4. Global asymptotic stability for *given* switching strategy?

Overview for Problem B: GAS for given switching strategy

1. multiple Lyapunov functions
2. state-dependent switching – single Lyapunov function
3. state-dependent switching – multiple Lyapunov function
4. piecewise linear systems – S-procedure

4.1 Multiple Lyapunov approach

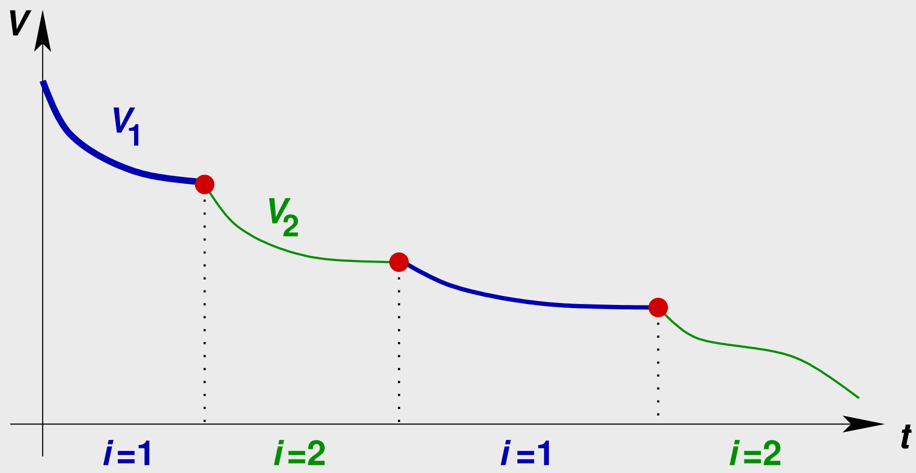
Switched system with $\dot{x} = f_i(x)$, $i = 1, 2$ are GAS with Lyapunov function $V_i(x)$

Assumption: no common Lyapunov function $\rightarrow$ not GUAS

Let switching times be given by t_k, $k = 0, 1, 2, \ldots$ and suppose that

$$V_{\sigma(t_{k-1})}(x(t_k)) = V_{\sigma(t_k)}(x(t_k)) \text{ for all } k = 1, 2, \ldots$$

V_σ is now continuous Lyapunov function $\Rightarrow$ switched system is GAS



4.2 Most general theorem

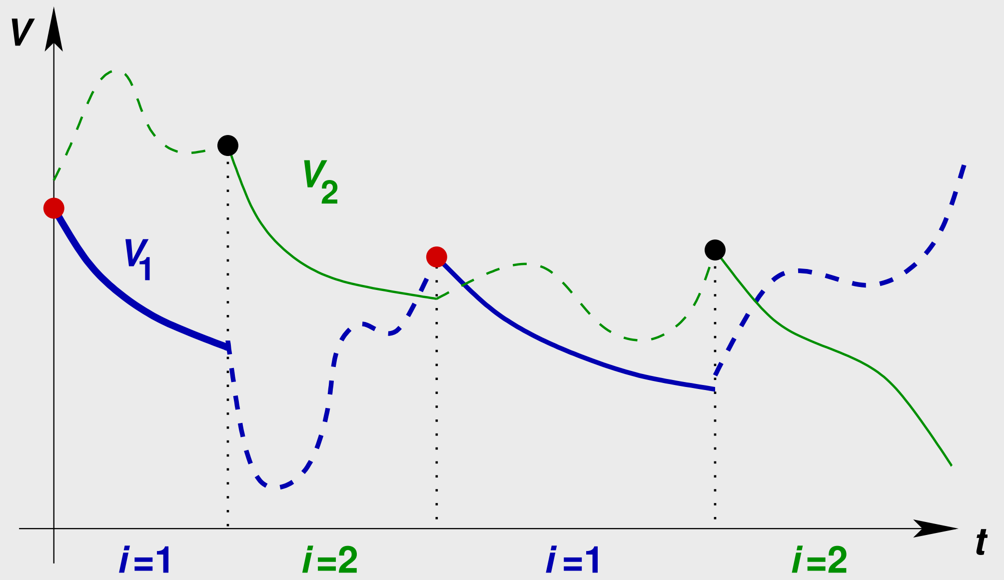
Theorem

Consider switched system with all submodels $\dot{x} = f_i(x)$ GAS with corresponding Lyapunov function V_i

