Modeling & Control of Hybrid Systems

Chapter 1 — Introduction

Overview

1. Hybrid automata
2. Examples of hybrid systems
3. Examples with Zeno behavior

1. Hybrid automata

"System"

- Example:

$$\dot{x}(t) = f(x(t), u(t))$$

  t: time
  x: state
  u: input

- More formal definition:

$$x(\sigma) = \phi(\tau, \sigma, x(\tau), u)$$

  τ: initial time
  σ: current time
  u: input function (over $[\tau, \sigma]$)
  ϕ: transition map

Classification

- Continuous-state / discrete-state
- Continuous-time / discrete-time
- Time-driven / event-driven
  - time-driven → state changes as time progresses, i.e., continuously (for CT), or at every tick of clock (for DT)
  - event-driven → state changes due to occurrence of event
    event:
    - start or end of an activity
    - asynchronous (occurrence times not necessarily equidistant)

Combinations → “hybrid”

Models for time-driven systems

- Continuous-time time-driven systems:

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$

- Discrete-time (or sampled) time-driven systems:

$$x(k+1) = f(x(k), u(k))$$
$$y(k) = g(x(k), u(k))$$

Models for event-driven systems

Automaton

Automaton is defined by triple $\Sigma = (\mathcal{Q}, \mathcal{U}, \phi)$ with

- $\mathcal{Q}$: finite or countable set of discrete states
- $\mathcal{U}$: finite or countable set of discrete inputs (“input alphabet”)
- $\phi : \mathcal{Q} \times \mathcal{U} \to P(\mathcal{Q})$: partial *transition function*.

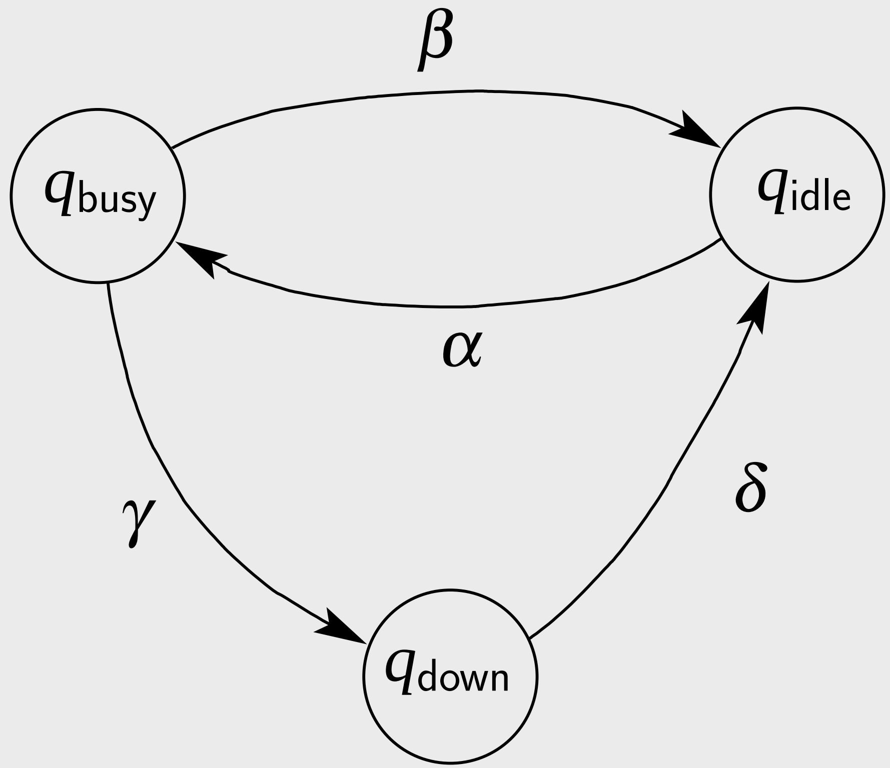
where $P(\mathcal{Q})$ is power set of $\mathcal{Q}$ (set of all subsets)

Finite automaton: $\mathcal{Q}$ and $\mathcal{U}$ finite

Evolution of automaton

- Given state $q \in \mathscr{Q}$ and discrete input symbol $u \in \mathscr{U}$, transition function ϕ defines collection of next possible states: $\phi(q,u) \subseteq \mathscr{Q}$
- If each set of next states has 0 or 1 element: $\longrightarrow$ “deterministic” automaton
- If some set of next states has more than 1 element: $\longrightarrow$ “non-deterministic” automaton

