STABILITY OF SWITCHED SYSTEMS

Daniel Liberzon

Coordinated Science Laboratory and
Dept. of Electrical & Computer Eng.,
Univ. of Illinois at Urbana-Champaign
U.S.A.

DISC HS, June 2003

SWITCHED vs. HYBRID SYSTEMS

Switched system:

$$\boxed{\dot{x} = f_\sigma(x)}$$

- $\dot{x} = f_p(x), \; p \in \mathcal{P}$ is a family of systems
- $\sigma : [0, \infty) \to \mathcal{P}$ is a switching signal

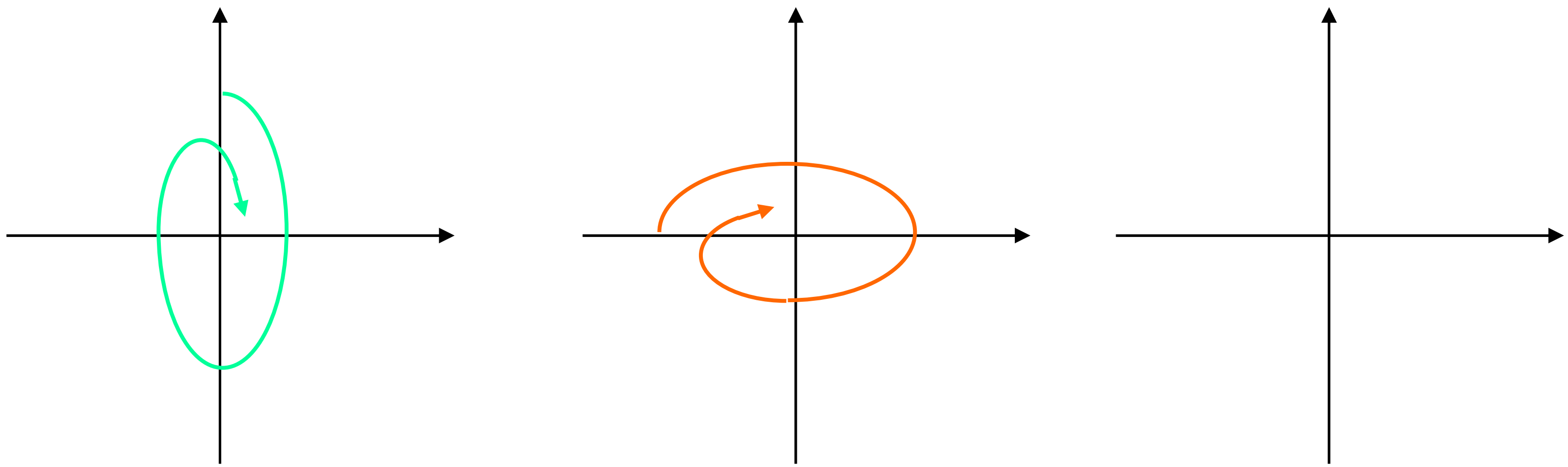
Switching can be:

- State-dependent or time-dependent
- Autonomous or controlled

Details of discrete behavior are “abstracted away”

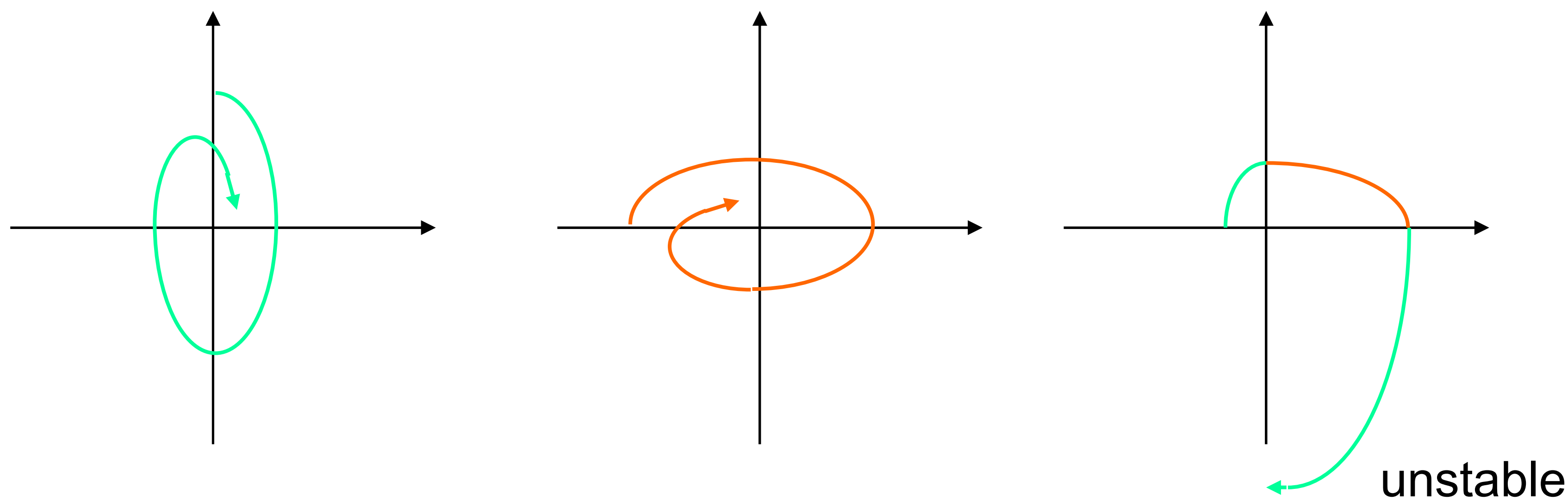
Properties of the continuous state: stability

