Modeling & Control of Hybrid Systems

Chapter 5 – Switched Control

Overview

1. Introduction & motivation for hybrid control
2. Stabilization of switched linear systems
3. Time-controlled switching & pulse width modulation
4. Sliding mode control
5. Stabilization by switching control
1. Introduction & motivation for hybrid control

Several “classical” control methods for continuous-time systems are hybrid:

- variable structure control
- sliding mode control
- relay control
- gain scheduling
- bang-bang time-optimal control
- fuzzy control

→ common characteristic: **switching**
1.1 Motivation for switched controllers

Theorem (Brockett’s necessary condition)

Consider system

\[ \dot{x} = f(x, u) \quad \text{with } x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ f(0, 0) = 0 \]

where \( f \) is smooth function

If system is asymptotically stabilizable (around \( x = 0 \)) using continuous feedback law \( u = \alpha(x) \),

then image of every open neighborhood of \((x, u) = (0, 0)\) under \( f \) contains open neighborhood of \( x = 0 \)
1.1 Motivation for switched controllers (continued)

- For non-holonomic integrator: \[
\begin{align*}
\dot{x} &= u \\
\dot{y} &= v \\
\dot{z} &= xv - yu
\end{align*}
\]
- Is asymptotically stabilizable (see later)
- Satisfies Brockett’s necessary condition?
  - if \( f_1 = f_2 = 0 \) then \( f_3 = 0 \)
  - hence, \((0, 0, \epsilon)\) cannot belong to image of \( f \) for any \( \epsilon \neq 0 \)
    \(\rightarrow\) image of open neighborhood of \((x, y, z; u, v) = (0, 0, 0; 0, 0)\)
    under \( f \) does \textit{not} contain open neighborhood of 
    \((x, y, z) = (0, 0, 0)\)
  - so non-holonomic integrator cannot be stabilized by \textit{continuous} feedback
  \(\rightarrow\) hybrid control schemes necessary to stabilize it!
1.2 Switching control/logic

controller 1

controller 2

controller $m-1$

controller $m$

supervisor

plant

$$u_1$$

$$u_2$$

$$u_{m-1}$$

$$u_m$$

$$y$$

$$u$$
1.2 Switching control/logic (continued)

\[ \Sigma \text{sup} \]

\[ \Sigma_{\text{ctrl}}(\sigma) \]

\[ \rightarrow \text{shared controller state variables} \]
1.2 Switching control/logic (continued)

Main problem: **Chattering** (i.e., very fast switching)

1. **Hysteresis switching logic**
   - let $h > 0$, let $\pi_\sigma$ be a performance criterion (to be minimized)
   - if supervisor changes value of $\sigma$ to $q$, then $\sigma$ is held *fixed* at $q$ until $\pi_p + h < \pi_q$ for some $p$
     $\rightarrow$ $\sigma$ is set equal to $p$
     $\Rightarrow$ threshold parameter $h > 0$ prevents infinitely fast switching
   - similar idea: **boundary layer** around switching surface in sliding mode control

2. **Dwell-time switching logic**
   once symbol $\sigma$ is chosen by supervisor it remains constant for at least $\tau > 0$ time units ($\tau$: “dwell time”)

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2. Stabilization of switched linear systems via suitable switching (Problem C)

\[ \dot{x} = A_i x, \quad i \in I := \{1, 2, \ldots, N\} \]

Find switching rule \( \sigma \) as function of time/state such that closed-loop system is asymptotically stable

2.1 Quadratic stabilization via *single* Lyapunov function

Select \( \sigma(x) : \mathbb{R}^n \to I := \{1, 2, \ldots, N\} \) such that closed-loop system has single quadratic Lyapunov function \( x^T P x \)

**One solution:** if some convex combination of \( A_i \) is stable:

\[ A := \sum \alpha_i A_i \quad (\alpha_i \geq 0, \sum \alpha_i = 1) \text{ is stable} \]

Select \( Q > 0 \) and let \( P > 0 \) be solution of \( A^T P + PA = -Q \)
Quadratic stabilization (continued)

• From $x^T(A^TP + PA)x = -x^TQx < 0$ it follows that

\[
\sum_i \alpha_i [x^T(A_i^TP + PA_i)x] < 0
\]

• For each $x$ there is at least one mode with $x^T(A_i^TP + PA_i)x < 0$ or stronger

\[
\bigcup_{i \in I} \{x \mid x^T(A_i^TP + PA_i)x \leq -\frac{1}{N}x^TQx\} = \mathbb{R}^n
\]

• Switching rule:

\[
i(x) := \arg \min x^T(A_i^TP + PA_i)x
\]

• Leads possibly to sliding modes. Alternative?
Alternative switching rule for quadratic stabilization

• Modified switching rule (based on hysteresis switching logic):
  – stay in mode $i$ as long as $x^T(A_i^T P + PA_i)x \leq -\frac{1}{2N} x^T Q x$
  – when bound reached, switch to a new mode $j$ that satisfies
    
    $$x^T(A_j^T P + PA_j)x \leq -\frac{1}{N} x^T Q x$$

• There is a lower bound on the duration in each mode!

