"** of length $T < \infty$

- **Only $K \geq 1$ future event times known**

- **Only probabilistic info. about event process known**

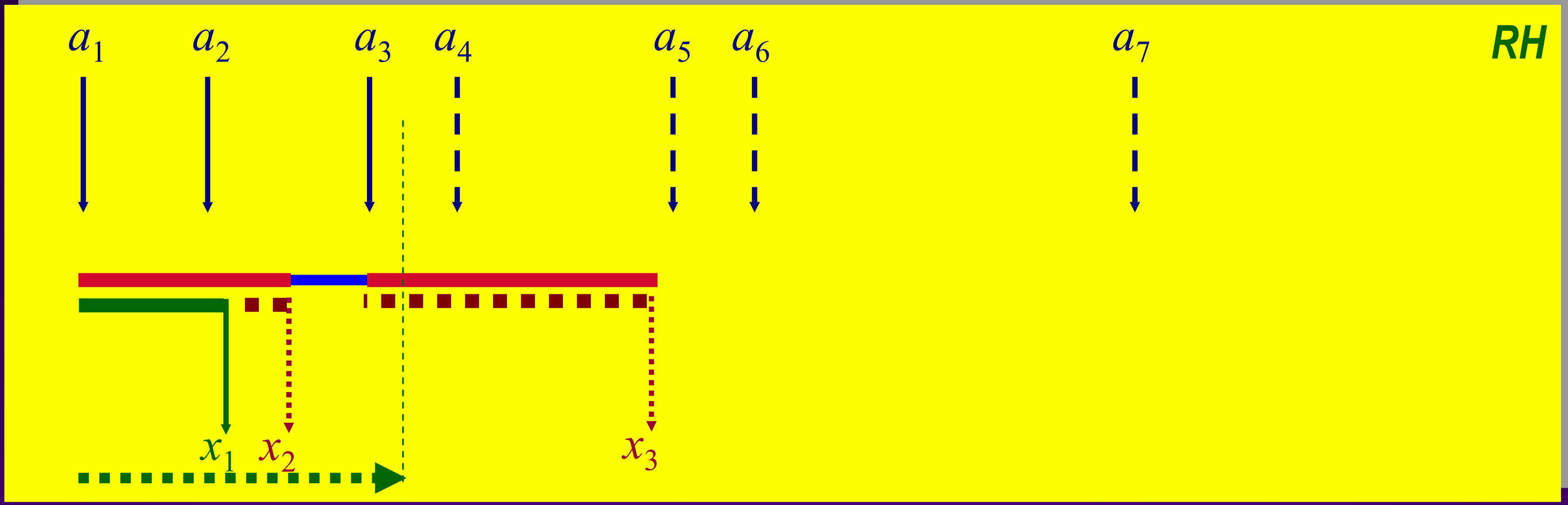
RECEDING HORIZON CONTROL

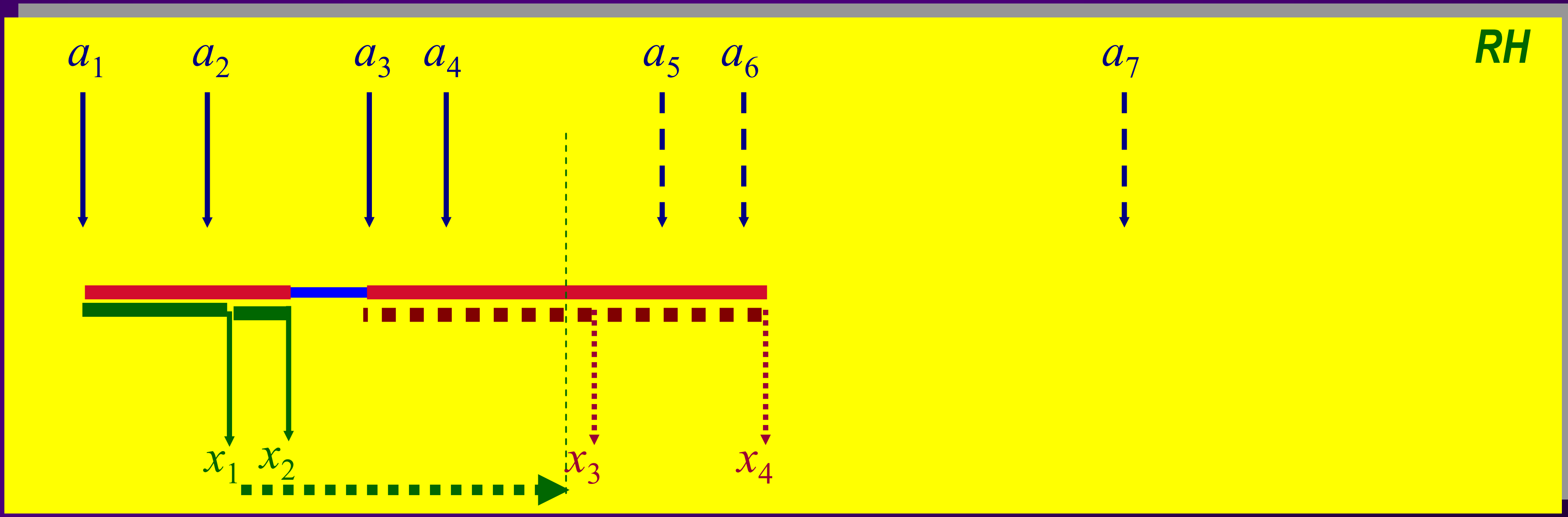




RECEDING HORIZON CONTROL

CONTINUED





RECEDING HORIZON CONTROL

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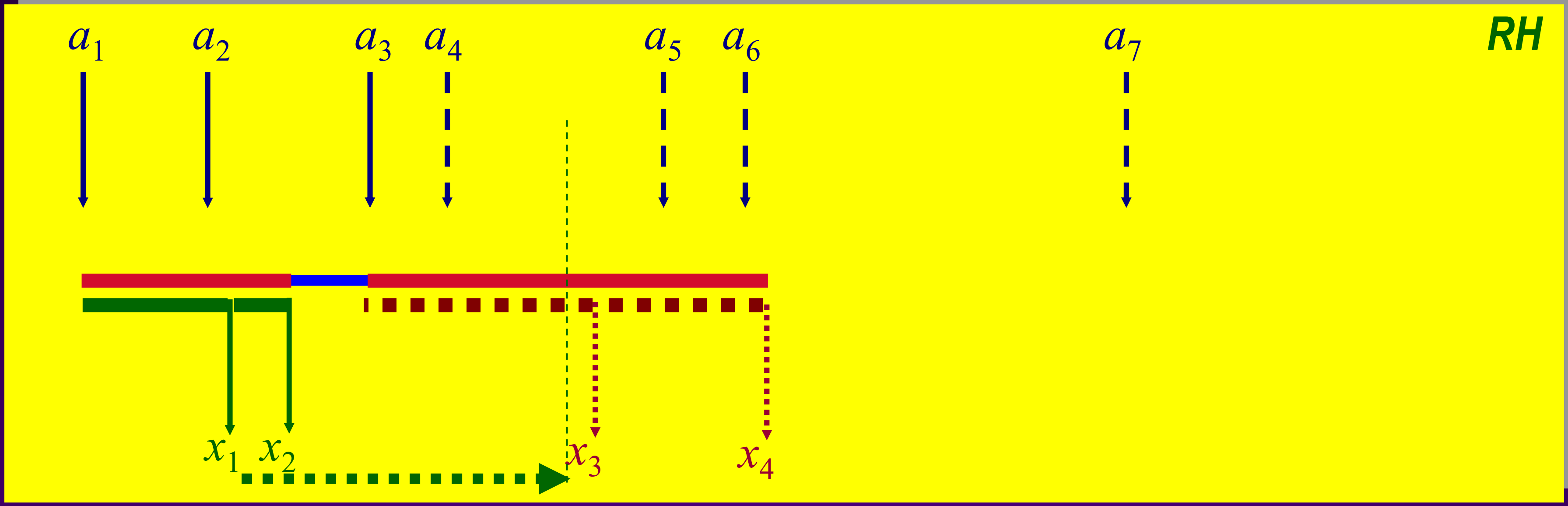

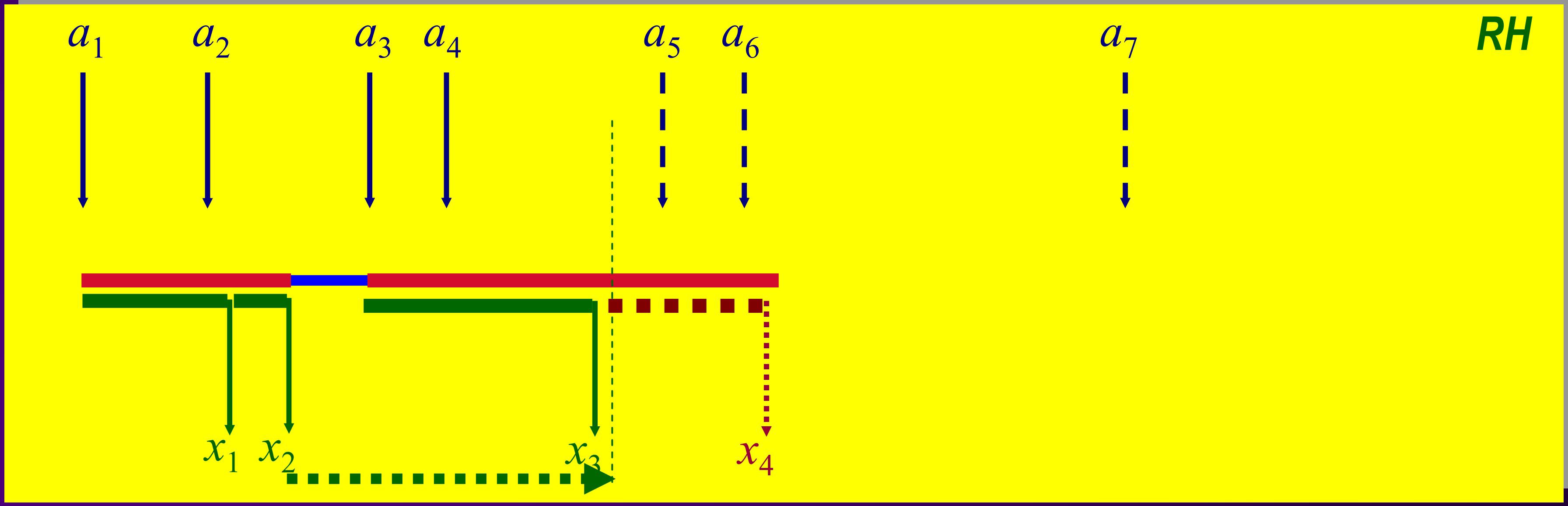

RECEDING HORIZON CONTROL

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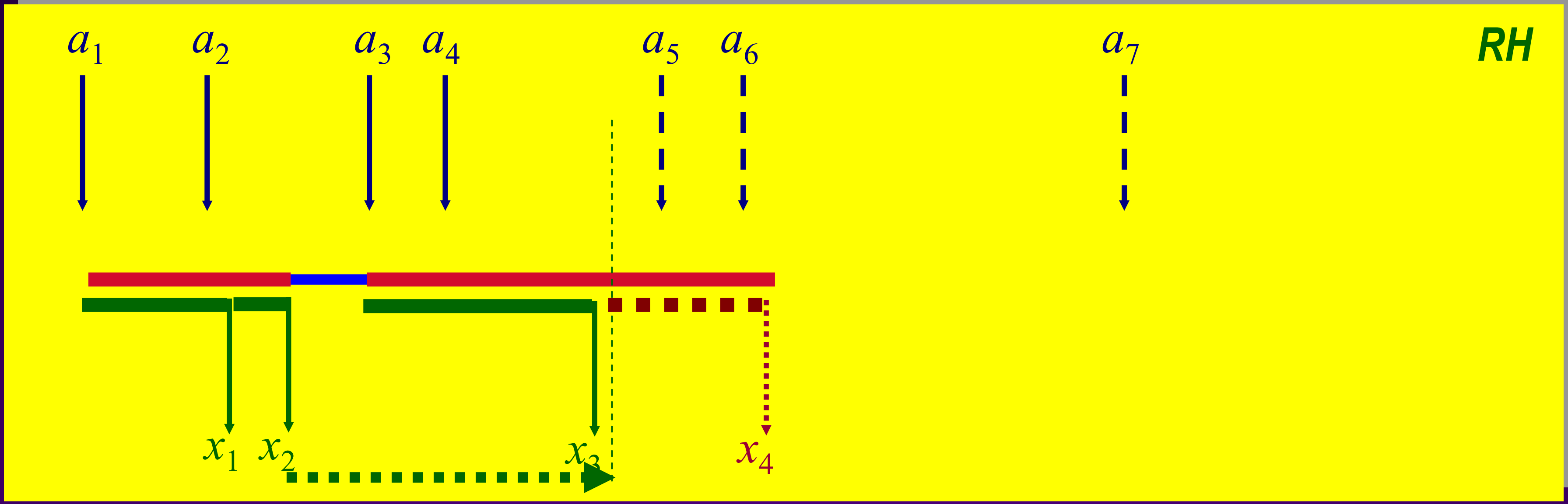