STABILITY ISSUE



Asymptotic stability of each subsystem is necessary for stability

STABILITY ISSUE



Asymptotic stability of each subsystem is necessary but not sufficient for stability

(This only happens in dimensions 2 or higher)

TWO BASIC PROBLEMS

- Stability for arbitrary switching
- Stability for constrained switching

TWO BASIC PROBLEMS

- Stability for arbitrary switching

- Stability for constrained switching

GLOBAL UNIFORM ASYMPTOTIC STABILITY

GUAS is Lyapunov stability

$$\forall \varepsilon \ \exists \delta \ \ |x(0)| \le \delta \Rightarrow |x(t)| \le \varepsilon \ \forall t \ge 0, \forall \sigma$$

plus asymptotic convergence

$$\forall \varepsilon, \delta \ \exists T \ \ |x(0)| \le \delta \Rightarrow |x(t)| \le \varepsilon \ \forall t \ge T, \forall \sigma$$

Reduces to standard GAS notion for non-switched systems

COMPARISON FUNCTIONS

class $\mathcal{K}$ function

$\beta(\cdot,\cdot)$ is of class $\mathcal{KL}$ if

- $\beta(\cdot,t) \in \mathcal{K}$ for each fixed t
- $\beta(r,t) \searrow 0$ as $t \to \infty$ for each r

Example: $\beta(r,t) = cre^{-\lambda t}, \;\; c,\lambda > 0$

GUES

$$\boxed{\text{GUAS:} \quad |x(t)| \le \beta(|x(0)|,t) \;\; \forall t \ge 0}$$

COMMON LYAPUNOV FUNCTION

Lyapunov theorem: $\dot{x} = f(x)$ is GAS iff $\exists$ pos def rad unbdd

C^1 function $V : \mathcal{R}^n \to \mathcal{R}$ s.t. $\frac{\partial V}{\partial x} f(x) < 0 \ \forall x \neq 0$

Similarly: $\dot{x} = f_\sigma(x)$ is GUAS iff $\exists V$ s.t.

$$\boxed{\frac{\partial V}{\partial x} f_p(x) \leq -W(x) \ \forall x, \forall p \in \mathcal{P}}$$

where W is positive definite

COMMON LYAPUNOV FUNCTION (continued)

$$\boxed{\frac{\partial V}{\partial x} f_p(x) \le -W(x) < 0 \;\; \forall x \neq 0, p \in \mathcal{P}}$$

Unless $\mathcal{P}$ is compact and f_p is continuous,

$\frac{\partial V}{\partial x} f_p(x) < 0 \;\; \forall x \neq 0, p \in \mathcal{P}$ is not enough

Example: $f_p(x) = -px, \;\; \mathcal{P} = (0, 1]$

$V(x) = \frac{x^2}{2}, \;\; \frac{\partial V}{\partial x} f_p(x) = -px^2 \to 0$ as $p \to 0$

$x(t) = e^{-\int_0^t \sigma(\tau) d\tau} x(0) \not\to 0$ if $\sigma \in L^1$

CONVEX COMBINATIONS

$$\boxed{\frac{\partial V}{\partial x} f_p(x) \leq -W(x) < 0 \;\; \forall x \neq 0, p \in \mathcal{P}}$$

Define $\quad f_{p,q,\alpha}(x) = \alpha f_p(x) + (1-\alpha) f_q(x)$

$$p, q \in \mathcal{P}, \;\; \alpha \in [0,1]$$

Corollary: $\dot{x} = f_{p,q,\alpha}(x)$ is GAS $\forall p, q, \alpha$

Proof:

$$\frac{\partial V}{\partial x} f_{p,q,\alpha}(x) = \alpha \frac{\partial V}{\partial x} f_p(x) + (1-\alpha) \frac{\partial V}{\partial x} f_q(x) \leq -W(x)$$

SWITCHED LINEAR SYSTEMS

$$\boxed{\dot{x} = A_\sigma x}$$

LAS for every σ

$$\Updownarrow$$

GUES

$$\Updownarrow$$

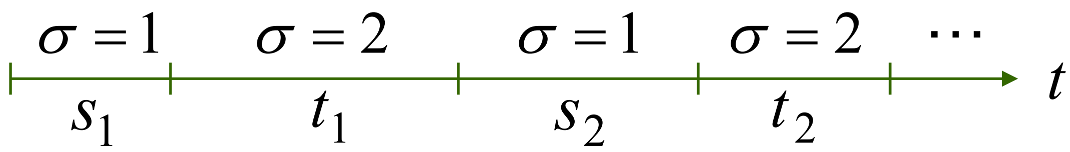
$\exists$ common Lyapunov function

but not necessarily quadratic:

$$V(x) = x^T P x, \quad A_p^T P + P A_p < 0 \ \forall p \in \mathcal{P} \quad \text{(LMIs)}$$

COMMUTING STABLE MATRICES => GUES

$$P = \{1, 2\}, \quad A_1 A_2 = A_2 A_1$$



$$x(t) = e^{A_2 t_k} e^{A_1 s_k} \cdots e^{A_2 t_1} e^{A_1 s_1} x(0)$$

$$= e^{A_2 (t_k + \ldots + t_1)} e^{A_1 (s_k + \ldots + s_1)} x(0) \to 0$$

$\exists$ quadratic common Lyap fcn:

$$A_1^T P_1 + P_1 A_1 = -I$$

$$A_2^T P_2 + P_2 A_2 = -P_1$$

LIE ALGEBRAS and STABILITY

Lie algebra: $g = \{A_p, p \in P\}_{\mathrm{LA}}$

Lie bracket: $[A_1, A_2] = A_1 A_2 - A_2 A_1$

$$\begin{array}{llcl}
g^1 = g, & g^{k+1} = [g, g^k] \subset g^k & & g \text{ is nilpotent if } \exists\, k \text{ s.t. } g^k = 0 \\
\parallel & \cup & & \Downarrow \\
g^{(1)} = g, & g^{(k+1)} = [g^{(k)}, g^{(k)}] \subset g^{(k)} & & g \text{ is solvable if } \exists\, k \text{ s.t. } g^{(k)} = 0
\end{array}$$

