Modeling & Control of Hybrid Systems
Chapter 4 – Stability

Overview:

1. Switched systems
2. Lyapunov theory for smooth and linear systems
3. Stability for any switching signal
4. Stability for a given switching signal
5. Summary

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Generic form for switched systems:

$$\dot{x} = f_\sigma(x)$$

Definitions

- $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$
- $\sigma : [0, \infty) \to \{1, 2, \ldots, N\}$

Assumptions

- f_σ are smooth vector fields

Switched Systems

[The class of systems considered in this lecture]

Generic form for switched systems:

$$\dot{x} = f_\sigma(x)$$

- N is the number of modes
- σ is the switching signal
- σ is a piece-wise constant function e.g. $\sigma(t), \sigma(x), \sigma(t, x)$

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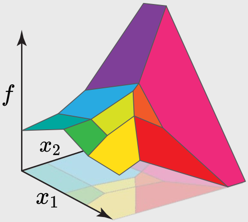
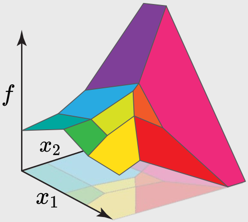


(Continuous) PWA function

Definitions

- $$f(x) = \begin{cases} A_1 x + a_1 & x \in \mathcal{X}_1 \\ A_2 x + a_2 & x \in \mathcal{X}_2 \\ \vdots \\ A_N x + a_N & x \in \mathcal{X}_N \end{cases}$$
- $\mathcal{X}_i := E_i x \geqslant e_i, \; i \in \{1, \dots, N\}$

Recap: (Continuous) PieceWise-Affine Systems



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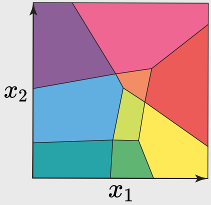

Sub-regions

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- $\mathcal{X}_i := E_ix \geqslant e_i, \; i \in \{1, \ldots, N\}$

Assumptions

- f is continuous
- $\bigcup_{i=1}^{N} \mathcal{X}_i = \mathbb{R}^n$
- $i \neq j \implies \text{interior}(\mathcal{X}_i) \cap \text{interior}(\mathcal{X}_j) = \emptyset$

Switched Linear Systems

Definitions

- $\dot{x} = A_\sigma x \quad x \in \mathcal{X}_\sigma$

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- $i \neq j \implies \text{interior}(\mathcal{X}_i) \cap \text{interior}(\mathcal{X}_j) = \emptyset$
- Switching is only state-dependant
- System is autonomous/unforced (no input)

Global Asymptotic Stability

Stability definitions:

- Asymptotic Stability (AS):

$$\lim_{t \to \infty} x(t) = 0$$

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A Find conditions of GAS for <u>any</u> switching signal (GUAS)

B Show the system is GAS for <u>a given</u> class of switching strategies

C Construct a switching signal to have GAS (stabilization problem → chp 5)

Definitions

- $\dot{x} = f(x)$

If (Assumptions)

- $f(0) = 0$

Lyapunov Stability Theorem
[Lecture notes: Theorem 4.3.4]

Definitions

- $\dot{x} = f(x)$
- $V : \mathbb{R}^n \to \mathbb{R}$

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- $|x| \to \infty \implies V(x) \to \infty$ (V is radially unbounded)
- $V(0) = 0 \wedge x \neq 0 \implies V(x) > 0$ (V is positive definite)
- $\forall x \neq 0 : \dot{V}(x) < 0$

note: $$\dot{V}(x) = L_f V(x) := \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x)$$

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[Lecture notes: Theorem 4.3.4]

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Then

- f is GAS for $x = 0$.