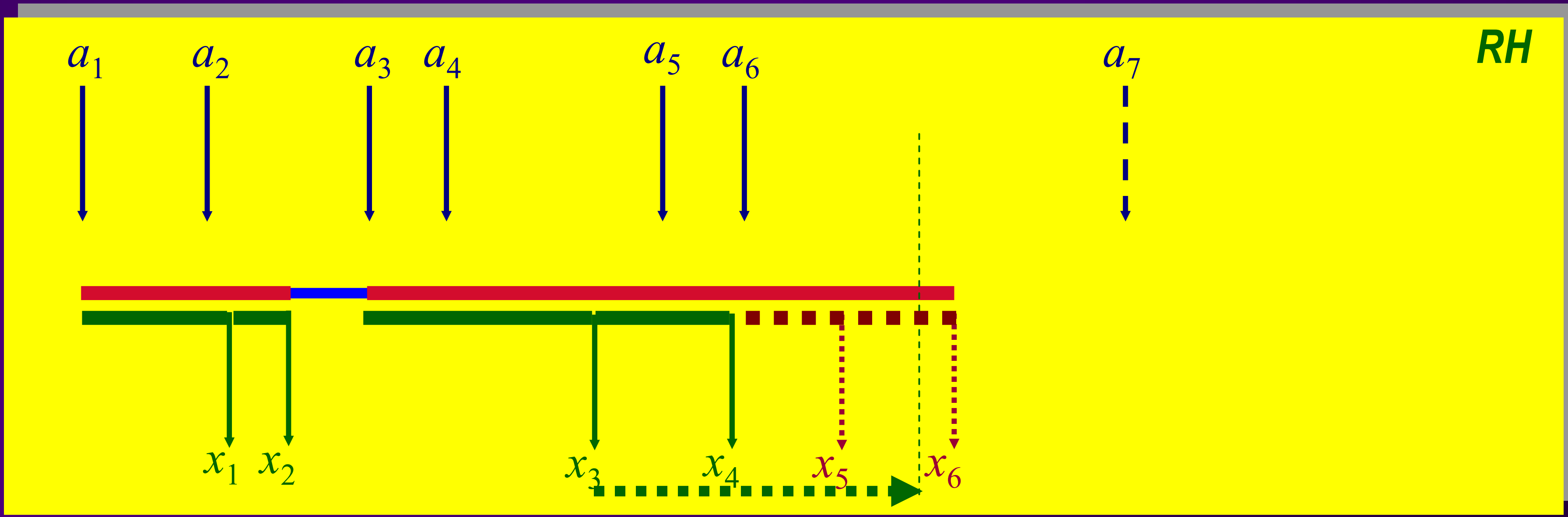

RECEDING HORIZON CONTROL

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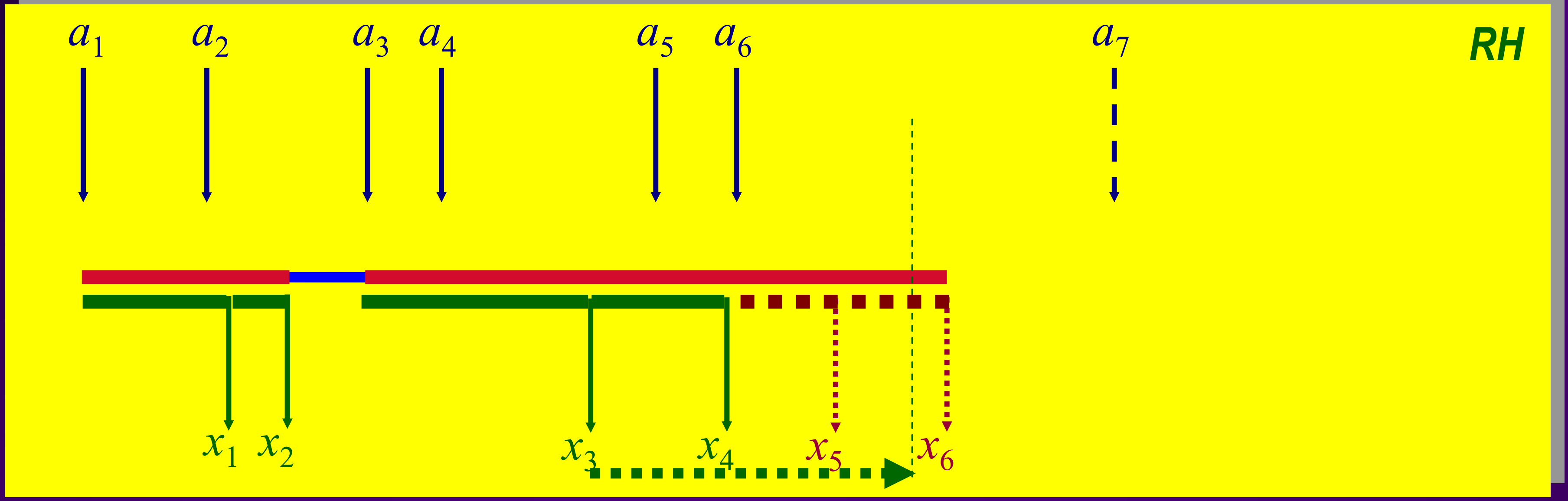
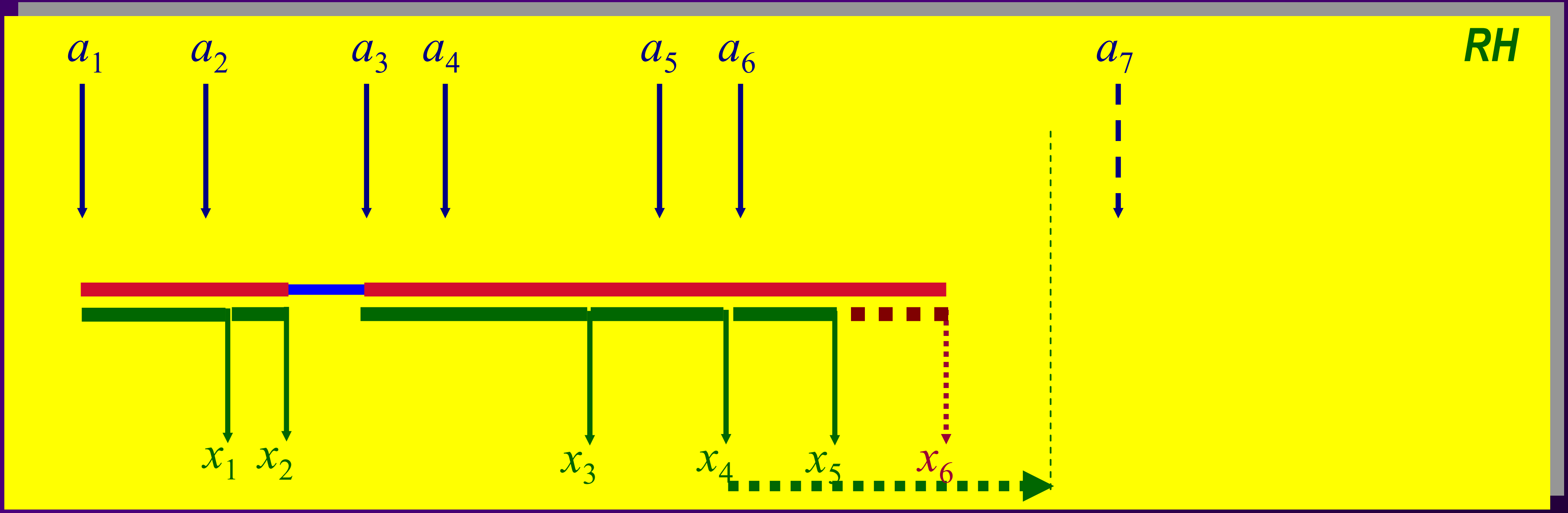
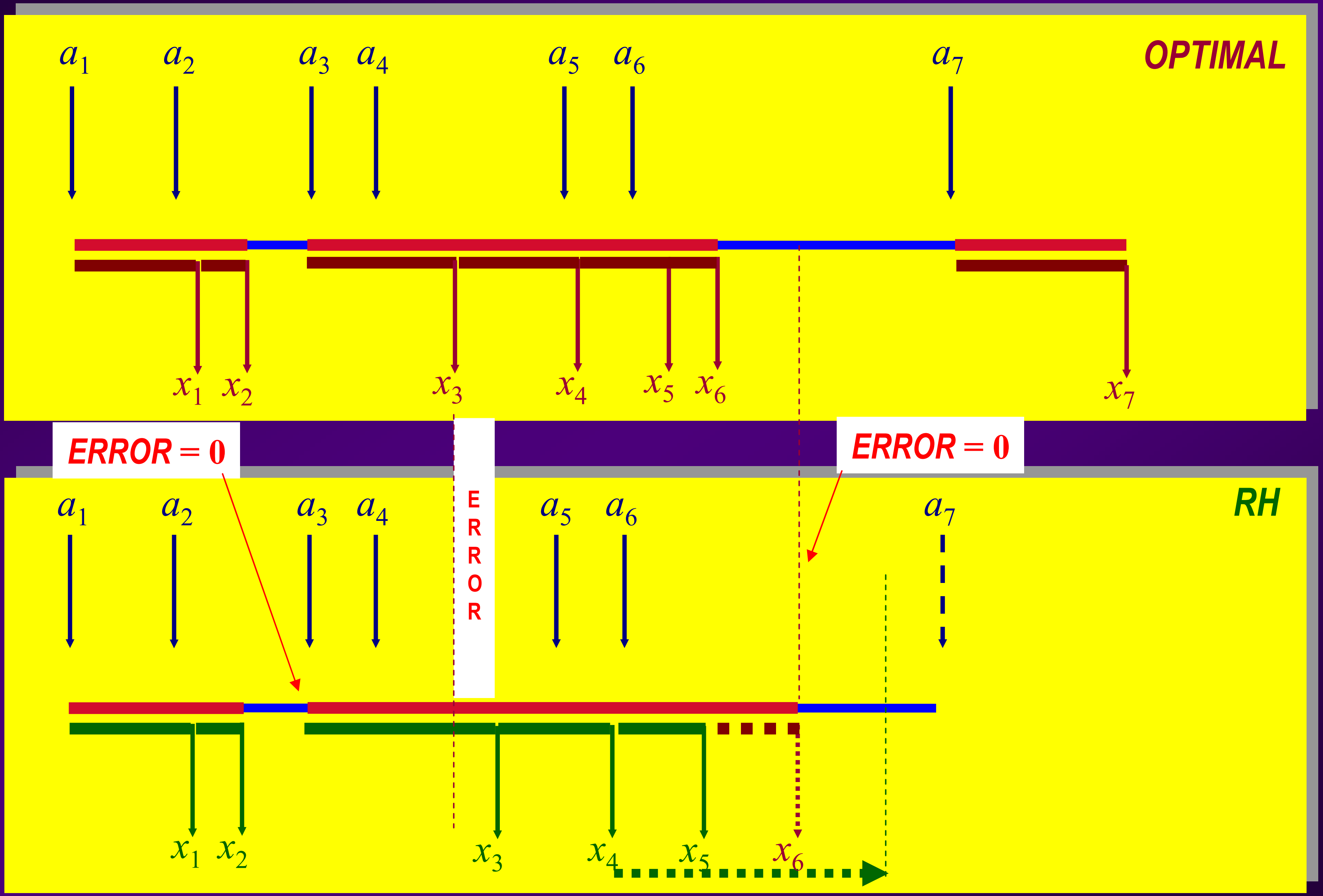