SOLVABLE LIE ALGEBRA => GUES

Lie's Theorem: g is solvable $\Rightarrow$ triangular form

$$A_p = \begin{pmatrix} \lambda_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

Example:

$$A_1 = \begin{pmatrix} -a_1 & b_1 \\ 0 & -c_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -a_2 & b_2 \\ 0 & -c_2 \end{pmatrix}$$

$\dot{x}_2 = -c_\sigma x_2 \Rightarrow x_2 \to 0$ exponentially fast

$\dot{x}_1 = -a_\sigma x_1 + b_\sigma \underset{\substack{\downarrow \\ 0}}{x_2} \Rightarrow x_1 \to 0$ exp fast

$\exists$ quadratic common Lyap fcn $x^T D x,\ D$ diagonal

MORE GENERAL LIE ALGEBRAS

Levi decomposition: $g = r \oplus s$

radical (max solvable ideal)

- s is compact => GUES, quadratic common Lyap fcn
- s is not compact => not enough info in Lie algebra

NONLINEAR SYSTEMS

- Commuting systems

$$[f_p, f_q] = 0 \implies \text{GUAS}$$

- Linearization (Lyapunov's indirect method)

$$A_p = \frac{\partial f_p}{\partial x}(0), \ \ p \in \mathcal{P}$$

- Nothing is known beyond this

REMARKS on LIE-ALGEBRAIC CRITERIA

- Checkable conditions
- Independent of representation
- In terms of the original data

- Not robust to small perturbations

SYSTEMS with SPECIAL STRUCTURE

- Triangular systems
- Feedback systems
    - passivity conditions
    - small-gain conditions
- 2-D systems

TRIANGULAR SYSTEMS

Recall: for linear systems, triangular => GUAS

For nonlinear systems, not true in general

Example:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= g_1(x_2) \end{aligned} \qquad\qquad \begin{aligned} \dot{x}_1 &= f_2(x_1, x_2) \\ \dot{x}_2 &= g_2(x_2) \end{aligned}$$

$$\dot{x}_2 = g_\sigma(x_2) \Rightarrow x_2 \to 0$$

For stability need to know $x_2 \to 0 \Rightarrow x_1 \to 0$

Not necessarily true

INPUT-TO-STATE STABILITY (ISS)

Linear systems:

$\dot{x} = Ax$ is AS $\Rightarrow$ $\dot{x} = Ax + Bu$ is ISS:

- u bounded $\Rightarrow x$ bounded
- $u \to 0 \Rightarrow x \to 0$

Nonlinear systems:

$$\dot{x} = -x + x^2 u$$

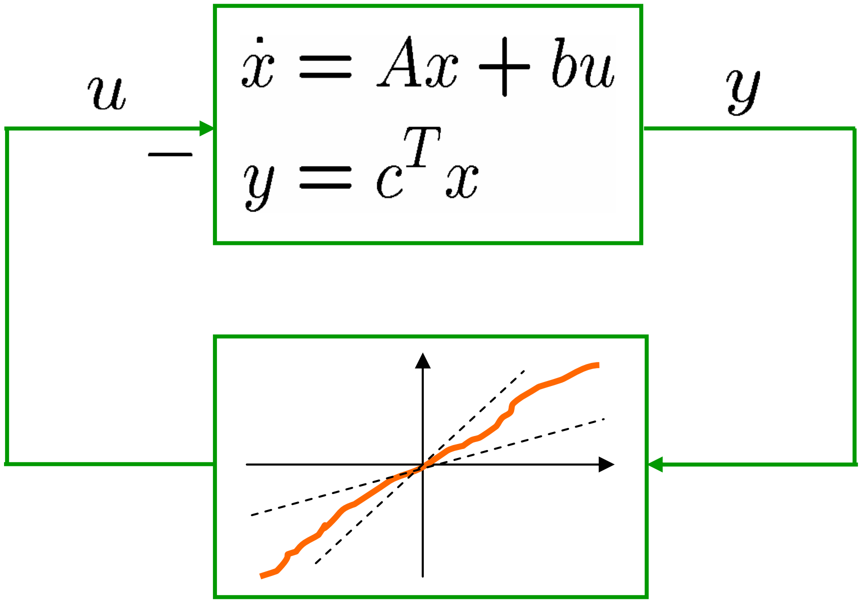
$u = 0 \Rightarrow x \to 0$ but u bdd $\not\Rightarrow x$ bdd, $u \to 0 \not\Rightarrow x \to 0$

$\dot{x} = f(x, u)$ is input-to-state stable (ISS) if

$$\boxed{|x(t)| \le \beta(|x(0)|, t) + \gamma\left(\|u\|_{[0,t]}\right)} \qquad \begin{array}{c} \beta \in \mathcal{KL} \\ \gamma \in \mathcal{K} \end{array}$$