Asymptotic Stability of Linear Systems

Definitions

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- $\dot{x} = Ax$
- $V(x) = x^T P x$

Conditions

- $P = P^T$
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- For every A that is Hurwitz (eigenvalues in the left half-plane):

$$\forall Q > 0 \quad \exists P > 0 \quad \text{s.t.} \quad A^T P + PA = -Q \text{ (Lyapunov equality)}$$

Definitions

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- $\mathrm{f(a)} = 0$

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Asymptotic stability

- $\forall \lambda = \mathrm{eig}(\mathrm{A})\ ,\ \mathrm{Re}(\lambda) < 0 \implies \mathrm{a}$ is **locally** asymptotically stable.

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Asymptotic Stability of Nonlinear Systems

Definitions

- $\dot{x} = f(x)$ where $f : D \to \mathbb{R}^n$
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Asymptotic stability

- $\forall\lambda = \mathrm{eig(A)}\ ,\ \mathrm{Re}(\lambda) < 0 \implies \mathrm{a}$ is **locally** asymptotically stable.
- $\exists\lambda = \mathrm{eig(A)}\ ,\ \mathrm{Re}(\lambda) > 0 \implies \mathrm{a}$ is **unstable**.

Question

- What if $\exists\lambda = \mathrm{eig(A)}$ s.t. $\mathrm{Re}(\lambda) = 0$?

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- Is $\dot{x} = f_\sigma(x)$ stable if each f_σ is stable?

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Example

$$\dot{x} = \begin{cases} A_1 x & x_1 x_2 < 0 \\ A_2 x & x_1 x_2 > 0 \end{cases}$$

where

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix} \quad \text{eigenvalues} = \{-1 \pm 31.6j\}$$

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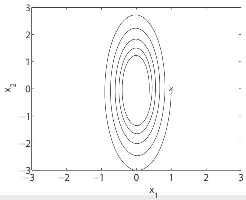
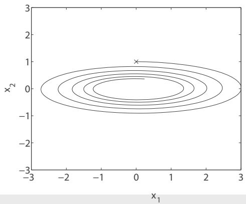


$A_1 x$



$A_2 x$

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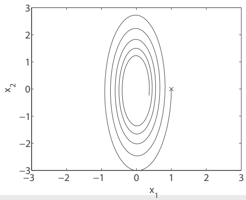

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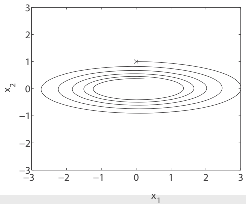

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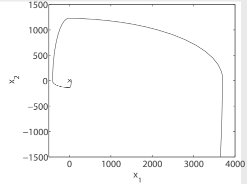

$\dot{x}$

Problem A: Global Uniform Asymptotic Stability

Find conditions of GAS for any switching signal

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GUAS Condition

- $V(0) = 0$
- $V(x) > 0 \qquad \forall x \neq 0$
- $\dot{V}(x) = L_{f_i} V(x) < 0 \qquad \forall x \neq 0 \,, \forall i \in \{1, 2, \ldots, N\}$

Converse Theorem

- if $\dot{x}$ is GUAS, then

$$\exists V : \mathbb{R}^n \to \mathbb{R}^+ \text{ s.t. } \dot{V}(x) = L_{f_i} V(x) < 0 \qquad \forall x \neq 0 \,, \forall i \in \{1, 2, \ldots, N\}$$

Problem A: Stability of Switched Linear Systems

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Stability Condition

- $\dot{V}(x) = L_{f_i} V(x) < 0 = x^T (A_i^T P + P A_i) x < 0 \qquad \forall x \neq 0, \forall i \in \{1, \ldots, N\}$

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- We need to find $P \succ 0$ s.t.

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which is a Linear Matrix Inequality (LMI).