RECEDING HORIZON CONTROL

CONTINUED



RECEDING HORIZON CONTROL

CONTINUED

RH

$\tilde{x}_t$ = RH Time of event t

$\tilde{u}_t$ = RH Control of event t

T = RH lookahead window length

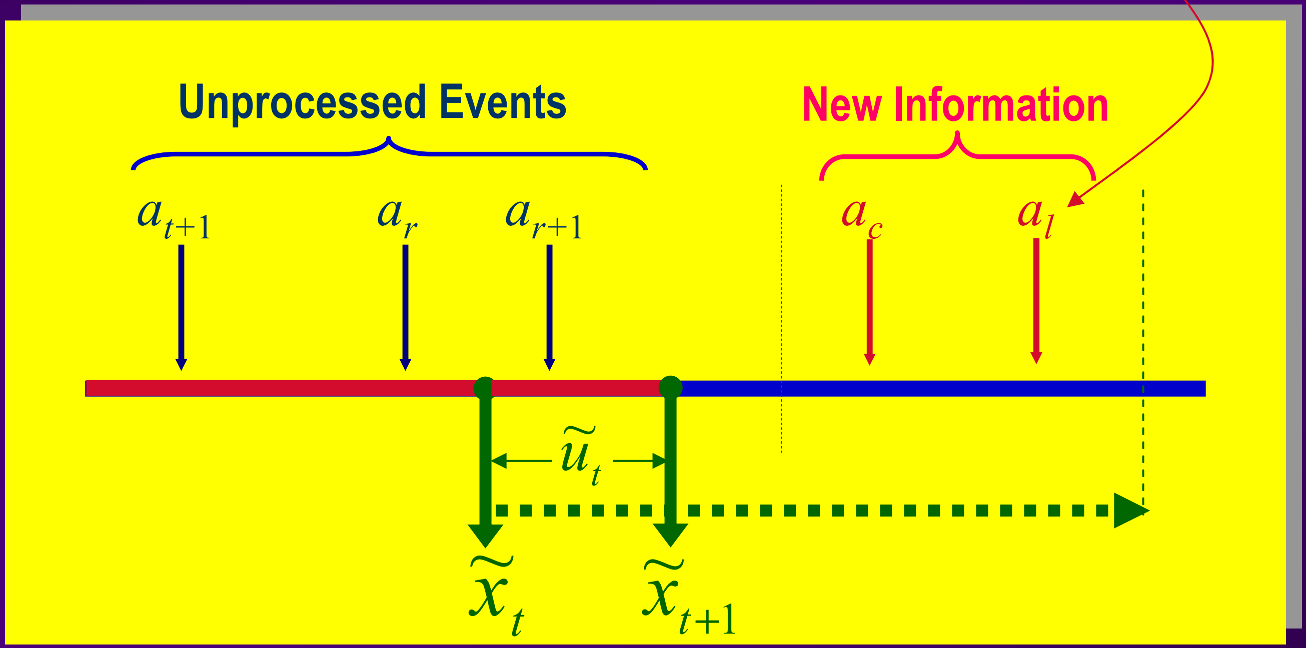
OPTIMAL

x_t^*

u_t^*

$T = \infty$

$$l = \arg\max_{i \geq t} \{a_i : a_i \leq \tilde{x}_t + T\}$$



RECEDING HORIZON CONTROL ALGORITHM

Original $Q(t+1,n)$ problem

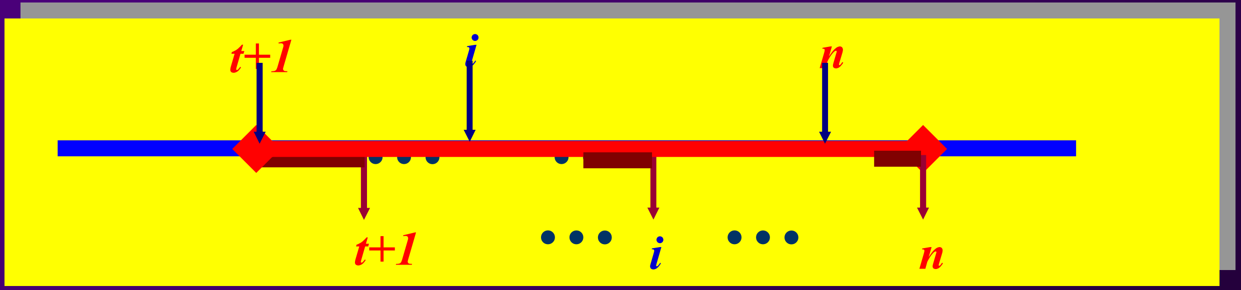
$$Q(t+1,n)=\min_{u_{t+1},\ldots,u_n}\left[\sum_{i=t+1}^{n}\left\{\theta_i(u_i)+\psi_i(a_{t+1}+\sum_{j=t+1}^{i}u_j)\right\}\right]$$

$$\text{s.t. } a_{t+1}+\sum_{j=t+1}^{i}u_j\geq a_{i+1},\quad i=t+1,\ldots,n-1,\quad u_i\geq 0$$

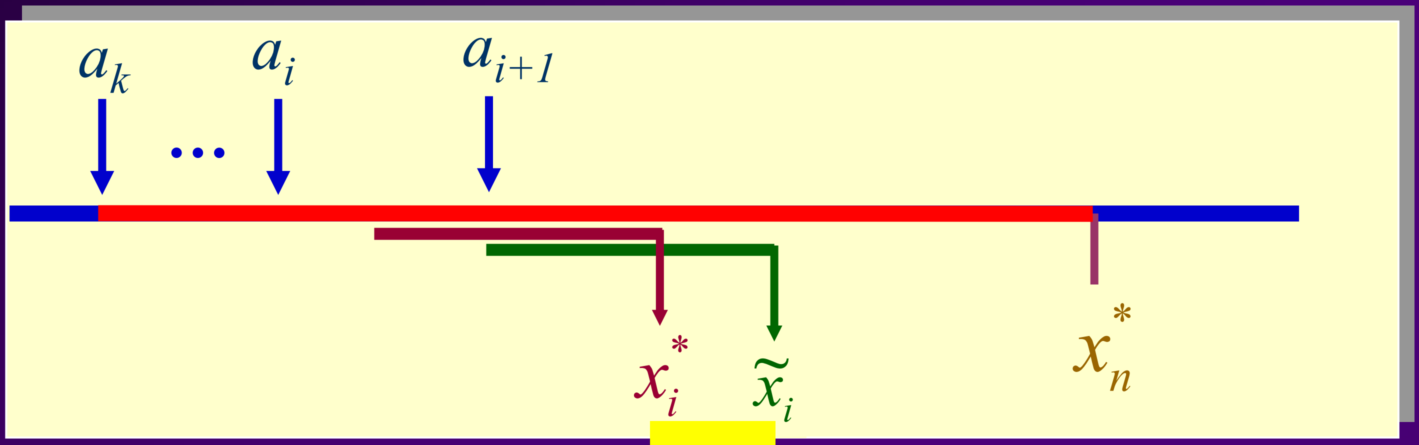
Replaces a_{t+1}

$$\tilde{Q}(t+1,n)=\min_{\tilde{u}_{t+1},\ldots,\tilde{u}_n}\left[\sum_{i=t+1}^{n}\left\{\theta_i(\tilde{u}_i)+\psi_i[\max(\tilde{x}_t,a_{t+1})+\sum_{j=t+1}^{i}\tilde{u}_j]\right\}\right]$$

$$\text{s.t. } \max(\tilde{x}_t,a_{t+1})+\sum_{j=t+1}^{i}\tilde{u}_j\geq a_{i+1},\quad i=t+1,\ldots,n-1,\quad \tilde{u}_j\geq 0$$



SOME RH CONTROLLER PROPERTIES



ERROR $\varepsilon_i = \tilde{x}_i - x_i^*$

PROPERTY 1: No critical event in $(k,n) \Rightarrow \tilde{x}_i \geq x_i^*$ for all $i = k,\ldots,n$

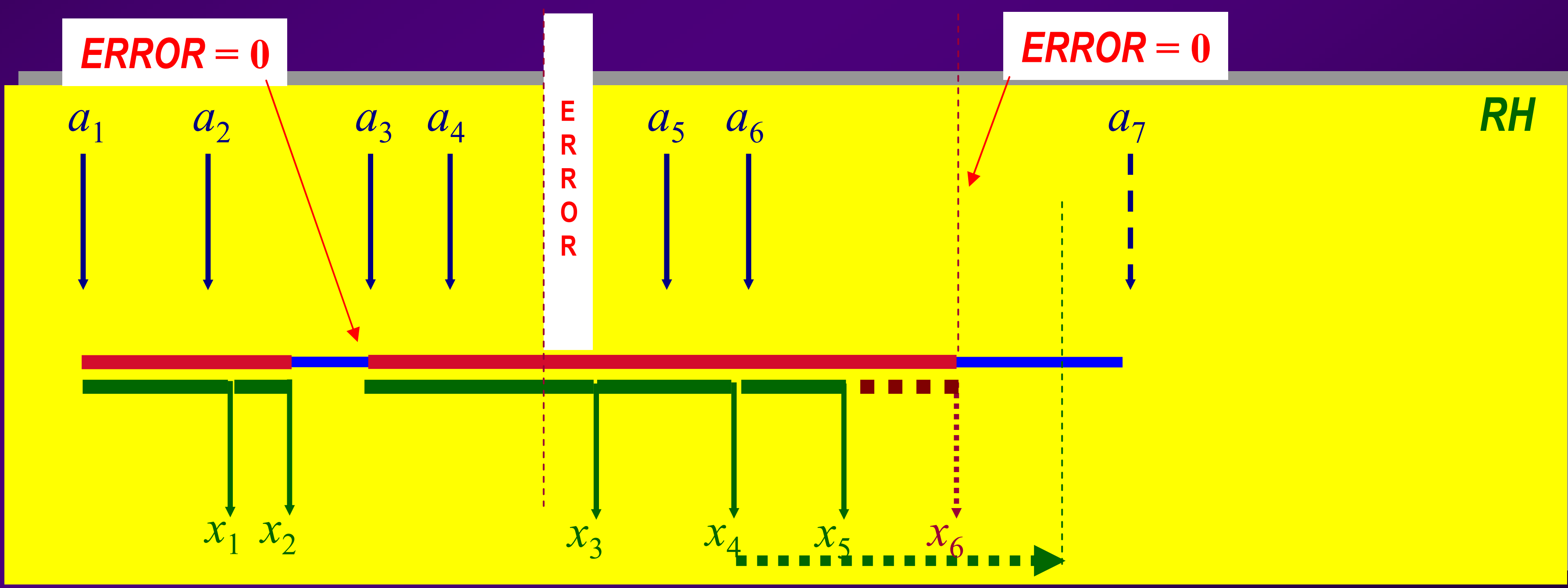
PROPERTY 2: $\tilde{x}_i > a_{i+1}$ for all $i = k,\ldots,n-1$

BUT:
Last (nth) event in BP can become critical in RH path !