For switched systems, triangular + ISS => GUAS

FEEDBACK SYSTEMS: ABSOLUTE STABILITY



A Hurwitz

$$g(s) = c^T(sI - A)^{-1}b$$

$$u = -\varphi_p(y)$$

$$k_1 y^2 \le y\varphi_p(y) \le k_2 y^2 \ \forall p$$

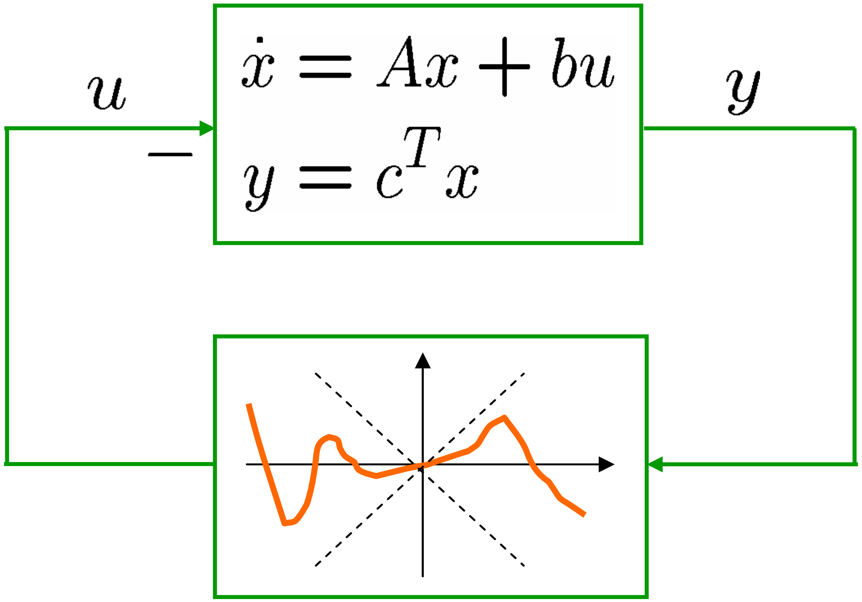
Circle criterion: $\exists$ quadratic common Lyapunov function $\Leftrightarrow$

$h(s) = \frac{1+k_2 g(s)}{1+k_1 g(s)}$ is strictly positive real (SPR): $Re\, h(i\omega) > 0$

For $k_1 = 0, k_2 = \infty$ this reduces to $g(s)$ SPR (passivity)

Popov criterion not suitable: V depends on φ_p

FEEDBACK SYSTEMS: SMALL-GAIN THEOREM



A Hurwitz

$$g(s) = c^T(sI - A)^{-1}b$$

$$u = -\varphi_p(y)$$

$$|\varphi_p(y)| \leq |y| \;\; \forall p$$

$$(k_1 = -1, k_2 = 1)$$

Small-gain theorem:

$\exists$ quadratic common Lyapunov function

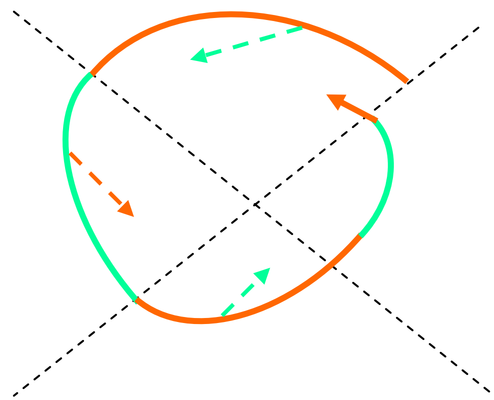
$$\Updownarrow$$

$$\|g\|_\infty = \max_\omega |g(i\omega)| < 1$$

TWO-DIMENSIONAL SYSTEMS

Necessary and sufficient conditions for GUES

known since 1970s



worst-case
switching

$$\boxed{\dot{x} = A_1 x, \ \ \dot{x} = A_2 x, \ \ x \in \mathcal{R}^2}$$

$\exists$ quadratic common Lyap fcn $\Longleftrightarrow$

convex combinations of $A_1, A_2, A_1^{-1}, A_2^{-1}$ Hurwitz

WEAK LYAPUNOV FUNCTION

Barbashin-Krasovskii-LaSalle theorem: $\dot{x} = f(x)$ is GAS

if $\exists$ pos def rad unbdd C^1 function $V : \mathcal{R}^n \to \mathcal{R}$ s.t.

- $\frac{\partial V}{\partial x} f(x) \leq 0 \;\; \forall x$ (weak Lyapunov function)
- $\dot{V}$ is not identically zero along any nonzero solution (observability with respect to $\dot{V}$)

Example:

$$\dot{x} = Ax, \quad V(x) = x^T P x$$

$$\left.\begin{array}{l} A^T P + PA \leq -C^T C \\ (A, C) \text{ observable} \end{array}\right\} \Rightarrow \text{GAS}$$

COMMON WEAK LYAPUNOV FUNCTION

Theorem: $\dot{x} = A_\sigma x$ is GAS if

- $A_p^T P + P A_p \le -C_p^T C_p \;\forall p, \quad P > 0$
- (A_p, C_p) observable for each p
- $\exists \tau > 0$ s.t. there are infinitely many switching intervals of length $\ge \tau$

Extends to nonlinear switched systems and nonquadratic common weak Lyapunov functions using a suitable nonlinear observability notion