Deterministic automaton



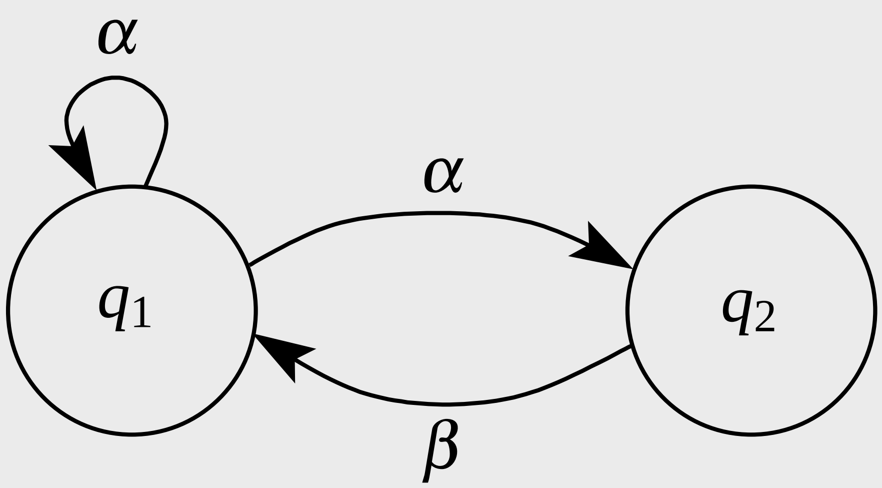
$\phi(q_{\text{busy}}, \beta) = \{q_{\text{idle}}\}$

$\phi(q_{\text{busy}}, \gamma) = \{q_{\text{down}}\}$

$\phi(q_{\text{idle}}, \alpha) = \{q_{\text{busy}}\}$

$\phi(q_{\text{down}}, \delta) = \{q_{\text{idle}}\}$

Non-deterministic automaton

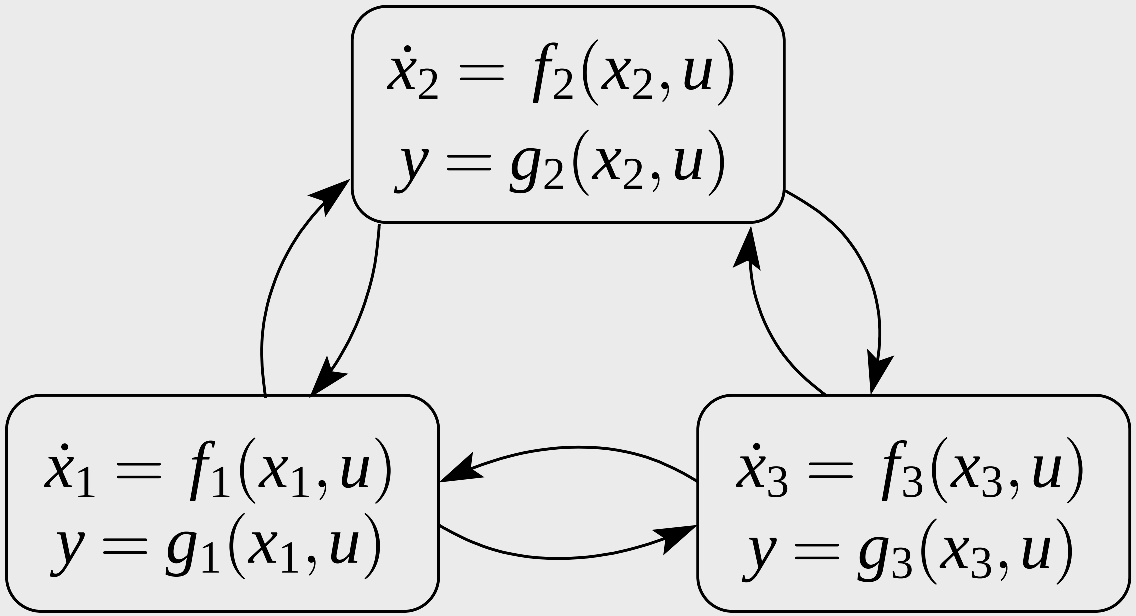


$$\phi(q_1, \alpha) = \{q_1, q_2\} \qquad \phi(q_2, \beta) = \{q_1\}$$

→ unmodeled dynamics

Hybrid system

- System can be in one of several modes
- In each mode: behavior described by system of difference or differential equations
- Mode switches due to occurrence of “events”



Hybrid system

- At switching time instant:
  $\longrightarrow$ possible state reset or state dimension change
- Mode transitions may be caused by
  - external control signal
  - internal control signal
  - dynamics of system itself (crossing of boundary in state space)

Models for hybrid systems

☞ • timed or hybrid Petri nets
- differential automata

☞ • hybrid automata
- Brockett's model

☞ • mixed logical dynamic models
- real-time temporal logics
- timed communicating sequential processes
- switched bond graphs
- predicate calculus

☞ • piecewise-affine models
- ...