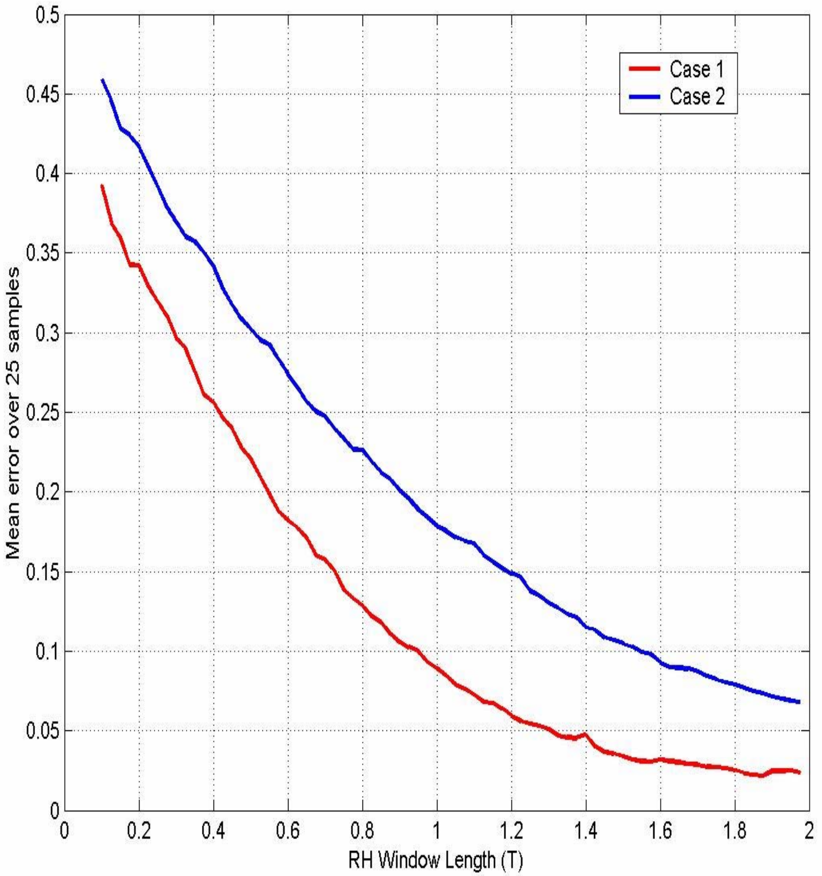
RH CONTROLLER *ERROR RESETTING* PROPERTY

PROPERTY 3: $k \le t < n,\ \ \tilde{x}_t + T \ge a_n \Rightarrow \begin{cases} 1. \quad \varepsilon_i \ge \varepsilon_{i+1} \text{ for } i = t+1,..,n \\ 2. \quad \varepsilon_j = 0 \text{ for some } j \Rightarrow \varepsilon_i = 0 \text{ for all } i > j \end{cases}$

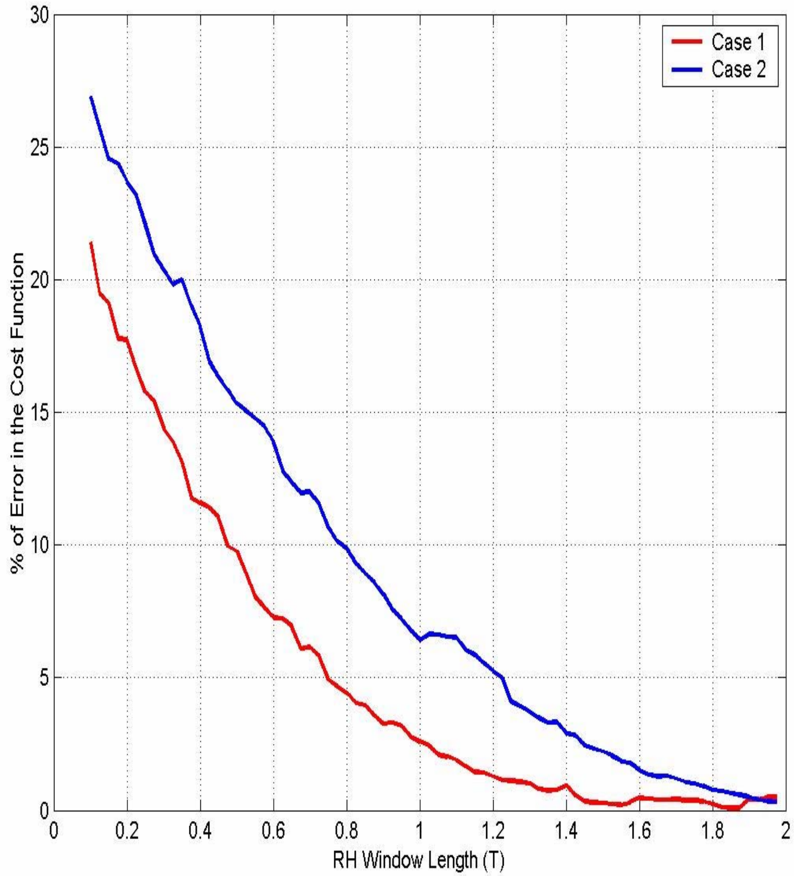
PROPERTY 4: $t \le k,\ \tilde{x}_t + T \ge a_n,\ \tilde{x}_{k-1} < a_k \Rightarrow \varepsilon_i = 0 \text{ for all } i = k,...,n$



Ensemble Avg. of Mean
ε(T) vs T
30 arrivals ~ U [0,15]

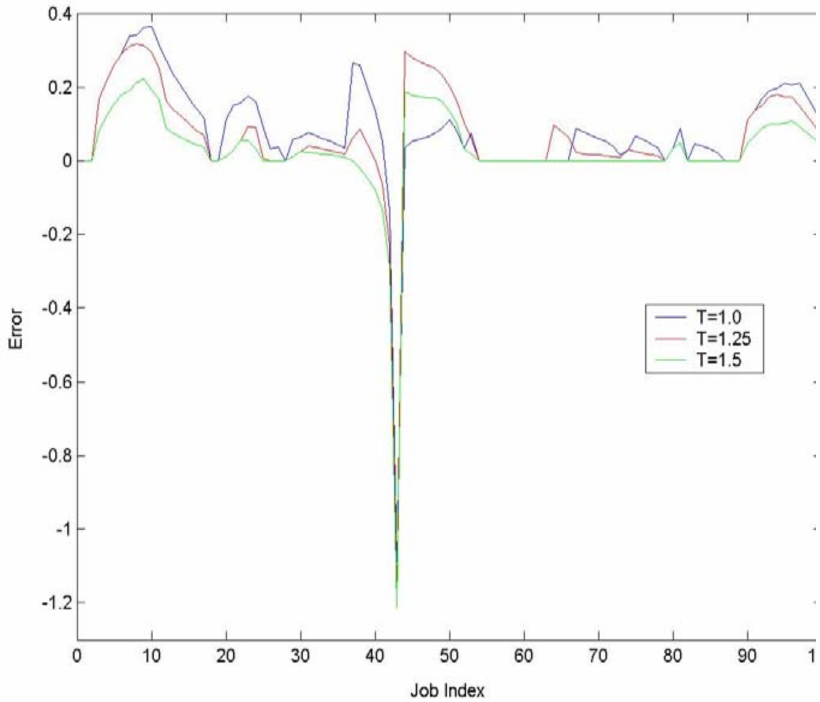


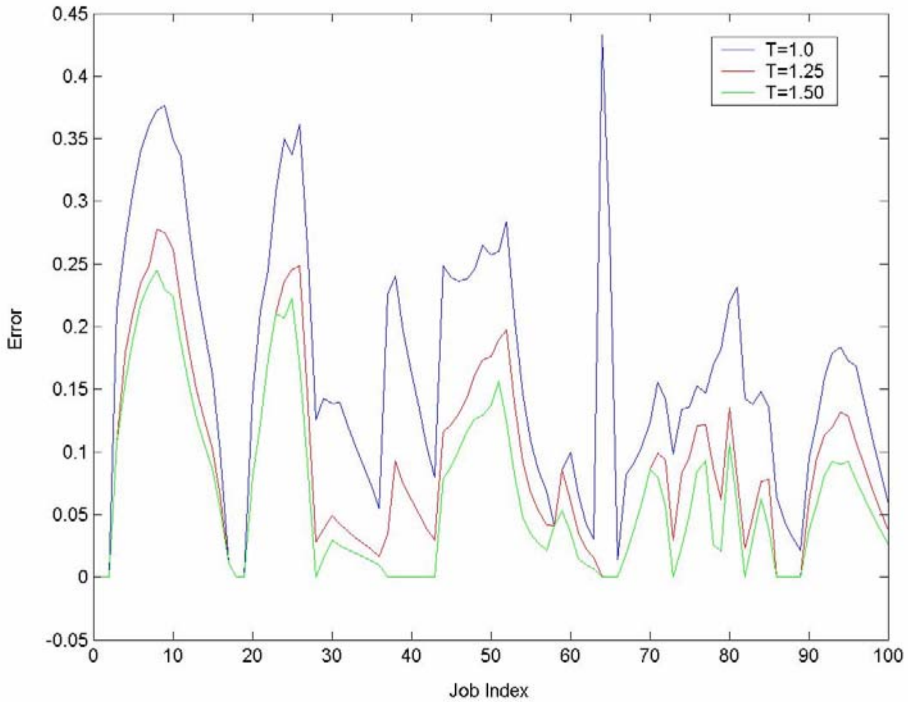
% of Error vs. T
N=100, Arr ~ U[0,50]



$\varepsilon_i(T)$ over Job index, *i* : Case 1

$\varepsilon_i(T)$ over Job index, *i* : Case 2





PARAMETRIC OPTIMIZATION

OPTIMIZATION OF *STOCHASTIC* HYBRID SYSTEMS

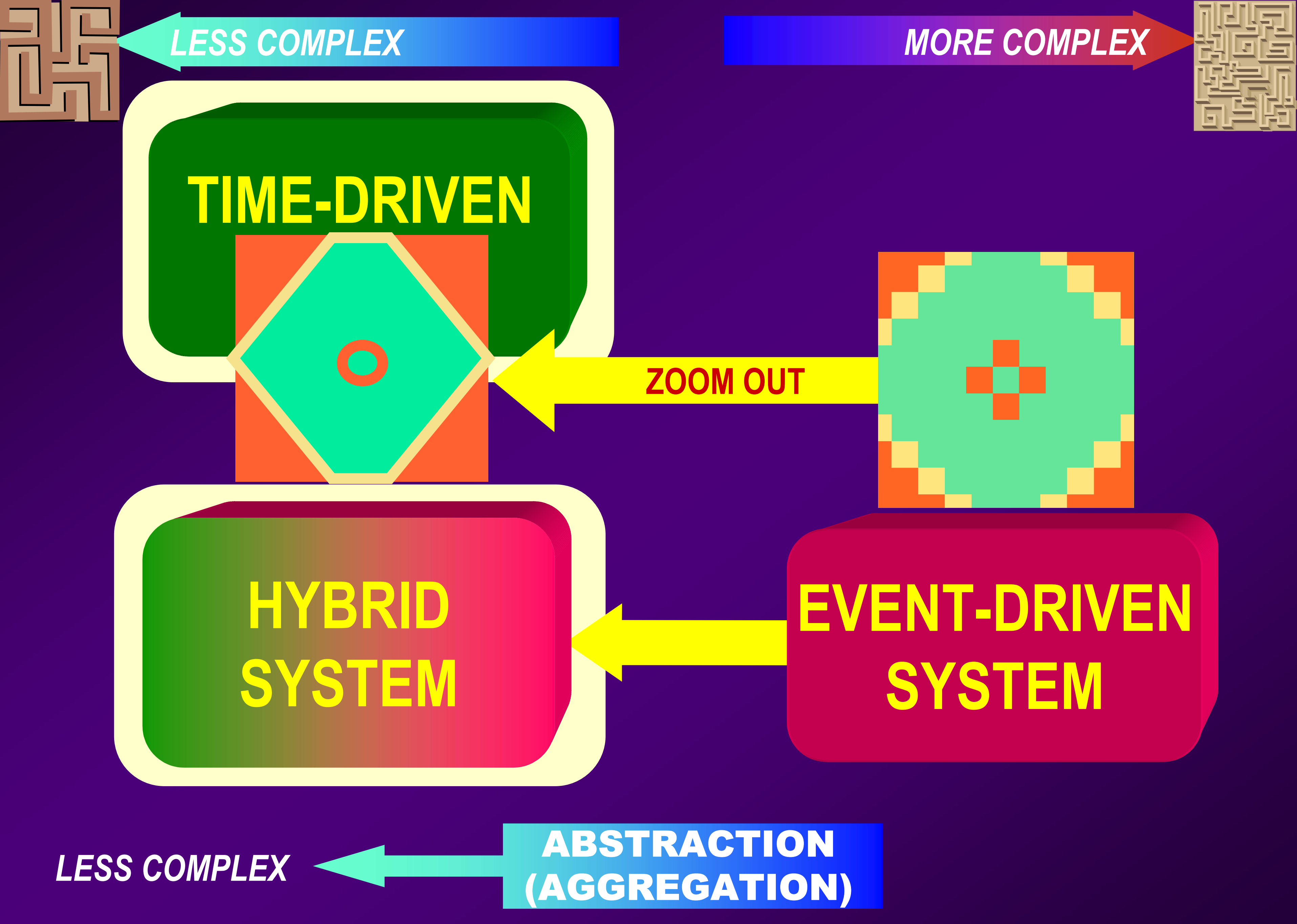
- Very hard *Stochastic Optimal Control* problems !
- *Parametric* optimization:
  use gradient estimates with on-line opt. algorithms

$$\theta_{n+1} = \theta_n - \eta_n H_n(\theta_n, \omega_n^{SFM}), \quad n = 0, 1, \ldots$$