Suppose that for every pair of switching times (t_k, t_l), $k < l$ with $\sigma(t_k) = \sigma(t_l) = i$ and $\sigma(t_m) \neq i$ for $t_k < t_m < t_l$, we have

$$V_i(x(t_l)) - V_i(x(t_k)) \leqslant -\rho(\|x(t_k)\|) < 0,$$

then switched system is GAS



4.3 State-dependent switchings: Single Lyapunov function

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 x_2 \leqslant 0 \\ A_2 x, & \text{if } x_1 x_2 > 0 \end{cases} \quad \text{with } A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}; \; A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

• No common quadratic Lyapunov function
• However, for $V(x) = x_1^2 + x_2^2$ it holds that $\dot{V} < 0$ along the nonzero solutions of the switched system, which implies GAS

Relaxation w.r.t. common Lyapunov function approach: Indeed, we only need

$$L_{A_1 x} V(x) < 0 \text{ if } x_1 x_2 \leqslant 0 \text{ and } L_{A_2 x} V(x) < 0 \text{ if } x_1 x_2 > 0$$

Hence, general set-up:

Find V such that $L_{f_i} V(x)$ is only negative where $\dot{x} = f_i(x)$ can be active

4.4 State-dependent switchings: Multiple Lyapunov function

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 \leqslant 0 \\ A_2 x, & \text{if } x_1 > 0, \end{cases} \quad \text{where } A_1 = \begin{pmatrix} -5 & -4 \\ -1 & -2 \end{pmatrix}; \ A_2 = \begin{pmatrix} -2 & -4 \\ 20 & -2 \end{pmatrix}$$

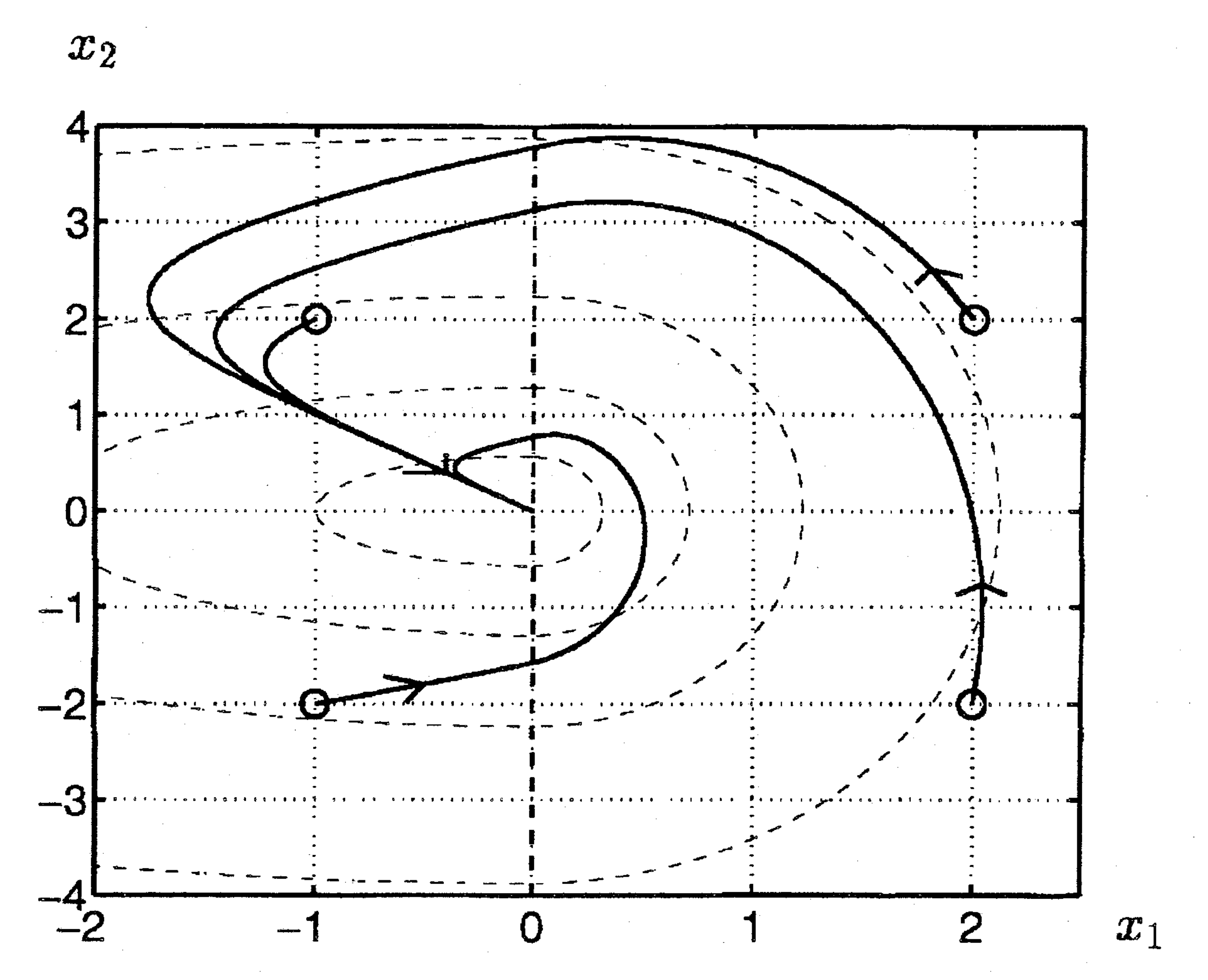
No common Lyapunov function and no quadratic function as in previous example

