Consider the following *continuous* PWA system:

$$\dot{x}(t) = A_i x(t) + B_i u(t) + a_i, \qquad \text{for } x \in X_i \tag{1}$$

Here, $\{X_i\}_{i\in I} \subseteq \mathbb{R}^n$ is a partition of the state space into a number of closed (possibly unbounded) polyhedral cells. The index set of the cells is denoted by I. Let $I_0 \subseteq I$ be the set of indices of the cells that contain the origin and $I_1 \subseteq I$ the set of indices of cells that do not contain the origin. It is assumed that $a_i = 0$, for all $i \in I_0$. Defining the following modified matrices:

$$\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}, \bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}, \tag{2}$$

we obtain:

$$\dot{\bar{x}}(t) = \bar{A}_i \bar{x}(t) + \bar{B}_i u(t), \qquad \text{for } x \in X_i \tag{3}$$

Next, we can construct matrices

$$\bar{E}_i = [E_i \;\; e_i], \qquad \bar{F}_i = [F_i \;\; f_i], \tag{4}$$

with $e_i = 0$ and $f_i = 0$ for $i \in I_0$ and such that

$$\bar{E}_i \bar{x} \geq 0, \qquad \text{if } x \in X_i, \;\; i \in I \tag{5}$$

$$\bar{F}_i \bar{x} = \bar{F}_j \bar{x}, \qquad \text{if } x \in X_i \cap X_j, \;\; i, j \in I \tag{6}$$

The construction of the constraint matrices $\bar{E}_i$ and $\bar{F}_i$ will be discussed later. Now consider symmetric matrices T, U_i and W_i, such that U_i and W_i have nonnegative entries. Moreover, define matrices $\bar{Q}_i$ as follows:

$$\bar{Q}_i = \begin{bmatrix} Q_i & 0 \\ 0 & 0 \end{bmatrix}, \tag{7}$$

with Q_i positive definite matrices. Now if $P_i = F_i^{\mathrm{T}} T F_i$, for $i \in I_0$, and $\bar{P}_i = \bar{F}_i^{\mathrm{T}} T \bar{F}_i$, for $i \in I_1$, satisfy ([1])

$$\begin{cases} 0 > A_i^{\mathrm{T}} P_i + P_i A_i + E_i^{\mathrm{T}} U_i E_i + Q_i, & \text{for all } i \in I_0, \\ 0 < P_i - E_i^{\mathrm{T}} W_i E_i, \end{cases} \tag{8}$$

$$\begin{cases} 0 > \bar{A}_i^{\mathrm{T}} \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{E}_i^{\mathrm{T}} U_i \bar{E}_i + \bar{Q}_i, & \text{for all } i \in I_1, \\ 0 < \bar{P}_i - \bar{E}_i^{\mathrm{T}} W_i \bar{E}_i, \end{cases} \tag{9}$$

then $x(t)$ tends to zero exponentially as $t \longrightarrow \infty$ for every continuous piecewise trajectory in $\cup_{i\in I} X_i$ satisfying (1) with $u \equiv 0$. As discussed before, for cells that contain the origin, a_i is zero. Therefore for these cells (which belong to the set I_0) we use the original matrix definitions (A_i and B_i) in (8).

Now in order to integrate the reference tracking problem into this framework, we define an additional state x_{e} as follows:

$$\dot{x}_{\mathrm{e}}(t) = x_{\mathrm{ref}}(t) - x(t) \tag{10}$$

with $x_{\mathrm{ref}}(t)$ the reference signal to be tracked. Thus, the dynamics are modified as follows:

$$\dot{x}_{\mathrm{t}} = \begin{bmatrix} \dot{x} \\ \dot{x}_{\mathrm{e}} \end{bmatrix} = \underbrace{\begin{bmatrix} A_i & 0 \\ -I & 0 \end{bmatrix}}_{A_{t,i}} \begin{bmatrix} x \\ x_{\mathrm{e}} \end{bmatrix} + \underbrace{\begin{bmatrix} B_i \\ 0 \end{bmatrix}}_{B_{t,i}} u + \begin{bmatrix} 0 \\ I \end{bmatrix} x_{\mathrm{ref}} + \begin{bmatrix} a_i \\ 0 \end{bmatrix} \tag{11}$$

and with the state augmentation technique introduced in (2), the dynamics can be reformulated as follows

$$\dot{\bar{x_t}} = \begin{bmatrix} \dot{x} \\ \dot{x}_e \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} A_i & 0 & a_i \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\bar{A}_{t,i}} \begin{bmatrix} x \\ x_e \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} B_i \\ 0 \\ 0 \end{bmatrix}}_{\bar{B}_{t,i}} u + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} x_{\text{ref}} \tag{12}$$

Now define the state feedback control law as

$$u(t) = K_i \bar{x_t}(t), \qquad \text{if } x \in X_i \tag{13}$$

If we replace u in (12) with the above feedback law, the closed-loop system will have $(\bar{A}_{t,i} + \bar{B}_{t,i} K_i)$ as its A matrix. Therefore, the design equations (8)–(9) will be modified as follows:

$$\begin{cases} 0 > (A_{t,i} + B_{t,i}K_i)^{\mathrm{T}} P_{t,i} + P_{t,i}(A_{t,i} + B_{t,i}K_i) + E_{t,i}^{\mathrm{T}} U_i E_{t,i} + Q_{t,i}, & \text{for all } i \in I_0 \\ 0 < P_{t,i} - E_{t,i}^{\mathrm{T}} W_i E_{t,i}, & \end{cases} \tag{14}$$