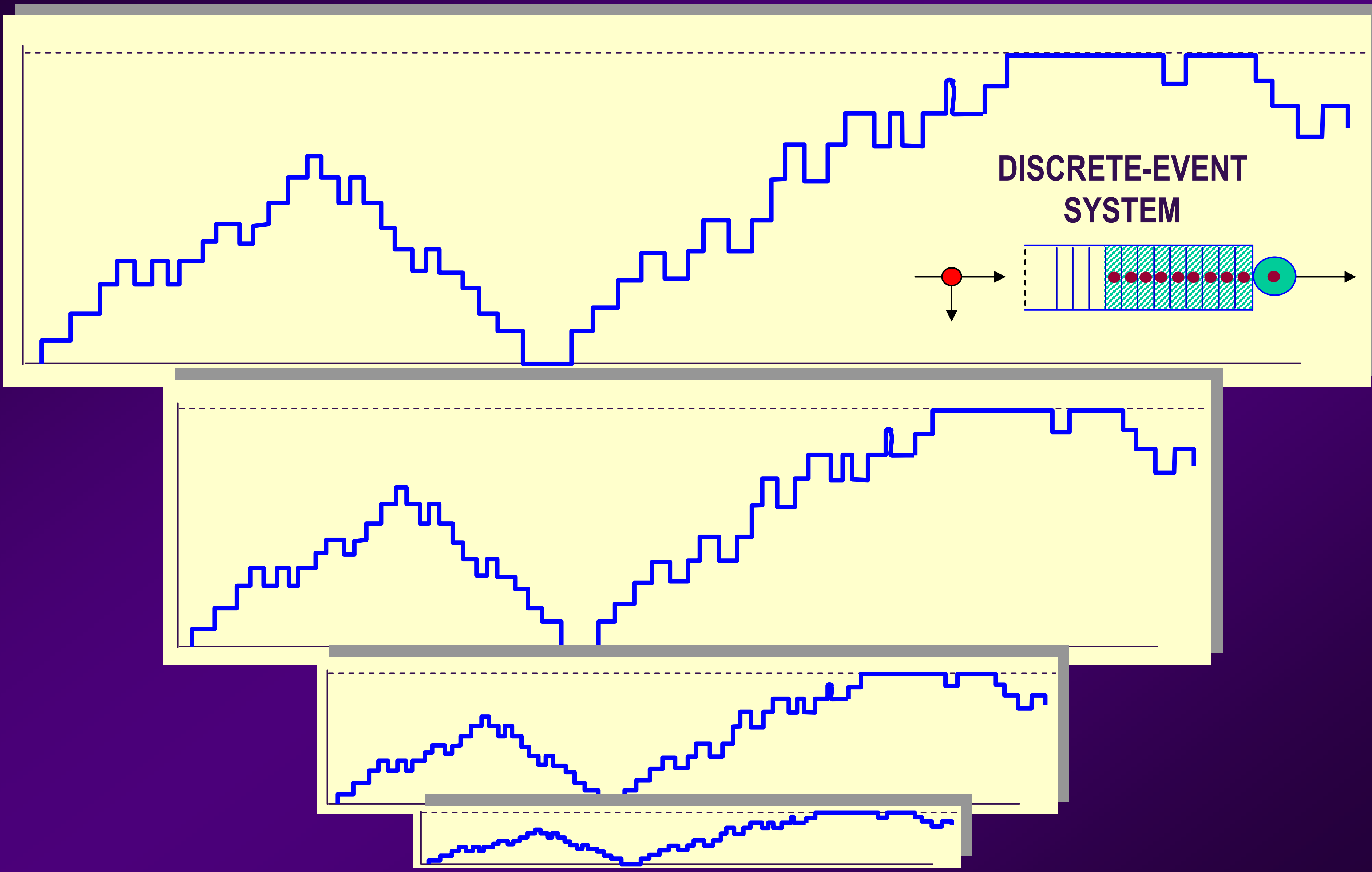
Adjust parameter…

…based on performance sensitivity estimates

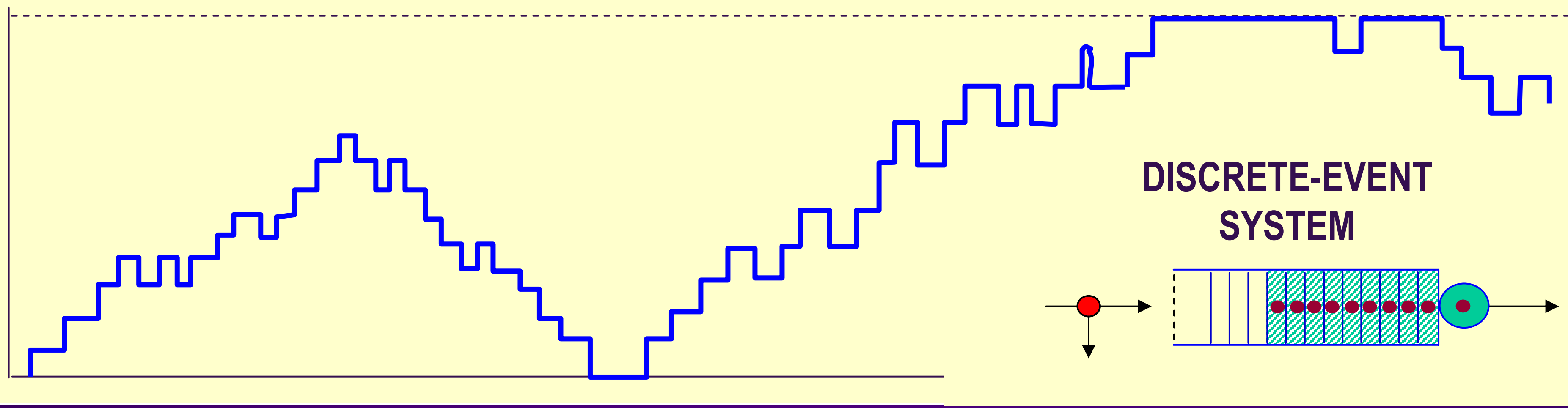
- Need efficient ways to estimate performance sensitivities

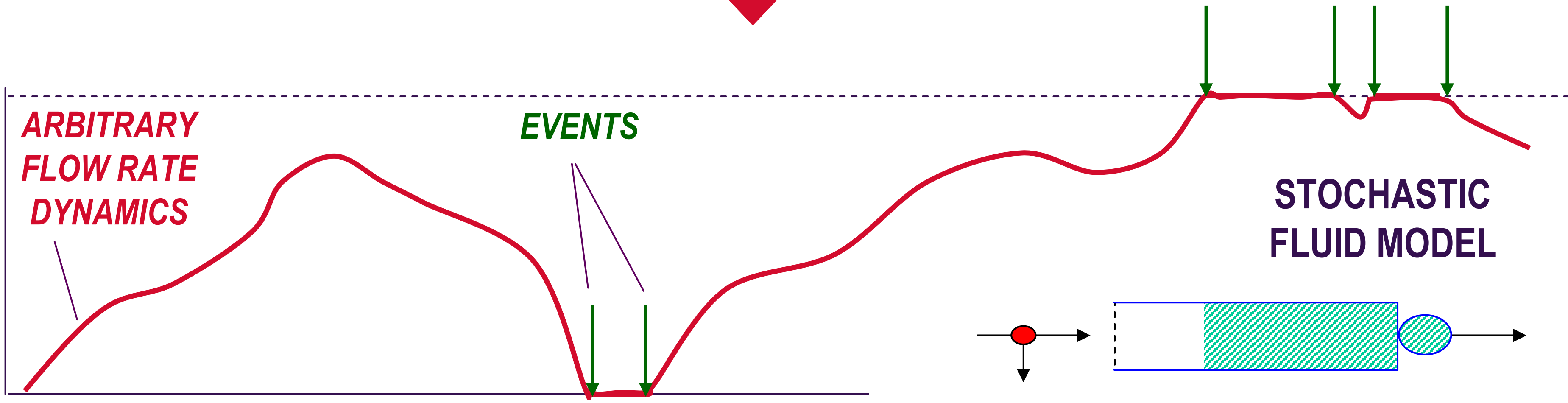

LESS COMPLEX
MORE COMPLEX
TIME-DRIVEN
ZOOM OUT
HYBRID SYSTEM
EVENT-DRIVEN SYSTEM
LESS COMPLEX
ABSTRACTION (AGGREGATION)

ABSTRACTION OF A DISCRETE-EVENT SYSTEM

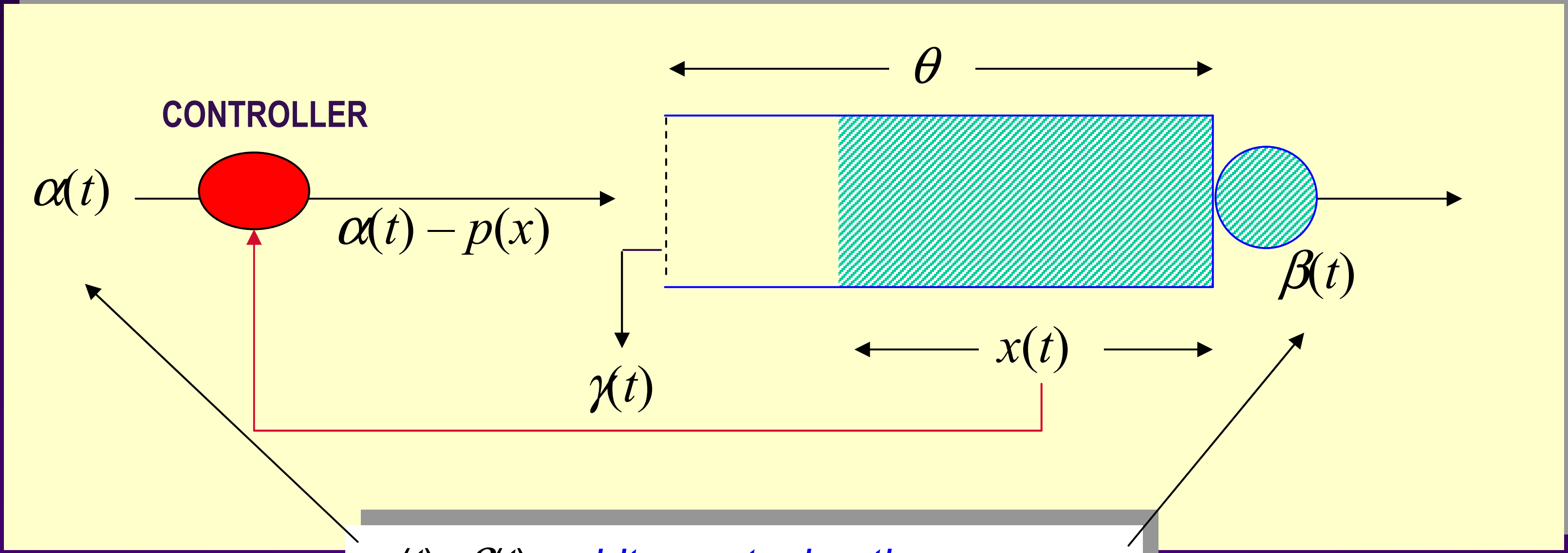


MODEL ABSTRACTION





STOCHASTIC FLOW MODEL (SFM)