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Quadratic Stability Conditions

- $\exists \epsilon > 0$ s.t. $\dot{V}(x) = L_{f_i} V(x) \leqslant -\epsilon |x|^2 \qquad \forall x \neq 0 \,, \forall i \in \{1, \ldots, N\}$

Problem A_{CQL}

- Find $P \succ 0$ s.t. $A_i^T P + PA_i < 0 \qquad \forall i \in \{1, \ldots, N\}$ (set of LMIs)

- We can also check if this problem is feasible

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Dual Theorem

- Problem $\mathrm{A_{CQL}}$ is infeasible if and only if

$$\exists R_i \succ 0 \text{ s.t. } \sum_{i=1}^{N} (A_i^T R_i + R_i A_i) > 0 \qquad \forall i \in \{1, \ldots, N\}$$

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- If problem A_{CQL} is infeasible, does this mean the system is unstable?

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- Now we have **a sufficient** condition for stability

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Example 4.4.6 in the lecture notes

$$\dot{x} = \begin{cases} A_1 x & x \in \mathcal{X}_1 \\ A_2 x & x \in \mathcal{X}_2 \end{cases} \quad \text{where} \quad A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \; A_2 = \begin{bmatrix} -1 & -10 \\ 0.1 & -1 \end{bmatrix}$$

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- Infeasibility test problem has the solution

$$R_1 = \begin{bmatrix} 0.2996 & 0.7048 \\ 0.7048 & 2.4704 \end{bmatrix} \; R_2 = \begin{bmatrix} 0.2123 & -0.5532 \\ -0.5532 & 1.9719 \end{bmatrix}$$

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- There is a **common ~~quadratic~~ Lyapunov function**

$$V(x) = \max_{i \in \{1,2,\ldots,k\}} (l_i^T x)^2$$

which is a **piece-wise quadratic** function

Explicit form for problem A_{CQL} [Theorem 4.4.4 in lecture notes]

Definitions

- $\dot{x} = A_\sigma x \;,\; \sigma \in \{1, \ldots, N\}$
- $V(x) = x^T P x \;,\; P \succ 0$

Conditions

- A_i is Hurwitz $\quad \forall i \in \{1, \ldots, N\}$
- $A_i A_j = A_j A_i \quad \forall i, j \in \{1, \ldots, N\}$
  (A_i commute pairwise)

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  - Step 3. Find $P_3 \succ 0$ s.t. $A_3^T P_3 + P_3 A_3 = -P_2$

  $\vdots$

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- The common Lyapunov function is found via this procedure:
  - Step 1. Find $P_1 \succ 0$ s.t. $A_1^T P_1 + P_1 A_1 = -I$
  - Step 2. Find $P_2 \succ 0$ s.t. $A_2^T P_2 + P_2 A_2 = -P_1$
  - Step 3. Find $P_3 \succ 0$ s.t. $A_3^T P_3 + P_3 A_3 = -P_2$

    $\vdots$
  - Step N. Find $P_N \succ 0$ s.t. $A_N^T P_N + P_N A_N = -P_{N-1}$

Problem A: Stability of Switched Linear Systems

Find conditions of GAS for any switching signal

Explicit form for problem A_{CQL} [Theorem 4.4.4 in lecture notes]

Definitions

- $\dot{x} = A_\sigma x \ , \ \sigma \in \{1, \dots, N\}$
- $V(x) = x^T P x \ , \ P \succ 0$

Conditions

- A_i is Hurwitz $\quad \forall i \in \{1, \dots, N\}$
- $A_i A_j = A_j A_i \quad \forall i, j \in \{1, \dots, N\}$ (A_i commute pairwise)

- The common Lyapunov function is found via this procedure:
  - Step 1. Find $P_1 \succ 0$ s.t. $A_1^T P_1 + P_1 A_1 = -I$
  - Step 2. Find $P_2 \succ 0$ s.t. $A_2^T P_2 + P_2 A_2 = -P_1$
  - Step 3. Find $P_3 \succ 0$ s.t. $A_3^T P_3 + P_3 A_3 = -P_2$

    ⋮
  - Step N. Find $P_N \succ 0$ s.t. $A_N^T P_N + P_N A_N = -P_{N-1}$
- Then $P = P_N$. [Proof: Exercise 4.4.5 in lecture notes]