Bisimilar control systems

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Modeling and Control of Hybrid Systems
Veldhoven, The Netherlands
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http://lcewww.et.tudelft.nl/~disc_hs/

Outline of this mini-course

Lecture 1 : Monday, June 23
Examples of hybrid systems, modeling formalisms

Lecture 2 : Monday, June 23
Transitions systems, temporal logic, refinement notions

Lecture 3 : Tuesday, June 24
Discrete abstractions of hybrid systems for verification

Lecture 4 : Tuesday, June 24
Discrete abstractions of continuous systems for control

Lecture 5 : Thursday, June 26
Bisimilar control systems

Continuous to continuous (Lecture 5)

Lecture 4

$$\begin{array}{c} T/\approx \\ \uparrow \; T \equiv T/\approx \\ T \quad y_{t+1} = F y_t + G v_t \end{array}$$

Exponential number states

Lecture 5

$$\begin{array}{c} T/\approx \quad y_{t+1} = F y_t + G v_t \\ \uparrow \; T \equiv T/\approx \\ T \quad x_{t+1} = A x_t + B u_t \end{array}$$

Bisimilar control systems

Goal

Abstracted Model

$$z'_1 = v$$
$$z'_2 = z_2$$

Observations

$$\langle z \rangle = \begin{cases} o_1 & \text{if } z_1 > 5 \wedge z_2 > 5 \wedge z_1 + z_2 < 10 \\ o_2 & \text{if } z_1 \le -5 \wedge z_2 \le -5 \wedge z_1 + z_2 > -10 \\ o_3 & \text{otherwise} \end{cases}$$

$\uparrow$ $z_1 = x_1$, $z_2 = x_4$ $\qquad$ $\uparrow$ $z_1 = x_1$, $z_2 = x_4$

Original Model

$$x'_1 = x_3$$
$$x'_2 = -x_2 + x_4 + u$$
$$x'_3 = x_1 + x_2$$
$$x'_4 = x_4$$

Observations

$$\langle x \rangle = \begin{cases} o_1 & \text{if } x_1 > 5 \wedge x_4 > 5 \wedge x_1 + x_4 < 10 \\ o_2 & \text{if } x_1 \le -5 \wedge x_4 \le -5 \wedge x_1 + x_4 > -10 \\ o_3 & \text{otherwise} \end{cases}$$

Game plan

$$\begin{tikzcd} T \arrow[r, leftrightarrow, "\text{Partition}"] & T/\approx \\ \Delta \arrow[u, "\text{Embedding}"] \arrow[r, leftrightarrow, "\text{Construction}"'] & \Delta/\approx \arrow[u] \end{tikzcd}$$

Lossless Embedding

$$T^{\Delta}_{\Sigma} = (Q, \Sigma, \rightarrow, O, \langle \cdot \rangle)$$

$$\uparrow$$

$$\Delta \quad \begin{array}{l} x_{k+1} = A x_k + B u_k \\ y_k = C x_k \end{array}$$

Transition System T^{Δ}_{U}

State set $Q = X = R^n$

Label set $\Sigma = U = R^m$

Observation set $O = Y = R^p$

Linear Observation Map $\langle x \rangle = Cx$

Transition Relation $\rightarrow \subseteq X \times U \times X$

$$x_1 \xrightarrow{u} x_2 \Leftrightarrow x_2 = A x_1 + B u$$

Loose Control…

$T_\Sigma^\Delta = (Q, \Sigma, \rightarrow, O, \langle\cdot\rangle)$

$\uparrow$

Δ $x_{k+1} = Ax_k + Bu_k$
$y_k = Cx_k$

Transition System T_1^Δ

State set $Q = X = R^n$

Label set $\Sigma = \{1\}$

Observation set $O = Y = R^p$

Linear Observation Map $\langle x \rangle = Cx$

Transition Relation $\rightarrow \subseteq X \times \{1\} \times X$

$x_1 \xrightarrow{1} x_2 \Leftrightarrow \exists u$ with $x_2 = Ax_1 + Bu$

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Keep time….

$T_\Sigma^\Delta = (Q, \Sigma, \rightarrow, O, \langle\cdot\rangle)$

$\uparrow$

Δ $x_{k+1} = Ax_k + Bu_k$
$y_k = Cx_k$

Transition System $T_{N_+}^\Delta$

State set $Q = X = R^n$

Label set $\Sigma = N_+$

Observation set $O = Y = R^p$

Linear Observation Map $\langle x \rangle = Cx$

Transition Relation $\rightarrow \subseteq X \times N_+ \times X$

$x_1 \xrightarrow{k} x_2 \Leftrightarrow$ $\exists u_0,\ldots,u_{k-1}$ with $x_2 = A^k x_1 + \sum_{i=0}^{k-1} A^{k-i-1} Bu_i$