TWO BASIC PROBLEMS

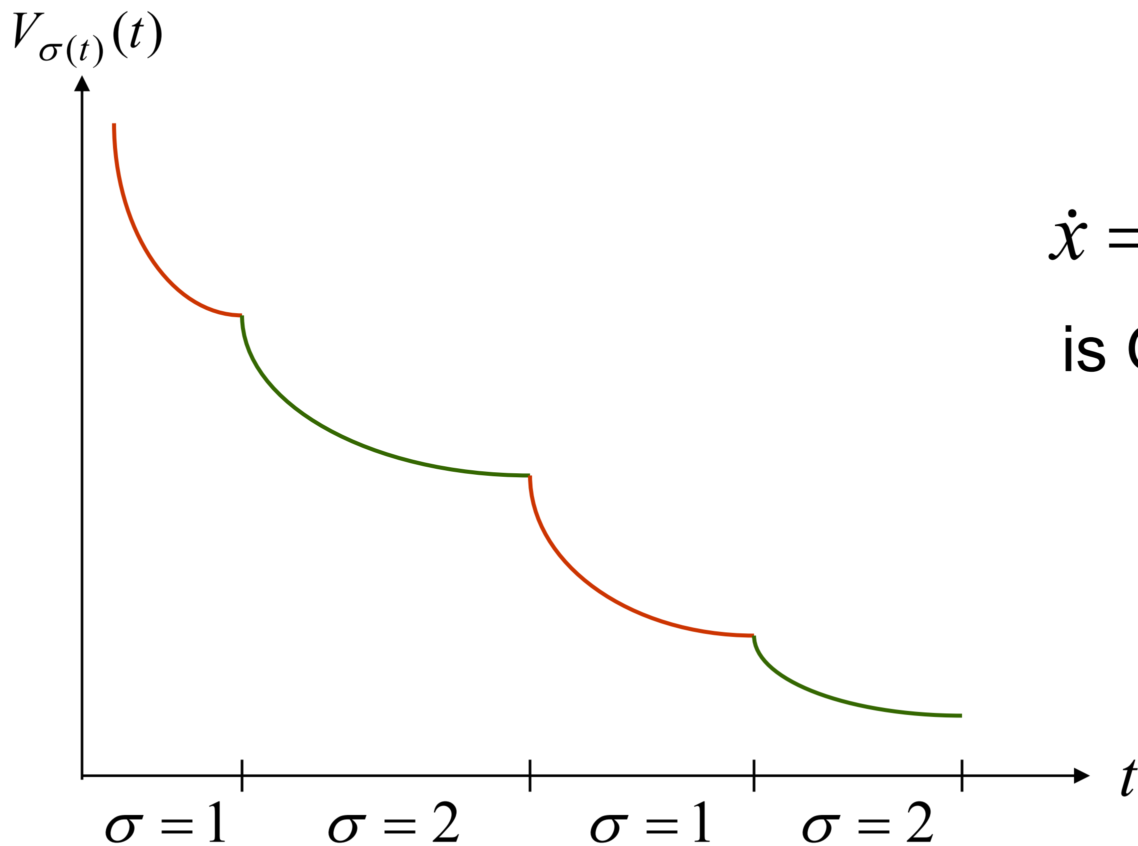
- Stability for arbitrary switching

- Stability for constrained switching

MULTIPLE LYAPUNOV FUNCTIONS

$$\dot{x} = f_1(x),\ \dot{x} = f_2(x)$$ – GAS

V_1, V_2 – respective Lyapunov functions

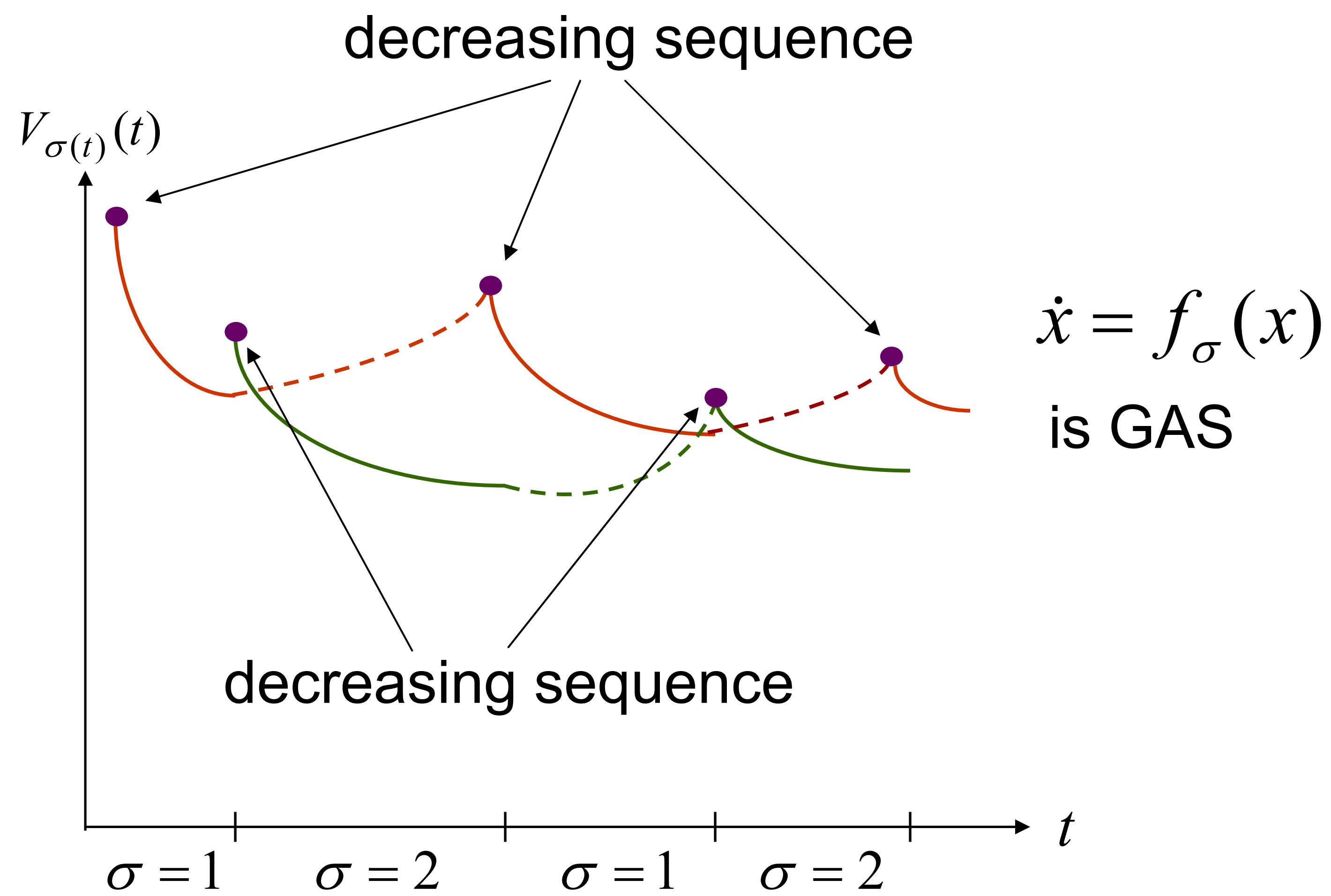


$$\dot{x} = f_\sigma(x)$$

is GAS

Very useful for analysis of state-dependent switching

MULTIPLE LYAPUNOV FUNCTIONS



DWELL TIME

The switching times $t_1, t_2, \ldots$ satisfy $t_{i+1} - t_i \geq \tau_D$

dwell time

$\dot{x} = f_1(x),\ \dot{x} = f_2(x)$ – GES

V_1, V_2 – respective Lyapunov functions

DWELL TIME

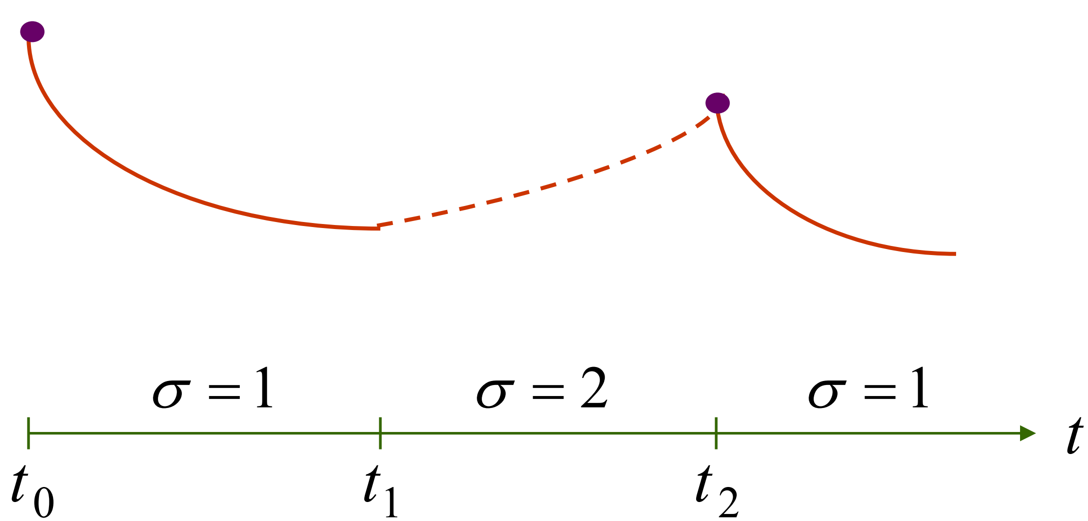
The switching times $t_1, t_2, \ldots$ satisfy $t_{i+1} - t_i \geq \tau_D$

$\dot{x} = f_1(x), \; \dot{x} = f_2(x)$ – GES

$$a_1 |x|^2 \leq V_1(x) \leq b_1 |x|^2, \qquad \frac{\partial V_1}{\partial x} f_1(x) \leq -\lambda_1 V_1(x)$$