Analysis techniques:

- formal verification
- computer simulation
- analytic techniques (for special subclasses)
- ...

$\Rightarrow$ no general modeling & analysis framework

modeling power $\leftrightarrow$ decision power

\+ computational complexity (NP-hard, undecidable)

$\Rightarrow$ special subclasses (Chapter 2)
hierarchical / modular approach

- **Undecidable problems**

  $\longrightarrow$ no algorithm at all can be given for solving the problem in general, i.e., no finite termination can be guaranteed

- **NP-complete and NP-hard problems**
  - *decision problem*: solution is either “yes” or “no”

    e.g., traveling salesman decision problem:

    Given a network of cities, intercity distances, and a number B, does there exist a tour with length $\leqslant B$?
  - *search problem*

    e.g., traveling salesman problem:

    Given a network of cities, intercity distances, what is the shortest tour?

P and NP-complete decision problems

- time complexity function $T(n)$: largest amount of time needed to solve problem instance of size n (*worst case!*)

- polynomial time algorithm:

$$T(n) \leqslant |p(n)| \qquad \text{for some polynomial } p$$

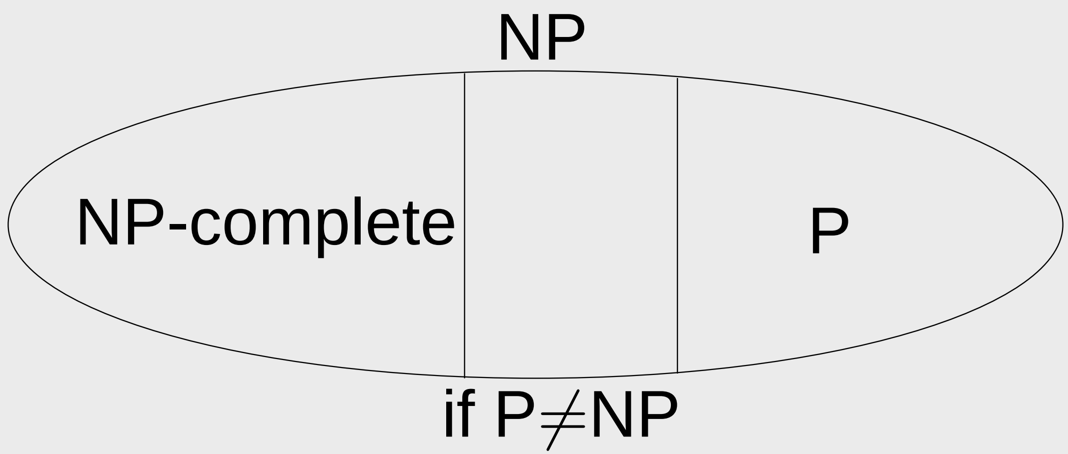
$\longrightarrow$ class P: solvable by polynomial time algorithm

- nondeterministic computer:
  - guessing stage (tour)
  - checking stage (compute length of tour + compare it with B)

$\longrightarrow$ class NP: "*nondeterministically polynomial*"
i.e., time complexity of checking stage is polynomial

P and NP-complete decision problems

- Each problem in NP can be solved in exponential time: $T(n) \leqslant 2^{n^k}$
- NP-complete problems: "hardest" class in NP:
  - any NP-complete problem solvable in polynomial time $\Rightarrow$ every problem in NP solvable in polynomial time
  - any problem in NP intractable $\Rightarrow$ NP-complete problems also intractable



NP-hard problems

- decision problem is NP-complete $\Rightarrow$ search problem is NP-hard
- NP-hard problems: at least as hard as NP-complete problems
  - NP-complete (decision problem)

    $\longrightarrow$ solvable in polynomial time *if and only if* P = NP
  - NP-hard (search problem)