Penn

Loose control and time…

$T_\Sigma^\Delta = (Q, \Sigma, \rightarrow, O, \langle\cdot\rangle)$

$\uparrow$

Δ $x_{k+1} = Ax_k + Bu_k$
$y_k = Cx_k$

Transition System T_τ^Δ

State set $Q = X = R^n$

Label set $\Sigma = \{\tau\}$

Observation set $O = Y = R^p$

Linear Observation Map $\langle x \rangle = Cx$

Transition Relation $\rightarrow \subseteq X \times \{\tau\} \times X$

$x_1 \xrightarrow{\tau} x_2 \Leftrightarrow$ $\exists k$ and $\exists u_0,\ldots,u_{k-1}$ with $x_2 = A^k x_1 + \sum_{i=0}^{k-1} A^{k-i-1} Bu_i$

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Finite Observations

$T_\Sigma^\Delta = (Q, \Sigma, \rightarrow, O, \langle\cdot\rangle)$

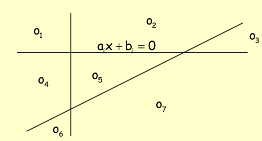
$\uparrow$

Δ $x_{k+1} = Ax_k + Bu_k$
$y_k = Cx_k$

All Transition Systems

Finite Observations $O = \{o_1, o_2, \ldots, o_p\}$

Polyhedral Map $\langle x \rangle : X \rightarrow O$



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Keep continuous time….

$T_\Sigma^\Delta = (Q, \Sigma, \rightarrow, O, \langle\cdot\rangle)$

$\uparrow$

Δ $x' = Ax + Bu$
$y = Cx$

Transition System $T_{R_+}^\Delta$

State set $Q = X = R^n$

Label set $\Sigma = R_+$

Observation set $O = Y = R^p$

Linear Observation Map $\langle x \rangle = Cx$

Transition Relation $\rightarrow \subseteq X \times R_+ \times X$

$x_1 \xrightarrow{t} x_2 \Leftrightarrow$ $\exists u_{[0,t]}$ with $x_2 = e^{At} x_1 + \int_0^t e^{A(t-s)} Bu(s) ds$

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Loose continuous time….

$T_\Sigma^\Delta = (Q, \Sigma, \rightarrow, O, \langle\cdot\rangle)$

$\uparrow$

Δ $x' = Ax + Bu$
$y = Cx$

Transition System T_τ^Δ

State set $Q = X = R^n$

Label set $\Sigma = \{\tau\}$

Observation set $O = Y = R^p$

Linear Observation Map $\langle x \rangle = Cx$

Transition Relation $\rightarrow \subseteq X \times \{\tau\} \times X$

$x_1 \xrightarrow{\tau} x_2 \Leftrightarrow$ $\exists t$ and $\exists u_{[0,t]}$ with $x_2 = e^{At} x_1 + \int_0^t e^{A(t-s)} Bu(s) ds$

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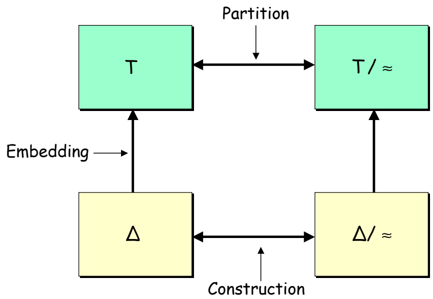
Partitions



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Respecting the observations

Two states are equivalent iff

$$\boxed{x_1 \approx x_2 \Leftrightarrow Hx_1 = Hx_2 \Leftrightarrow x_1 - x_2 \in Ker(H)}$$

for some surjective map z=Hx. Simulation S=(x,Hx)

Partition is observation preserving iff
Linear observations :

$$Ker(H) \subseteq Ker(C)$$

Finite, polyhedral observations :

$$Ker(H) \subseteq Ker(a_i)$$

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Respecting the controlled transitions

Respecting the transitions depends on the embedding.

Consider the transition system T_U^Δ

$$\begin{array}{ccc} x_1 & \xrightarrow{u} & x_1' = Ax_1 + Bu \\ \approx & & \approx \\ x_2 & \xrightarrow{u} & x_2' = Ax_2 + Bu \end{array}$$

Proposition : Partition respects the transitions iff

$$AKer(H) \subseteq Ker(H)$$

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Respecting the timed transitions

Consider the control-abstract transition system T_1^Δ

$$\begin{array}{ccc} x_1 & \xrightarrow{1} & x_1' = Ax_1 + Bu \quad \text{for some } u \\ \approx & & \approx \\ x_2 & \xrightarrow{1} & x_2' = Ax_2 + Bu' \quad \text{for some } u' \end{array}$$