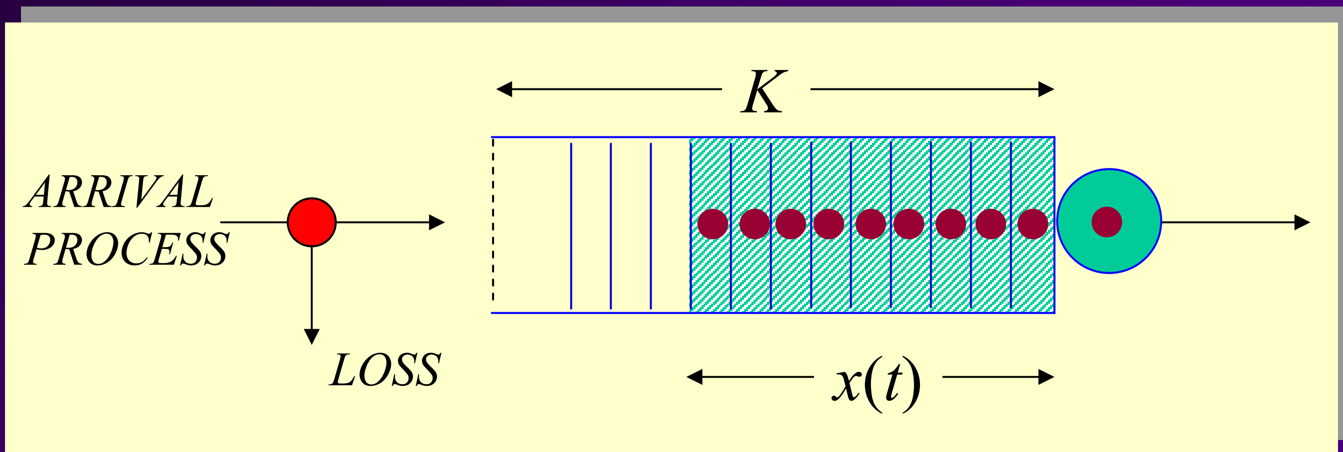
$\alpha(t)$, $\beta(t)$: *arbitrary stochastic processes (piecewise continuously differentiable)*

$$\frac{dx}{dt} = \begin{cases} 0 & x(t)=0,\ \lambda(t)-p(0)\le 0 \\ 0 & x(t)=\theta,\ \lambda(t)-p(\theta)\ge 0 \\ \lambda(t)-p(x(t)) & \text{otherwise} \end{cases}$$

$\lambda(t) = \alpha(t) - \beta(t)$

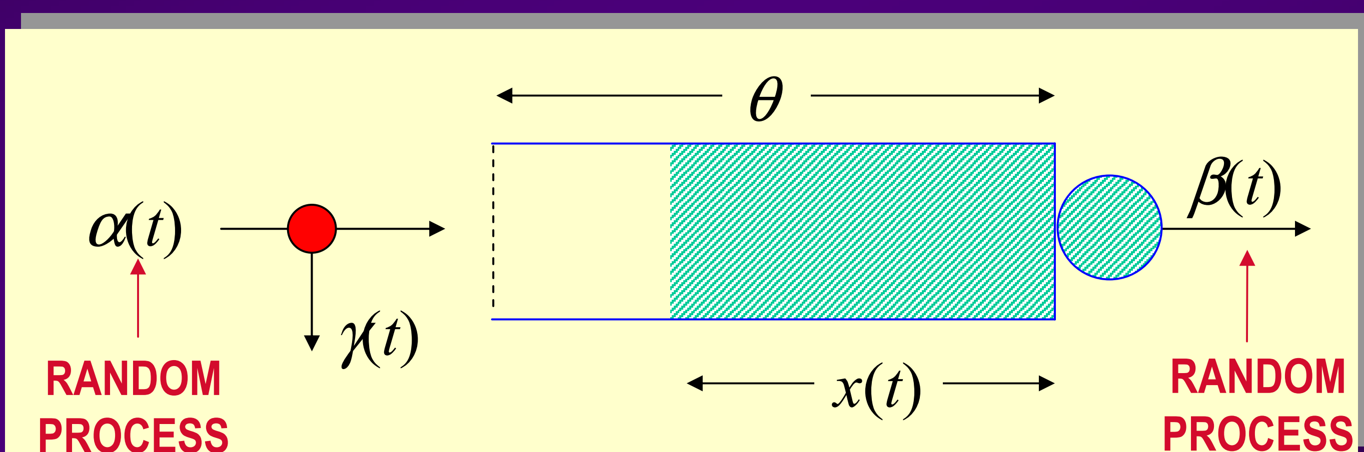
feedback

MOTIVATING EXAMPLE: *THRESHOLD-BASED BUFFER CONTROL*



"REAL" SYSTEM

$$J_T(K) = \bar{Q}_T(K) + R \bullet \bar{L}_T(K)$$

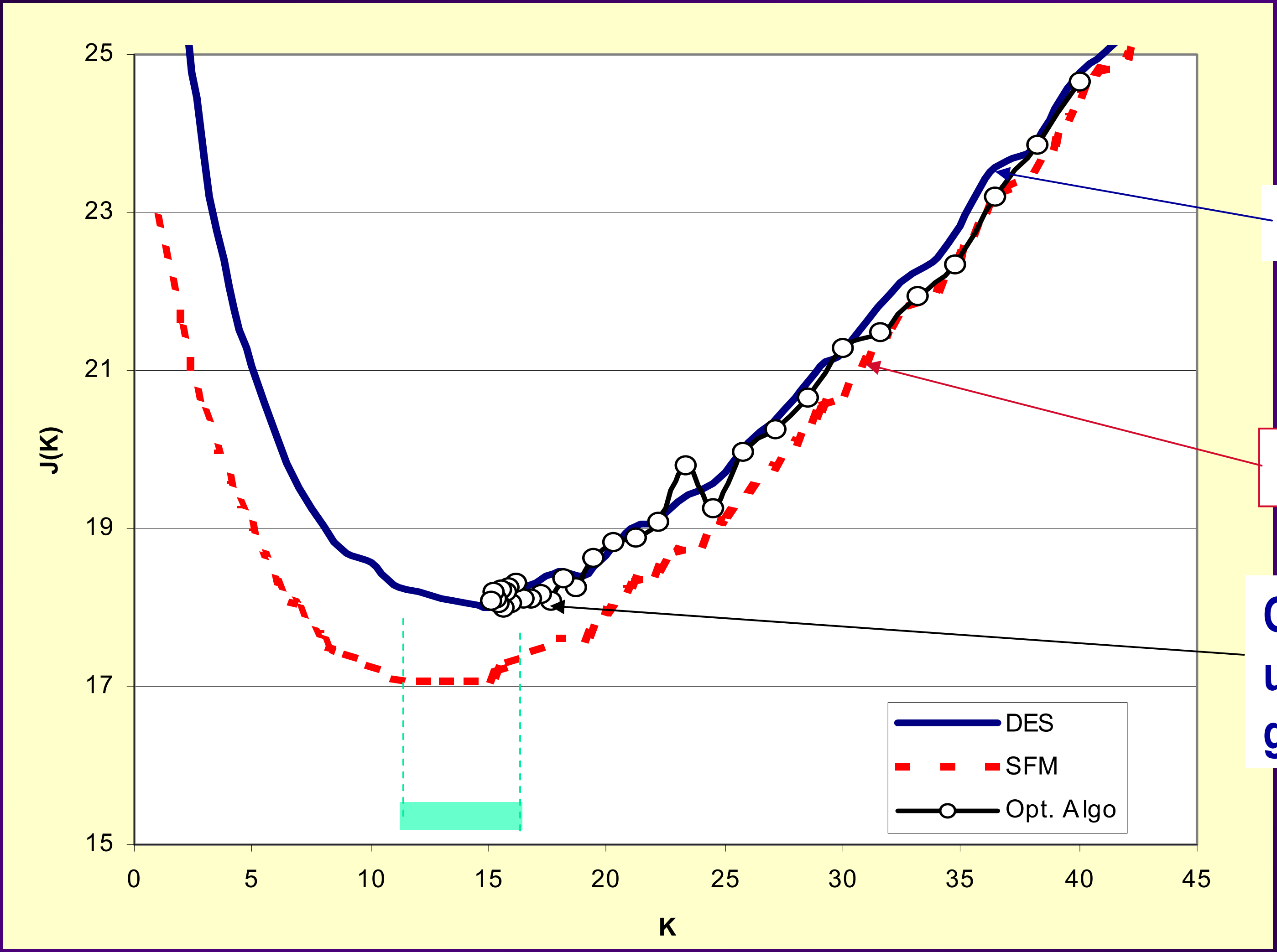


Stochastic
Flow
Model
(SFM)

$$J_T^{SFM}(\theta) = \bar{Q}_T^{SFM}(\theta) + R \bullet \bar{L}_T^{SFM}(\theta)$$

MOTIVATING EXAMPLE

CONTINUED



“Real” System

SFM

Optim. Algorithm using SFM-based gradient estimates

Cassandras et al, IEEE Trans. on AC, 2002

OPTIMIZATION OF SFMs

- Very hard *Stochastic Optimal Control* problems !
- *Parametric* optimization:
  use gradient estimates with on-line opt. algorithms

$$\theta_{n+1} = \theta_n - \eta_n H_n(\theta_n, \omega_n^{SFM}), \quad n = 0, 1, \ldots$$

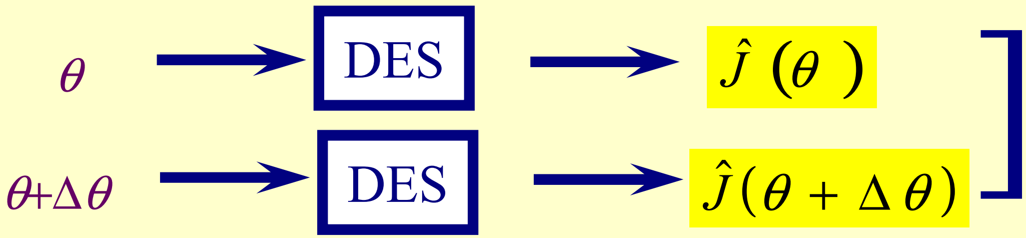
Adjust parameter…

…based on performance sensitivity estimates

- Need efficient ways to estimate performance sensitivities

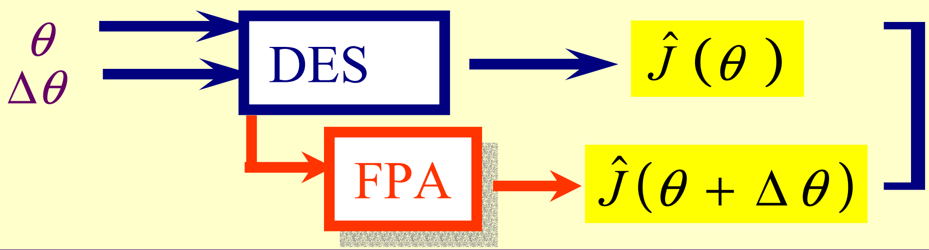
INFINITESIMAL PERTURBATION ANALYSIS (IPA)

"Brute Force" Sensitivity Estimation:



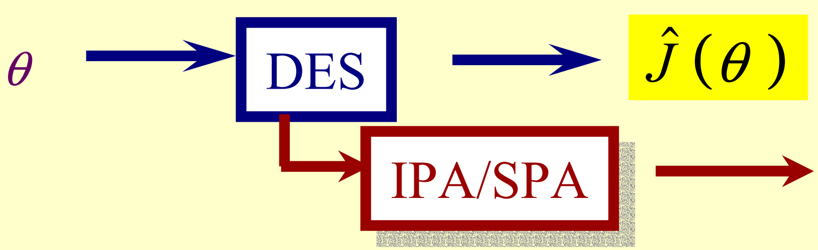
$$\Rightarrow \left[\frac{dJ}{d\theta}\right]_{est} = \frac{\hat{J}(\theta+\Delta\theta)-\hat{J}(\theta)}{\Delta\theta}$$

Finite Perturbation Analysis (FPA):



$$\Rightarrow \left[\frac{dJ}{d\theta}\right]_{est} = \frac{\hat{J}(\theta+\Delta\theta)-\hat{J}(\theta)}{\Delta\theta}$$

Infinitesimal or *Smoothed* Perturbation Analysis (IPA, SPA):



$$\left[\frac{dJ}{d\theta}\right]_{est}$$

INFINITESIMAL PERTURBATION ANALYSIS (IPA)

OBJECTIVES:

- Obtain sample performance derivatives that depend **ONLY** on observed sample path data:

Performance

$$L'_T(\theta) \equiv \frac{dL_T(\theta)}{d\theta}$$

Parameter

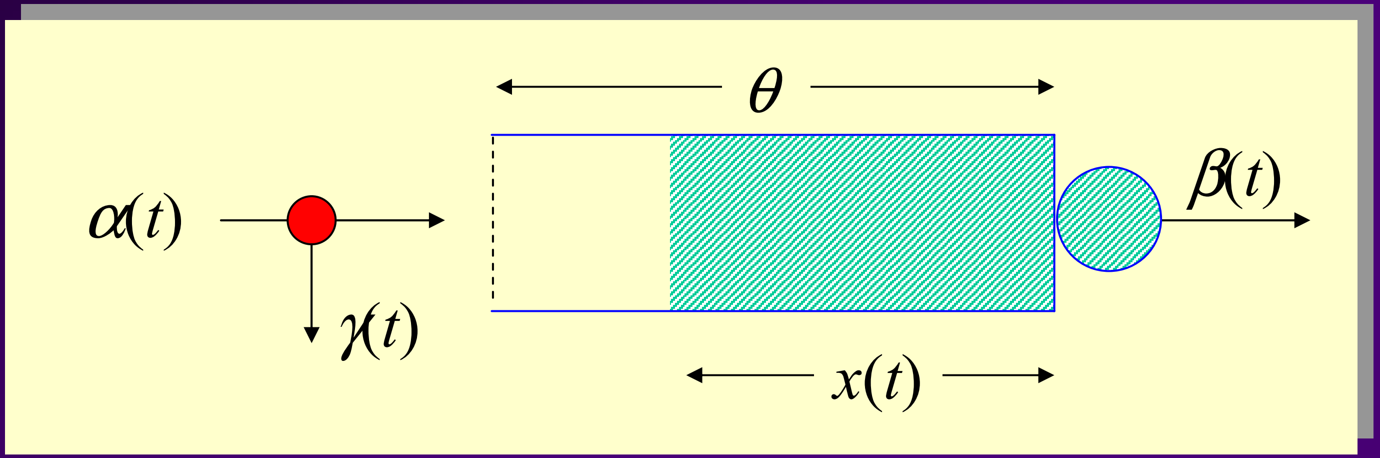
- Prove unbiasedness:

$$\frac{dE[L_T(\theta)]}{d\theta} = E\left[\frac{dL_T(\theta)}{d\theta}\right]$$

- Then, use gradient estimates to drive on-line opt. algorithms:

$$\theta_{n+1} = \theta_n + \eta_n L'_T(\theta_n), \quad n = 0,1,\ldots$$

THRESHOLD-BASED BUFFER CONTROL



$$J_T(\theta) = Q_T(\theta) + RL_T(\theta)$$

WORK

$$Q_T(\theta) = \int_0^T x(\theta;t)dt$$

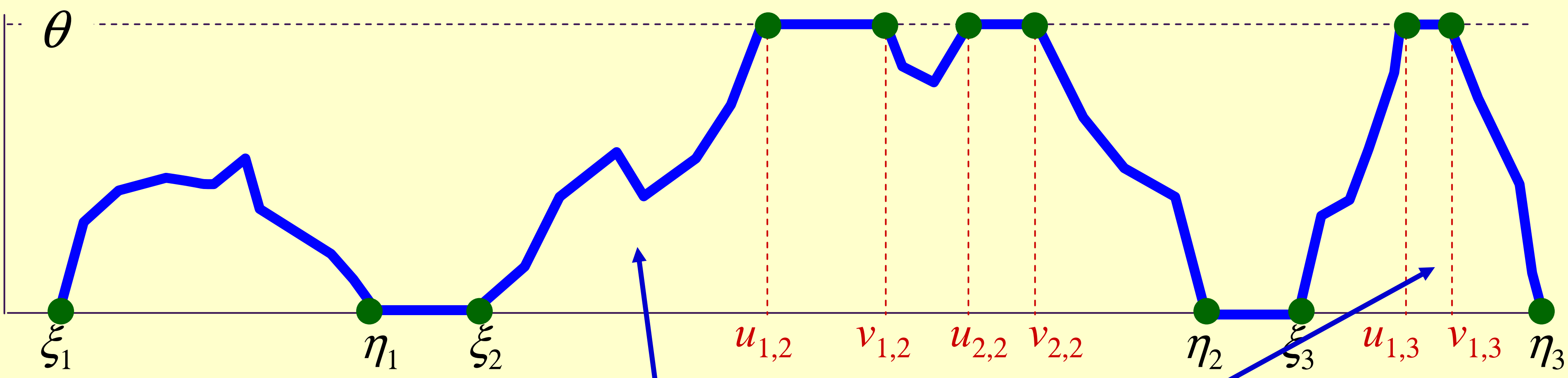
LOSS

$$L_T(\theta) = \int_0^T \gamma(\theta;t)dt$$

Assume: $\{\alpha(t)\}$ and $\{\beta(t)\}$ are right-continuous piecewise continuously differentiable, independent of threshold θ

THRESHOLD-BASED BUFFER CONTROL

CONTINUED



BUFFERING PERIOD: $\mathcal{B}_k = (\xi_k, \eta_k(\theta)), \quad k = 1,\ldots,K$

OVERFLOW PERIOD: $\mathcal{F}_{i,k} = [u_{i,k}(\theta), v_{i,k}]$, $i = 1,\ldots,M$

SET OF BPs WITH AT LEAST ONE OVERFLOW:

$\Phi(\theta) := \{k \in \{1,\ldots,K\} \ : \ x(t) = \theta, \ \alpha(t) - \beta(t) > 0$ for some $t \in (\xi_k, \eta_k(\theta))\}$

$$B(\theta) = |\Phi(\theta)|$$

IPA – MAIN RESULTS

Cassandras et al, 2002

THEOREM 1 (Loss IPA derivative):

$$L_T'(\theta) = -B(\theta)$$

- Simple *count* of Buffering Periods with at least one overflow
- Nonparametric (independent of θ and any model assumption)

THEOREM 2 (Work IPA derivative):

$$Q_T'(\theta) = \sum_{k \in \Phi(\theta)} \left[\eta_k(\theta) - u_{k,1}(\theta) \right]$$

- Simple *timer* for each Buffering Period with at least one overflow
- Nonparametric (independent of θ and any model assumption)

IPA – MAIN RESULTS

CONTINUED

THEOREM 1 (Loss IPA derivative):

$$L'_T(\theta) = -B(\theta)$$

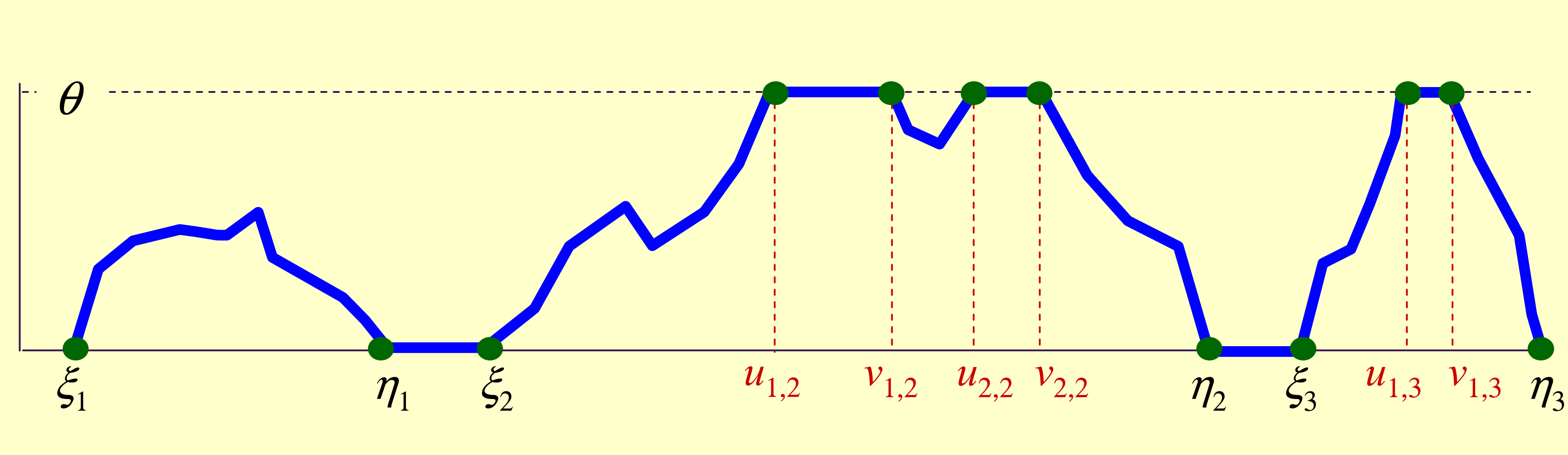
Proof:

$$L_T(\theta) = \sum_{k\in\Phi(\theta)} \int_{\xi_k}^{\eta_k(\theta)} \gamma(\theta;t)dt$$

For a typical BP:

$$\lambda(\theta) = \int_{\xi}^{\eta(\theta)} \gamma(\theta;t)dt$$

$$\lambda(\theta) = \sum_{m=1}^{M} \int_{u_m(\theta)}^{v_m} [\alpha(t) - \beta(t)]dt$$



IPA – MAIN RESULTS

CONTINUED

$$\lambda(\theta) = \sum_{m=1}^{M} \int_{u_m(\theta)}^{v_m} [\alpha(t) - \beta(t)]dt$$

$$\Rightarrow \lambda'(\theta) = -\sum_{m=1}^{M} [\alpha(u_m(\theta)) - \beta(u_m(\theta))] u'_m(\theta) \quad (*)$$

For $(\xi, u_1(\theta))$:

$$\int_{\xi}^{u_1(\theta)} [\alpha(t) - \beta(t)]dt = \theta$$

$$\Rightarrow [\alpha(u_1(\theta)) - \beta(u_1(\theta))] u'_1(\theta) = 1$$

For $(v_{m-1}(\theta), u_m(\theta))$:

$$\int_{v_{m-1}}^{u_m(\theta)} [\alpha(t) - \beta(t)]dt = 0$$

$$\Rightarrow [\alpha(u_m(\theta)) - \beta(u_m(\theta))] u'_m(\theta) = 0$$

From (*): $\lambda'(\theta) = -1$

UNBIASEDNESS

For any sample function $\mathcal{L}(\theta)$,
the sample derivative $\mathcal{L}'(\theta)$ is an unbiased estimator of $\dfrac{dE[\mathcal{L}(\theta)]}{d\theta}$
under the following conditions:

1. The sample derivative $\mathcal{L}'(\theta)$ exists for all θ
2. For all θ, $\mathcal{L}(\theta)$ is Lipschitz continuous w.p. 1
   and the Lipschiz constant has a finite first moment

[Rubinstein and Shapiro, 1993]

UNBIASEDNESS

CONTINUED

Let $N(T)$ be the number of events in $[0, T]$.

THEOREM:

1. If $E[N(T)] < \infty$**, then** $L_T'(\theta)$ **is an unbiased estimator of** $\dfrac{dE[L_T(\theta)]}{d\theta}$

2. $Q_T'(\theta)$ **is an unbiased estimator of** $\dfrac{dE[Q_T(\theta)]}{d\theta}$

Proof: **Show that** $|\Delta L_T(\theta)| \leq N(T)|\Delta\theta|$

$$|\Delta Q_T(\theta)| = \left| \int_0^T \Delta x(\theta, \Delta\theta; t)dt \right| \leq T|\Delta\theta|$$

ON-LINE CONTROL AND OPTIMIZATION

"Real" Discrete Event System:

$$K_{n+1} = K_n - \eta_n H_n(K_n, \omega_n^{DES}), \quad n = 0, 1, \ldots$$

Adjust "real" threshold

Gradient estimate from "real" DES data

SFM:

$$\theta_{n+1} = \theta_n - \eta_n H_n(\theta_n, \omega_n^{SFM}), \quad n = 0, 1, \ldots$$

Gradient estimate from SFM simulated data

ON-LINE CONTROL AND OPTIMIZATION

CONTINUED

But SFM-based gradient estimator can be used in "real" system, since it does not involve any SFM-specific data !