$$\begin{cases} 0 > (\bar{A}_{t,i} + \bar{B}_{t,i}K_i)^{\mathrm{T}} \bar{P}_{t,i} + \bar{P}_{t,i}(\bar{A}_{t,i} + \bar{B}_{t,i}K_i) + \bar{E}_{t,i}^{\mathrm{T}} U_i \bar{E}_{t,i} + \bar{Q}_{t,i}, & \text{for all } i \in I_1 \\ 0 < \bar{P}_{t,i} - \bar{E}_{t,i}^{\mathrm{T}} W_i \bar{E}_{t,i}, & \end{cases} \tag{15}$$

with $P_{t,i} = F_{t,i}^{\mathrm{T}} T F_{t,i}$, for $i \in I_0$, and $\bar{P}_{t,i} = \bar{F}_{t,i}^{\mathrm{T}} T \bar{F}_{t,i}$, for $i \in I_1$. Clearly, we have to re-define the matrices $\bar{E}_{t,i}$ and $\bar{F}_{t,i}$ using the new state vector $\bar{x_t}$. But note that the cells are still defined based on x.

After all, the feedback gains K_i are determined by finding a solution for (14)–(15) (by solution we mean values for matrices T_i, W_i, U_i, and K_i that satisfy (14)–(15); remember that T_i, U_i, and W_i are symmetric, and furthermore U_i and W_i have nonnegative elements). This is a nonlinear feasibility problem that can be solved using optimization tools in MATLAB[1].

Constraint handling:
If we have constraints on the control input u of the following form

$$u_{\mathrm{L}} \leq u \leq u_{\mathrm{H}}, \tag{16}$$

we can integrate them in our design approach as follows. Note that the control input is in fact a state feedback controller of the following form:

$$u = Kx \tag{17}$$

Therefore, we need to have

$$u_{\mathrm{L}} \leq Kx \leq u_{\mathrm{H}} \tag{18}$$

Before proceeding, we have to make the assumption that the state vector x is constrained in the following region of admissible states:

$$0 \leq x \leq x^* \tag{19}$$

This assumption is consistent with the system under study (the ACC system in the practical assignment) in which the speed of the following car is limited between zero and a certain maximum speed determined in the Step 1 of the assignment. Furthermore, we introduce the following decomposition for K:

$$K = K^+ - K^-, \tag{20}$$

[1]Furthermore, the problem can be recast as a linear matrix inequalities (LMI) problem but we skip this step in the current report.

where K^+ and K^- are matrices with nonnegative elements:

$$K^+ \geq 0,\ K^- \geq 0 \tag{21}$$

With this definition we will be able to multiply (19) with K^+ and K^- and come up with the following inequalities:

$$0 \leq K^+ x \leq K^+ x^* \tag{22}$$
$$-K^- x^* \leq -K^- x \leq 0 \tag{23}$$

This yields the following:

$$-K^- x^* \leq Kx \leq K^+ x^* \tag{24}$$

Hence, in order to satisfy (16) it is necessary and sufficient that

$$K^+ x^* \leq u_{\rm H} \tag{25}$$
$$K^- x^* \leq -u_{\rm L} \tag{26}$$

Note that since we have assumed $0 \leq x \leq x^*$, we can conclude (24) from (22).

Hence, in our design approach we use the constraints (25)–(26) together with (20)–(21) in order to guarantee (18) and therefore, variables k^+ and k^- are considered as variables too. Also note that we have to determine x^* based on the model of the system (similar to what we did in step 1 of the assignment).

Hints on finding E and F matrices:

In this section, we show how matrices E and F are determined for a simple PWA system. For the general case the interested reader is referred to [1]. Assume that a scalar PWA system consists of two affine pieces $a_i x + b_i$, $x \in X_i$, $i \in \{1,2\}$. Moreover, assume that $0 \leq x \leq \alpha$ for X_1 and $\alpha \leq x \leq \beta$ for X_2 $(\alpha, \beta > 0)$. The boundary of two regions is specified by $x = \alpha$.

Now define the matrices V and $\bar{V}$ as follows:

$$V = [v_0 \cdots v_2] \tag{27}$$
$$\bar{V} = [\bar{v}_0 \cdots \bar{v}_2] \tag{28}$$

with $v_0 = 0$, and $\bar{v}_k = [v_k, 1]^{\rm T}$. Then each $\bar{x} = [x, 1]^{\rm T}$, with $x \in X_i$, has a unique representation as a convex combination of the elements $\bar{v}_k$ as long as v_k belongs to X_i (note that we assume each X_i is bounded with finite number of corner points (vertices), therefore each point in X_i can be represented by a convex combination of the vertices v_k of X_i). As a hint, in our case v_1 and v_2 are in fact α and β, respectively. Moreover, for each cell X_i, we define a matrix $Y_i \in \mathbb{R}^{3\times 2}$. The kth row of Y_i is zero for all k such that $v_k \notin X_i$ and the remaining rows of Y_i are equal to the rows of an identity matrix. Now we can define the matrices $\bar{E}_i$ and $\bar{F}_i$ as:

$$\bar{F}_i = [0 \ \ I_2] Y_i (\bar{V} Y_i)^{-1}, \tag{29}$$

$$\bar{E}_i = Y_i^{\rm T} \begin{bmatrix} 0 \\ \bar{F}_i \end{bmatrix}, \tag{30}$$

with I_2 the identity matrix of order 2. Moreover, for the region X_1 which contains the origin, the last columns of the matrices $\bar{E}_1$ and $\bar{F}_1$ will become zero. Therefore, as discussed before, for the state feedback design, we can use E_1 and F_1 obtained from eliminating the last columns of $\bar{E}_1$ and $\bar{F}_1$.

References

[1] A. Rantzer and M. Johansson. Piecewise linear quadratic optimal control. *Automatic Control, IEEE Transactions on*, 45(4):629–637, 2000.