Proposition* : Partition respects the transitions iff

$$AKer(H) \subseteq Ker(H) + R(B)$$

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Similarly

Consider the control-abstract transition system $T_{N_+}^\Delta$

$$\begin{array}{ccc} x_1 & \xrightarrow{k} & x_1' \\ \approx & & \approx \\ x_2 & \xrightarrow{k} & x_2' \end{array}$$

Proposition* : Partition respects the transitions iff

$$AKer(H) \subseteq Ker(H) + R(B)$$

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Respecting the untimed transitions

Consider the time-abstract transition system T_τ^Δ

$$\begin{array}{ccc} x_1 & \xrightarrow{\tau} & x_1' \\ \approx & & \approx \\ x_2 & \xrightarrow{\tau} & x_2' \end{array}$$

Proposition* : Partition respects the transitions iff

$$AKer(H) \subseteq Ker(H) + R(A,B)$$

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Timed, continuous transitions

Consider the time-abstract transition system $T_{\mathbb{R}_+}^{\Delta}$

$$\begin{array}{ccc} x_1 & \xrightarrow{t} & x_1' \\ \approx & & \approx \\ x_2 & \xrightarrow{t} & x_2' \end{array}$$

Proposition* : Partition respects the transitions iff

$$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + R(A,B)$$

Untimed, continuous transitions

Consider the time-abstract transition system T_{τ}^{Δ}

$$\begin{array}{ccc} x_1 & \xrightarrow{\tau} & x_1' \\ \approx & & \approx \\ x_2 & \xrightarrow{\tau} & x_2' \end{array}$$

Proposition* : Partition respects the transitions iff

$$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + R(A,B)$$

Summary

In addition to preserving the observations…

Embedding	**Condition**
T_1^{Δ} $T_{N_+}^{\Delta}$	$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + R(B)$
T_{τ}^{Δ}	$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + R(A,B)$
$T_{\mathbb{R}_+}^{\Delta}$	$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + R(A,B)$
T_{τ}^{Δ}	$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + R(A,B)$

Coarsest Bisimulation

Find map z=Hx which abstracts as much as possible.
Thus Ker(H) must be maximal but also…

Preserves observations

$$\mathrm{Ker}(H) \subseteq \mathrm{Ker}(C)$$

Preserves transitions of T_1^{Δ}

$$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + R(B)$$

Other variations for other embeddings…

Coarsest Bisimulation Algorithm

Maximal controlled invariant subspace computation

$$V_0 = \mathrm{Ker}(C)$$

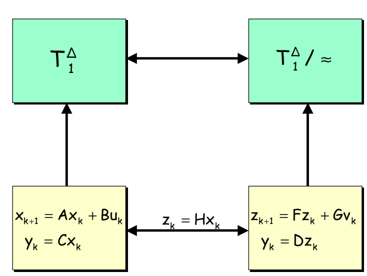
$$V_{k+1} = V_{k-1} \cap A^{-1}(V_{k-1} + R(B))$$

Then $V^* = V_n$ is the maximal desired subspace

Once V* is computed, then pick map z=Hx such that

Ker(H)=V*

Constructing the abstraction



Construction

Construction of the generator of system $T_1^\Delta / \approx$

$$\begin{array}{ccl} x_1 & \xrightarrow{1} & x_2 = Ax_1 + Bu \quad \text{for some } u \\ {\scriptstyle z_1 = Hx_1}\downarrow & & \downarrow{\scriptstyle z_2 = Hx_2} \\ z_1 & \xrightarrow{1} & z_2 = Fz_1 + Gv \quad \text{for some } v \end{array}$$

Equivalently, for any x, u, there must exist a v such that

$$HAx + HBu = FHx + Gv$$

H-related control systems

Consider discrete-time or continuous-time linear systems

(X) $x' = Ax + Bu$

(Z) $z' = Fz + Gv$

where z=Hx is surjective. Then (Z) is H-related to (X) if for all x,u there exists v such that

$$H(Ax+Bu) = FHx + Gv$$

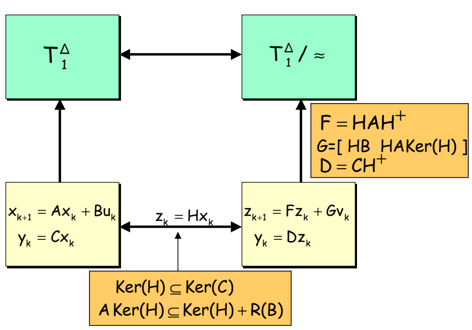
Proposition* : Given x'=Ax+Bu and onto map y=Hx choose

$F = HAH^+$

$G = [\, HB \quad HAKer(H) \,]$

Then z'=Fz+Gv is H-related to x'=Ax+Bu

Bisimilar Linear Systems



Goal achieved

Abstracted Model

$z'_1 = v$

$z'_2 = z_2$

Observations

$$\langle z \rangle = \begin{cases} o_1 & \text{if } z_1 > 5 \wedge z_2 > 5 \wedge z_1 + z_2 < 10 \\ o_2 & \text{if } z_1 \le -5 \wedge z_2 \le -5 \wedge z_1 + z_2 > -10 \\ o_3 & \text{otherwise} \end{cases}$$