• No conservatism for 2 modes (necessary & sufficient for this case):

  **Theorem:** If there exists a quadratically stabilizing state-dependent switching law for the switched linear system with $N = 2$, then matrices $A_1$ and $A_2$ have a stable convex combination
2.2 Stabilization via multiple Lyapunov functions (Problem C)

**Main idea:** Find function $V_i(x) = x^T P_i x$ that decreases for $\dot{x} = A_i x$ in some region

Define $\mathcal{X}_i := \{x \mid x^T [A_i^T P_i + P_i A_i] x < 0\}$

If $\bigcup_i \mathcal{X}_i = \mathbb{R}^n$, try to switch to satisfy multiple Lyapunov criterion to guarantee asymptotic stability.

**Find** $P_1$ and $P_2$ such that they satisfy the coupled conditions:

- $x^T (P_1 A_1 + A_1^T P_1) x < 0$ when $x^T (P_1 - P_2) x \geq 0$, $x \neq 0$

  and

- $x^T (P_2 A_2 + A_2^T P_2) x < 0$ when $x^T (P_2 - P_1) x \geq 0$, $x \neq 0$

Then $\sigma(t) = \arg \max\{V_i(x(t)) \mid i = 1, 2\}$ is stabilizing ($V_\sigma$ will be continuous)
2.3 S-procedure

**S-procedure** If there exist $\beta_1$, $\beta_2 \geq 0$ such that

$$-P_1 A_1 - A_1^T P_1 + \beta_1 (P_2 - P_1) > 0$$
$$-P_2 A_2 - A_2^T P_2 + \beta_2 (P_1 - P_2) > 0$$

then $\sigma(t) = \arg \max_i \{V_i(x(t)) \mid i = 1, 2\}$

→ only finds switching sequence (discrete inputs)!

What if also continuous inputs are present?
2.4 Stabilization of switched linear systems with continuous inputs

Switched linear system with inputs:

\[ \dot{x} = A_i x + B_i u, \quad i \in I = \{1, \ldots, N\} \]

Now both \( \sigma : [0, \infty) \rightarrow I \) and feedback controllers \( u = K_i x \) are to be determined.

**Case 1**: Determine \( K_i \) such that closed loop is stable under arbitrary switching (assuming we *know* mode)!

**Case 2**: Determine both \( \sigma : [0, \infty) \rightarrow I \) and \( K_i \)

**Case 3** (for PWL systems): Given \( \sigma \) as function of state, determine \( K_i \)
Case 1: Stabilization of switched linear system under arbitrary switching

\[ \dot{x} = A_i x + B_i u, \quad i \in I = \{1, \ldots, N\} \]

Sufficient condition: find common *quadratic* Lyapunov function \( V(x) = x^T P x \) for some positive definite matrix \( P \) and \( K_1, \ldots, K_N \)

\[
(A_i + B_i K_i)^T P + P (A_i + B_i K_i) < 0 \text{ for all } i = 1, \ldots, N \text{ and } P > 0
\]

→ LMIs (also for Cases 2 and 3)

→ state-based switching in this section, ... next ...
3. Time-controlled switching & pulse width modulation

If dynamical system switches between several subsystems → stability properties of total system may be quite different from those of subsystems

\[ T = 0, x = x_0 \]

\[
\begin{align*}
\text{mode 1} & : \\
\dot{x} &= A_1 x \\
\dot{T} &= 1 \\
T &\leq \frac{1}{2} \varepsilon \\
T &= 0
\end{align*}
\]

\[
\begin{align*}
\text{mode 2} & : \\
\dot{x} &= A_2 x \\
\dot{T} &= 1 \\
T &\geq \frac{1}{2} \varepsilon \\
T &\leq \frac{1}{2} \varepsilon
\end{align*}
\]
3.1 Time-controlled switching