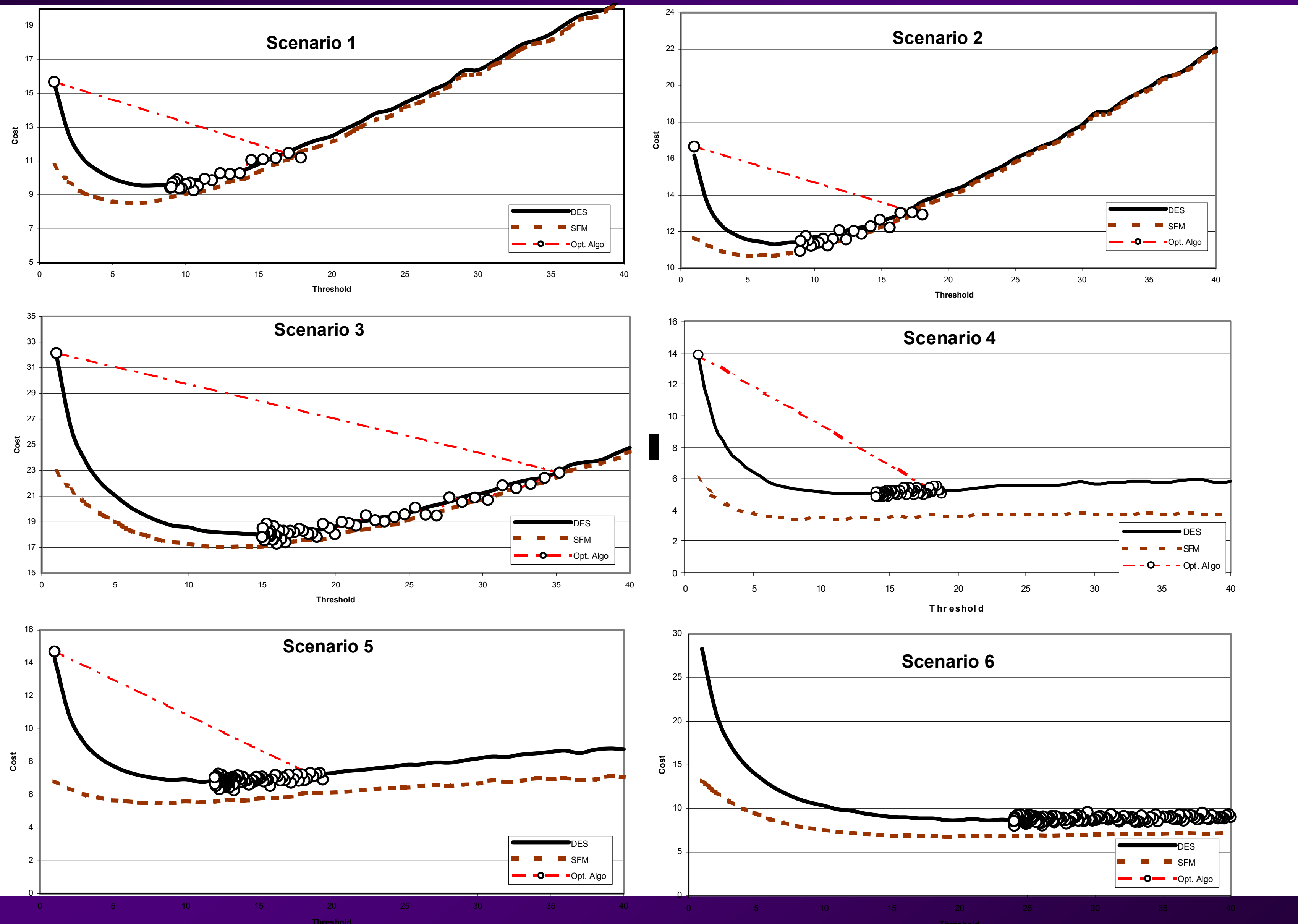
$$\frac{dE[L_T(\theta)]}{d\theta} \qquad \frac{dE[Q_T(\theta)]}{d\theta}$$

$$K_{n+1} = K_n - \eta_n H_n(K_n, \omega_n^{DES}), \quad n = 0, 1, \ldots$$

SFM-based FORM,
but "real" DES data:

- Event detection
- Counters, Timers

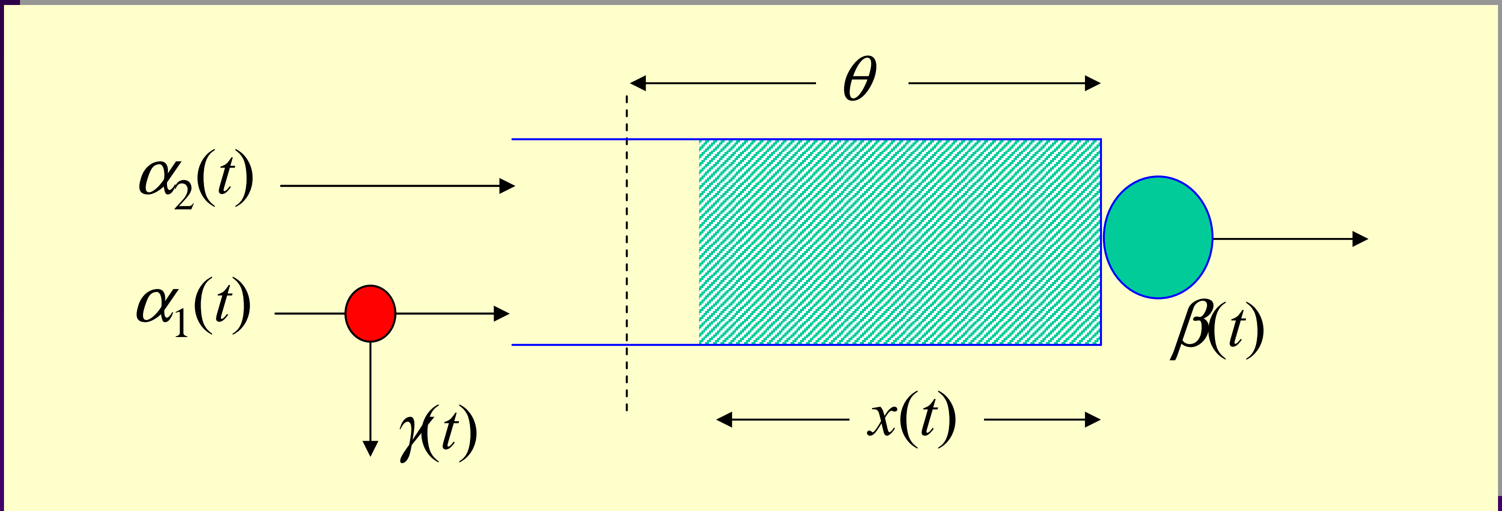
EXAMPLES



Scenario 1
Scenario 2
Scenario 3
Scenario 4
Scenario 5
Scenario 6
Cost
Threshold
DES
SFM
Opt. Algo

IPA FOR MULTICLASS SFM

Sun et al, IEEE Trans. on AC, 2003



- **Similar results, except**

$$L_T'(\theta) = -|\Phi(\theta)| + \sum_{k \notin \Phi(\theta)} \lambda_k'(\theta)$$

Simple *count* of BPs with at least one Partial Loss period

Depends only on *flow rates* at some event times (*known or measured on line*)

IPA FOR MULTICLASS SFM

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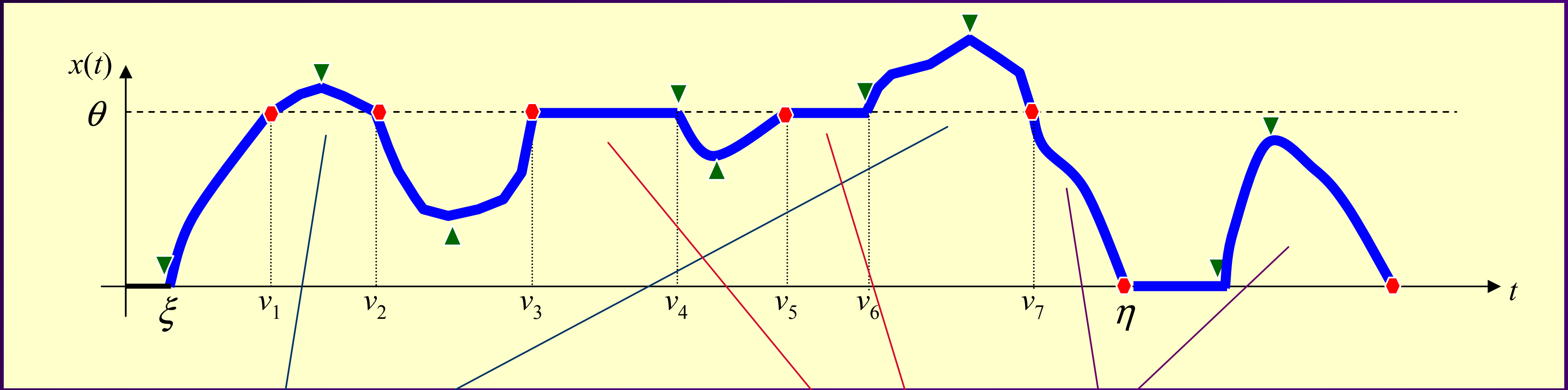


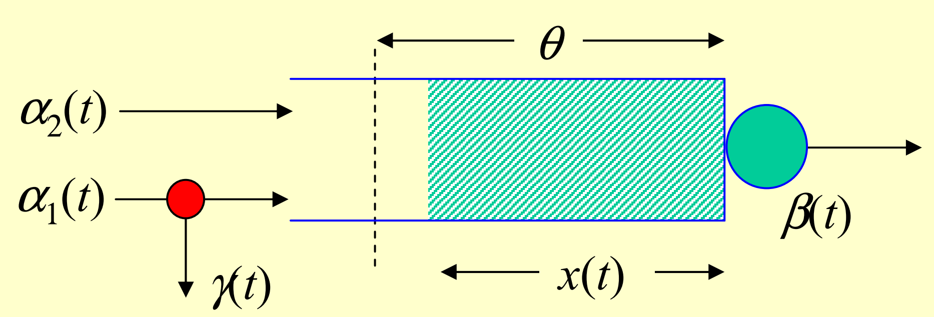
NO LOSS PERIODS

FULL LOSS PERIODS

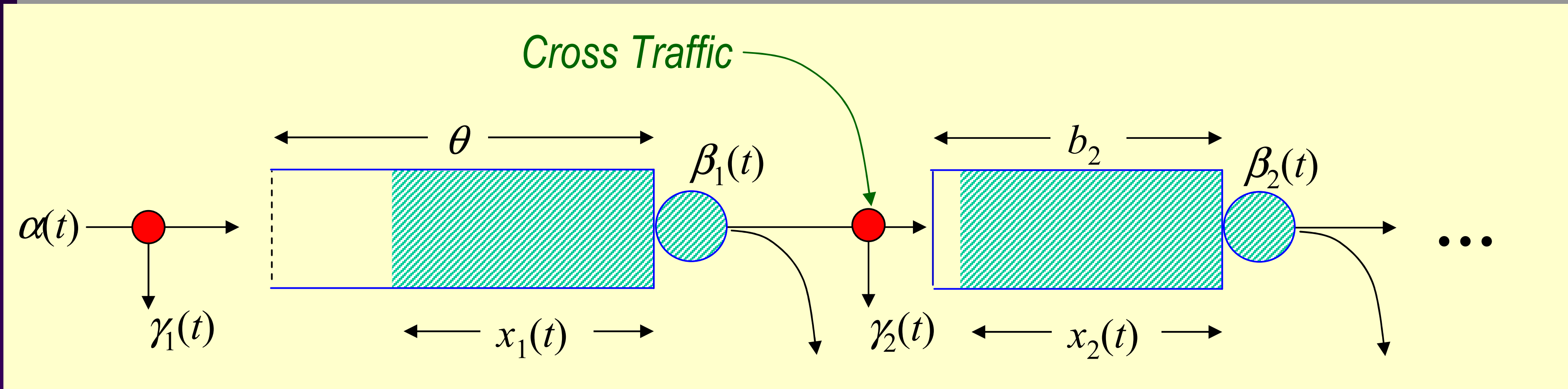
$\gamma(t) = \alpha_1(t)$

PARTIAL LOSS PERIODS

$\gamma(t) = \alpha_1(t)+\alpha_2(t)-\beta(t), \quad \alpha_2(t)-\beta(t) < 0$



WHAT ABOUT *NETWORKS* OF SFMs ?



- **Similar IPA -- still need only counters, timers, some flow info.**
- **IPA shows that *CONGESTION IS A LARGELY LOCAL PHENOMENON***
  - **Perturbation propagation occurs with low probability**
  - **Local control eliminates congestion**

FUTURE DIRECTIONS

- Input traffic with feedback (e.g., TCP) – *what if current queue length is used to determine input flow* ?
- What are the “right” performance metrics ?
- Applications to SENSOR NETWORKS
- Applications to NETWORK SECURITY (Denial-of-Service attacks)