$z_1 = x_1$, $z_2 = x_4$

Original Model

$x'_1 = x_3$

$x'_2 = -x_2 + x_4 + u$

$x'_3 = x_1 + x_2$

$x'_4 = x_4$

Observations

$$\langle x \rangle = \begin{cases} o_1 & \text{if } x_1 > 5 \wedge x_4 > 5 \wedge x_1 + x_4 < 10 \\ o_2 & \text{if } x_1 \le -5 \wedge x_4 \le -5 \wedge x_1 + x_4 > -10 \\ o_3 & \text{otherwise} \end{cases}$$

Preserving Controllability

Theorem* : Given onto map z=Hx and linear system

x'=Ax+Bu (X)

construct the canonical H-related system

z'=Fz+Gv (Z)

with $F = HAH^+$ and $G=[\, HB \quad HAKer(H)\,]$. Assume that

$Ker(H) \subseteq R(A,B)$

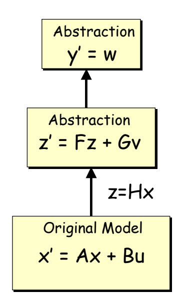
Then (X) is controllable if and only if (Z) is controllable.

Reachable sets satisfy $R(A,B) = H^{-1}(R(F,G))$

This leads to a hierarchical controllability algorithm…

*G.J. Pappas, G. Lafferriere, and S. Sastry, Hierarchically Consistent Control Systems, IEEE Transactions on Automatic Control, June 2000

Hierarchical Controllability Algorithm

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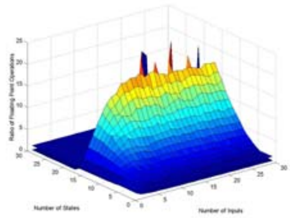
Hierarchical Algorithm

0 Initially x'=Ax+Bu,0=<k<n
1 If rank(B) is
   0 : System uncontrollable
   n : System controllable
2 Find H with
   Ker(H) = Im[B AB…A^kB]
3 Compute
   A:=HAH+
   B:=[HB HAKer(H)]
4 Go to 2
```

Hierarchical Controllability Algorithm

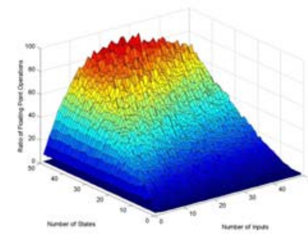
Comparison with Kalman rank test



$$\text{Ratio} = \frac{\text{flops of other tests}}{\text{flops of our algorithm}}$$

Hierarchical Controllability Algorithm

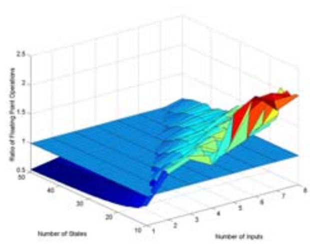
Comparison with Popov-Belevitch-Hautus (PBH) test



Algorithm with k=0 recovers best known algorithm

Hierarchical Controllability Algorithm

Comparison of our Algorithm with k=0 and k=1



Higher Lie brackets help for underactuated systems

Property Preserving Abstractions

Control is critical for abstraction!

Property	**Condition**
Controllability	$\mathrm{Ker}(H) \subseteq R(A,B)$
Output Controllability	$A\,\mathrm{Ker}(H) \subseteq \mathrm{Ker}(H) + R(A,B)$
Trajectories	$\mathrm{Ker}(H) \subseteq R(B)$
Stabilizability	$\mathrm{Ker}(H) \subseteq X^{-} + R(A,B)$

Some take home messages

New (hybrid) models, but also new (hybrid) questions
- Partial synchronization of continuous systems

Logic is entering our world
- Temporal logic for complicated specifications
- First-order logic for syntactically specifying hybrid systems

Algorithmic approaches to analysis and controller design
- Is your design method computationally feasible?
- Is your design method computationally efficient?
- Focus on tool development

Decidability boundary for hybrid problems is mature
- Complexity boundary is not

(Bi)simulation relations are very useful
- Theoretically : As a system theoretic concept
- Practically : As a complexity reduction mechanism

Some future directions

Stochastic hybrid systems

Equivalence (model reduction) of hybrid systems

Approximate but efficient algorithms for analysis and design

Understanding compositionality and concurrency

Hybrid (heterogeneous) systems in a broader context

A unified systems theory

An invitation

Hybrid Systems : Computation and Control
University of Pennsylvania
March 25-27, 2004

http://www.seas.upenn.edu/hybrid/HSCC04/