$$a_2 |x|^2 \leq V_2(x) \leq b_2 |x|^2, \qquad \frac{\partial V_2}{\partial x} f_2(x) \leq -\lambda_2 V_2(x)$$

Need: $V_1(t_2) < V_1(t_0)$



DWELL TIME

The switching times $t_1, t_2, \ldots$ satisfy $t_{i+1} - t_i \geq \tau_D$

$\dot{x} = f_1(x), \ \dot{x} = f_2(x)$ – GES

$$a_1 |x|^2 \leq V_1(x) \leq b_1 |x|^2, \qquad \frac{\partial V_1}{\partial x} f_1(x) \leq -\lambda_1 V_1(x)$$

$$a_2 |x|^2 \leq V_2(x) \leq b_2 |x|^2, \qquad \frac{\partial V_2}{\partial x} f_2(x) \leq -\lambda_2 V_2(x)$$

Need: $V_1(t_2) < V_1(t_0)$

must be < 1

$$V_1(t_2) \leq \frac{b_1}{a_2} V_2(t_2) \leq \frac{b_1}{a_2} e^{-\lambda_2 \tau_D} V_2(t_1)$$

$$\leq \frac{b_1}{a_2}\frac{b_2}{a_1} e^{-\lambda_2 \tau_D} V_1(t_1) \leq \frac{b_1}{a_2}\frac{b_2}{a_1} e^{-(\lambda_1+\lambda_2)\tau_D} V_1(t_0)$$