- \( x(t_0 + 1/2\varepsilon) = \exp(1/2\varepsilon A_1)x_0 = x_0 + \frac{\varepsilon}{2}A_1x_0 + \frac{\varepsilon^2}{8}A_1^2x_0 + \cdots \)

\[
x(t_0 + \varepsilon) = \exp(1/2\varepsilon A_2)\exp(1/2\varepsilon A_1)x_0
\]

\[
= (I + \frac{\varepsilon}{2}A_2 + \frac{\varepsilon^2}{8}A_2^2 + \cdots)(I + \frac{\varepsilon}{2}A_1 + \frac{\varepsilon^2}{8}A_1^2 + \cdots)x_0
\]

\[
= (I + \varepsilon[\frac{1}{2}A_1 + \frac{1}{2}A_2] + \frac{\varepsilon^2}{8}[A_1^2 + A_2^2 + 2A_2A_1] + \cdots)x_0.
\]

- Compare with

\[
\exp[\varepsilon(\frac{1}{2}A_1 + \frac{1}{2}A_2)] = I + \varepsilon[\frac{1}{2}A_1 + \frac{1}{2}A_2] + \frac{\varepsilon^2}{8}[A_1^2 + A_2^2 + A_1A_2 + A_2A_1] + \cdots
\]

\( \rightarrow \) same for \( \varepsilon \approx 0 \)

- So for \( \varepsilon \to 0 \) solution of switched system tends to solution of

\[
\dot{x} = (\frac{1}{2}A_1 + \frac{1}{2}A_2)x \quad \text{("averaged" system)}
\]

- Possible that \( A_1, A_2 \) stable, whereas \( \frac{1}{2}A_1 + \frac{1}{2}A_2 \) unstable, or vice versa
Example

\[ T = 0, \ x = x_0 \]

\[ \dot{x} = A_1 x \]
\[ \dot{T} = 1 \]
\[ T \leq \frac{1}{2} \varepsilon \]
\[ T := 0 \]

\[ \dot{x} = A_2 x \]
\[ \dot{T} = 1 \]
\[ T \geq \frac{1}{2} \varepsilon \]

\[ T := 0 \]

- Consider

\[ A_1 = \begin{bmatrix} -0.5 & 1 \\ 100 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -100 \\ -0.5 & -1 \end{bmatrix} \]

- \( A_1, A_2 \) unstable, but matrix \( \frac{1}{2}(A_1 + A_2) \) is stable
  → switched system should be stable if frequency of switching is sufficiently high

- Minimal switching frequency found by computing eigenvalues of the mapping \( \exp\left(\frac{1}{2} \varepsilon A_2\right) \exp\left(\frac{1}{2} \varepsilon A_1\right) \) (Why?)
Example (continued)

\[ \rightarrow \text{maximal value of } \varepsilon: 0.04 (=50 \text{ Hz}) \]
Example (continued)

for $\varepsilon = 0.02$
3.2 Pulse width modulation

- Assume mode 1 followed during $h\varepsilon$, and mode 2 during $(1 - h)\varepsilon$
  → behavior of system is well approximated by system

$$\dot{x} = (hA_1 + (1 - h)A_2)x$$

- Parameter $h$ might be considered as control input

- If $h$ varies, should be on time scale that is much slower than the time scale of switching

- If mode 1 is “power on” and mode 2 is “power off”, then $h$ is known as duty ratio

- Power electronics: fast switching theoretically provides possibility to regulate power without loss of energy
  → used in power converters (e.g., Boost converter)
3.2 Pulse width modulation (continued)

- System: \( \dot{x} = f(x, u), \quad u \in \{0, 1\} \)
- Duty cycle: \( \Delta \) (fixed)
- \( u \) is switched exactly one time from 1 to 0 in each cycle
- Duty ratio \( \alpha \): fraction of duty cycle for which \( u = 1 \)

\[
\begin{align*}
  u(\tau) &= 1 \quad \text{for} \ t \leq \tau < t + \alpha \Delta \\
  u(\tau) &= 0 \quad \text{for} \ t + \alpha \Delta \leq \tau < t + \Delta
\end{align*}
\]

- Hence, \( x(t + \Delta) = x(t) + \int_t^{t+\alpha\Delta} f(x(\tau), 1) d\tau + \int_{t+\alpha\Delta}^{t+\Delta} f(x(\tau), 0) d\tau \)

- Ideal averaged model (\( \Delta \to 0 \)):

\[
\dot{x}(t) = \lim_{\Delta \to 0} \frac{x(t + \Delta) - x(t)}{\Delta} = \alpha f(x(t), 1) + (1 - \alpha) f(x(t), 0)
\]