However, consider 2 quadratic Lyapunov functions $V_i(x) = x^\top P_i x$ with

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}$$

V_i is Lyapunov function for $\dot{x} = A_i x$

V_σ (with $\sigma = 1$ if $x_1 \leqslant 0$ and $\sigma = 2$ when $x_1 > 0$) is continuous and strictly decreasing


x_2
4
3
2
1
0
−1
−2
−3
−4
−2
−1
0
1
2
x_1

4.5 More general set-up for piecewise linear systems

$$\dot{x} = A_i x \text{ if } x \in \mathscr{X}_i$$

Several relaxations possible w.r.t. common *quadratic* Lyapunov function:

- One can require that derivative $L_{f_i(x)}V(x)$ of $V(x) = x^T P x$ is only negative in region where subsystem is active
- One can use multiple Lyapunov functions, say $V_i(x) = x^T P_i x$, for each submodel and "connect them" in a suitable way
- One can require that the Lyapunov function $V_i(x) = x^T P_i x$ is only positive definite in its active region

4.6 Relaxation: S-procedure

Aim: $V(x) = x^T P x$, $P > 0$ such that $x^T[A_i^T P + PA_i]x < 0$ for $0 \neq x \in \mathscr{X}_i$

Find: $S_i(x)$ based on $\mathscr{X}_i$ with $S_i(x) \geqslant 0$ when $x \in \mathscr{X}_i$

Next: search for $\beta \geqslant 0$ satisfying

$$x^T A_i^T P x + x^T P A_i x + \beta S_i(x) < 0 \text{ for all } x$$

Result: Since $S_i(x)$ might be negative outside $\mathscr{X}_i$, so less conservative than $A_i^T P + PA_i < 0$ (i.e., $x^T A_i^T P x + x^T P A_i x < 0$ for all x)

Computationally interesting: $S_i(x) = x^T S_i x$, then LMI:

$$\text{Find } \beta_i \geqslant 0 \text{ and } P > 0 \text{ such that } A_i^T P + PA_i + \beta_i S_i < 0$$

+ other relaxations (cf. lecture notes)

5. Summary

- Stability of submodels $\not\Rightarrow$ stability!
- Problem A: GAS for arbitrary switchings:
  - common Lyapunov function approach
  - piecewise linear: common quadratic Lyapunov function
- Problem B: GAS for specific switchings
  - multiple Lyapunov function: hard to verify in general case
  - state-dependent switching
    - decrease of Lyapunov function only in active region
    - multiple Lyapunov function (continuous over boundary)
  - Piecewise linear systems:
    - S-procedure: nice tool to get LMI