AVERAGE DWELL TIME

$$N_\sigma(T,t) \le N_0 + \frac{T-t}{\tau_{AD}}$$

# of switches on (t,T) average dwell time

$N_0 = 0$ – no switching: cannot switch if $T - t < \tau_{AD}$

$N_0 = 1$ – dwell time: cannot switch twice if $T - t < \tau_{AD}$

$$\boxed{\dot{x} = f_\sigma(x)}$$

AVERAGE DWELL TIME

$$N_\sigma(T,t) \le N_0 + \frac{T-t}{\tau_{AD}}$$

# of switches on (t,T)

average dwell time

$$\boxed{\dot{x} = f_\sigma(x)}$$

$$\alpha_1(|x|) \le V_p(x) \le \alpha_2(|x|)$$

$$\frac{\partial V_p}{\partial x} f_p(x) \le -\lambda V_p(x)$$

$$V_p(x) \le \mu V_q(x), \quad p,q \in P$$

=>

$\dot{x} = f_\sigma(x)$

is GAS

if $\boxed{\tau_{AD} > \frac{\log \mu}{\lambda}}$

SWITCHED LINEAR SYSTEMS

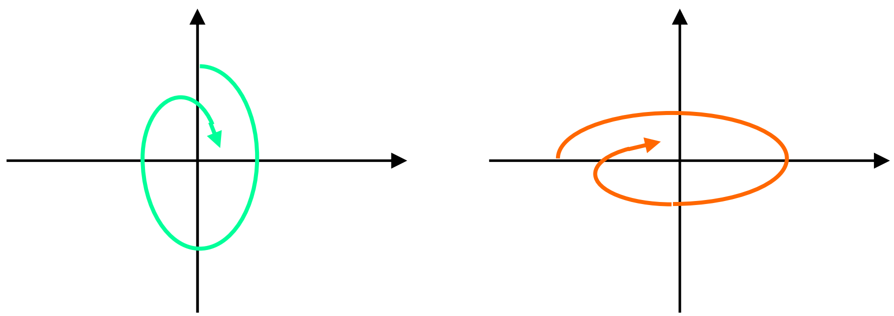
$$\boxed{\dot{x} = A_\sigma x}$$

- GUES over all σ with large enough τ_{AD}
- Finite induced norms for

$$\boxed{\begin{aligned} \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x \end{aligned}}$$

- The case when some subsystems are unstable

STATE-DEPENDENT SWITCHING

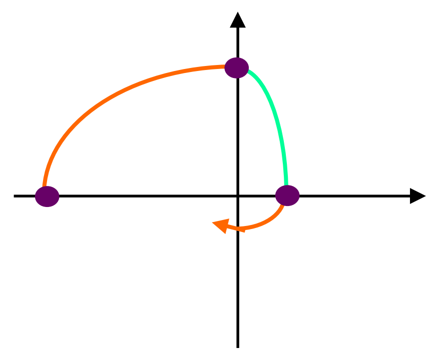


Switched system

unstable for some σ

$\Rightarrow$ no common V

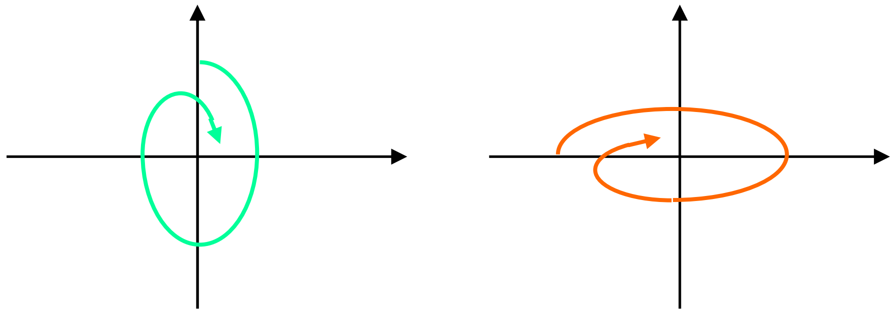
But switched system is stable for (many) other σ



switch on the axes

$V(x) = x^T x$ is a Lyapunov function

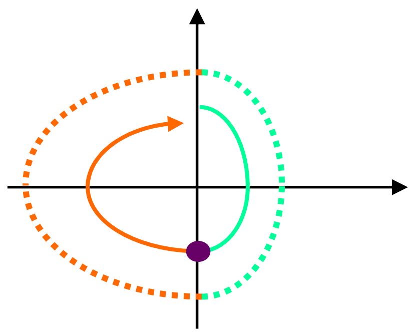
STATE-DEPENDENT SWITCHING



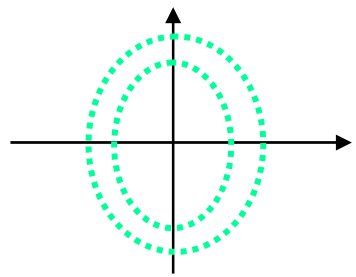
Switched system unstable for some σ

$\Rightarrow$ no common V

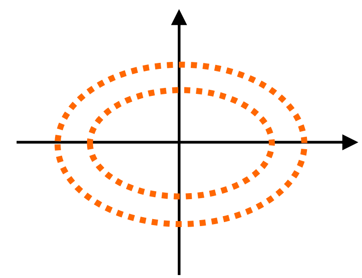
But switched system is stable for (many) other σ