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4. Sliding mode control

- Consider $\dot{x}(t) = f(x(t), u(t))$ with $u$ scalar
- Suppose switching feedback control scheme:
  \[ u(t) = \begin{cases} 
  \phi_+(x(t)) & \text{if } h(x(t)) > 0 \\
  \phi_-(x(t)) & \text{if } h(x(t)) < 0 
  \end{cases} \]

- Surface $\{x \mid h(x) = 0\}$ is called *switching surface*

- Let $f_+(x) = f(x, \phi_+(x))$ and $f_-(x) = f(x, \phi_-(x))$, then
  \[ \dot{x} = \frac{1}{2}(1 + v)f_+(x) + \frac{1}{2}(1 - v)f_-(x), \quad v = \text{sgn}(h(x)) \]

- Use solutions in Filippov’s sense if “chattering”
4. Sliding mode control (continued)

- Assume “desired behavior” whenever constraint $s(x) = 0$ is satisfied
- Set $\{x \mid s(x) = 0\}$ is called *sliding surface*
- Find control law $u$ such that

  \[ \frac{1}{2} \frac{d}{dt} s^2 \leq -\alpha |s| \]

  where $\alpha > 0$

  \[ \rightarrow \text{squared “distance” to sliding surface decreases along all system trajectories} \]
Properties of sliding mode control

- Quick succession of switches may occur
  - increased wear, high-frequency vibrations
  - embed sliding surface in thin boundary layer
    - smoothen discontinuity by replacing sgn by steep sigmoid function
    - Note: modifications may deteriorate performance of closed-loop system

- Main advantages of sliding mode control:
  - conceptually simple
  - robustness w.r.t. uncertainty in system data

- Possible disadvantage:
  - excitation of unmodeled high-frequency modes
5. Stabilization by switching control

- For multi-model *linear* systems
  → use techniques for quadratic stabilization using single or multiple Lyapunov function

- Stabilization of non-holonomic systems using hybrid feedback control (e.g., non-holonomic integrator)
  → rather ad hoc
     not structured
     complicated analysis and proofs
Stabilization of non-holonomic integrator

• System: \( \dot{x} = u, \quad \dot{y} = v, \quad \dot{z} = xv - yu \)

• Sliding mode control: \( u = -x + y \text{sgn}(z) \)
  \( v = -y - x \text{sgn}(z) \)

• Switching surface: \( z = 0 \)

• Lyapunov function for \((x, y)\) subspace: \( V(x, y) = \frac{1}{2}(x^2 + y^2) \)
  \( \Rightarrow \dot{V} = -x^2 + xy \text{sgn}(z) - y^2 - xy \text{sgn}(z) = -(x^2 + y^2) = -2V \)
  \( \Rightarrow x, y \to 0 \)

• \( \dot{z} = xv - yu = -(x^2 + y^2) \text{sgn}(z) = -2V \text{sgn}(z) \)
  So \(|z|\) will decrease and reach \(0\) provided that
  \[ 2 \int_{0}^{\infty} V(\tau) d\tau > |z(0)| \]
  \( \to z \) will reach \(0\) in finite time
Stabilization of non-holonomic integrator (continued)

- Since \( V(t) = V(0)e^{-2t} = \frac{1}{2}(x^2(0) + y^2(0))e^{-2t} \)
  condition for system to be asymptotically stable is
  \[
  \frac{1}{2}(x^2(0) + y^2(0)) \geq |z(0)|
  \]
  \[\rightarrow\] defines parabolic region \( \mathcal{P} = \{(x, y, z) \mid 0.5(x^2 + y^2) \leq |z|\} \)

- If initial conditions do not belong to \( \mathcal{P} \) then sliding mode control asymptotically stabilizes system

- If initial state is inside \( \mathcal{P} \):
  - first use control law (e.g., nonzero constant control) to steer system outside \( \mathcal{P} \)
  - then use sliding mode control
  \[\rightarrow\] hybrid control scheme
6. Summary

- Problem C: stabilization $\rightarrow$ construct switching signal $\sigma$
  - single Lyapunov function $\rightarrow$ find convex combination that is stable
  - multiple Lyapunov functions $\rightarrow$ “max”-switching law, S-procedure
  - with continuous inputs $\rightarrow$ also find state feedback ($K_i$) $\rightarrow$ LMIs
- Pulse width modulation
- Sliding mode control
- Stabilization of non-holonomic integrator