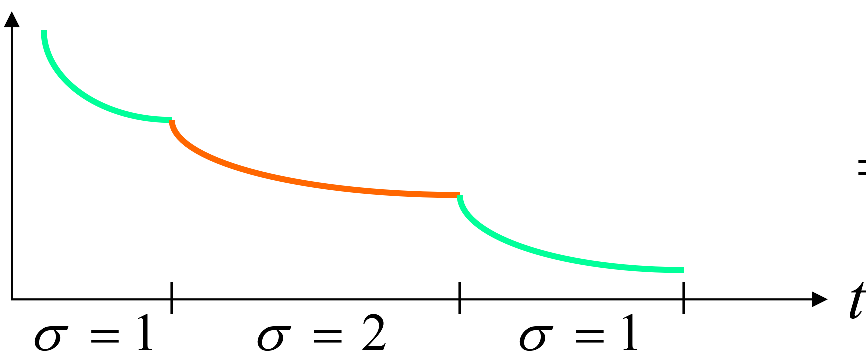
Switch on y-axis



level sets of V_1



level sets of V_2

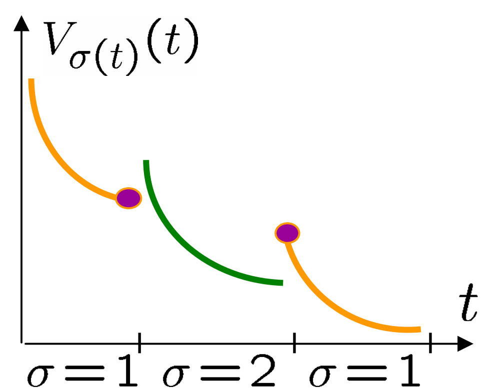


=> GAS

MULTIPLE WEAK LYAPUNOV FUNCTIONS

Theorem: $\dot{x} = A_\sigma x$ is GAS if

- $A_p^T P_p + P_p A_p \leq -C_p^T C_p \ \forall p, \quad P_p > 0$
  (each $V_p(x) = x^T P_p x$ is a weak Lyapunov function)
- (A_p, C_p) observable for each p
- $\exists \tau > 0$ s.t. there are infinitely many switching intervals of length $\geq \tau$
- For every pair of switching times $t_i < t_j$ s.t. $\sigma(t_i) = \sigma(t_j) = p$ have $V_p(x(t_j)) \leq V_p(x(t_{i+1}))$



STABILIZATION by SWITCHING

$\dot{x} = A_1 x, \ \dot{x} = A_2 x$ – both unstable

Assume: $A = \alpha A_1 + (1-\alpha) A_2$ stable for some $\alpha \in (0,1)$

$$A^T P + PA < 0$$

STABILIZATION by SWITCHING

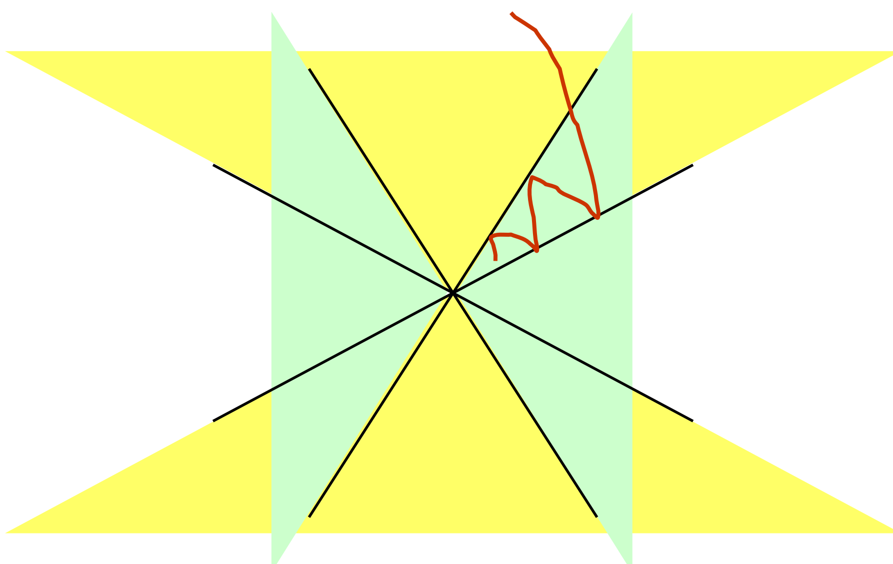
$$\dot{x} = A_1 x\,,\ \ \dot{x} = A_2 x \ \ -$$ both unstable

Assume: $A = \alpha A_1 + (1-\alpha) A_2$ stable for some $\alpha \in (0,1)$

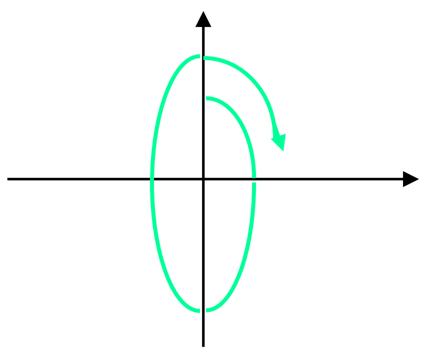
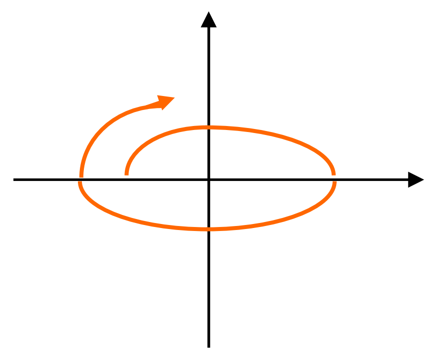
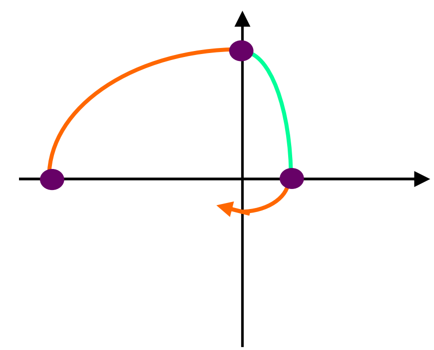
$$\alpha (A_1^T P + P A_1) + (1-\alpha)(A_2^T P + P A_2) < 0$$

So for each $x \neq 0$:

either $x^T (A_1^T P + P A_1)\, x < 0$ or $x^T (A_2^T P + P A_2)\, x < 0$



UNSTABLE CONVEX COMBINATIONS

Can also use multiple Lyapunov functions

LMIs

REFERENCES

Branicky, DeCarlo, Hespanha