    $\longrightarrow$ cannot be solved in polynomial time *unless* P = NP

Examples of NP-hard and undecidable problems

- Consider simple hybrid system:

$$x(k+1) = \begin{cases} A_1 x(k) & \text{if } c^{\mathsf{T}} x(k) \geqslant 0 \\ A_2 x(k) & \text{if } c^{\mathsf{T}} x(k) < 0 \end{cases}$$

  $\longrightarrow$ deciding whether system is stable or not is NP-hard

- Given two Petri nets, do they have the same reachability set?
  $\longrightarrow$ undecidable

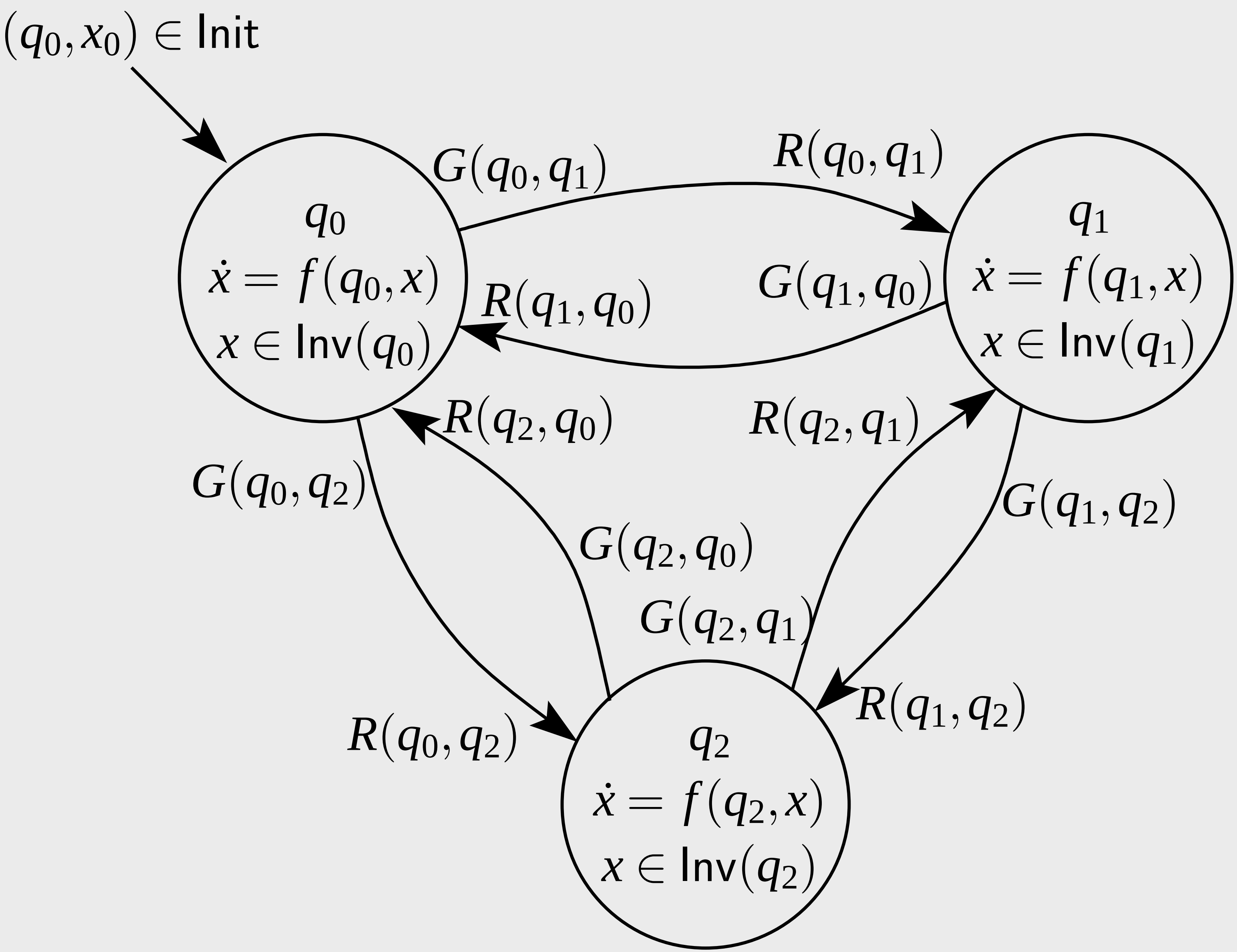
Hybrid automaton

Hybrid automaton H is collection $H = (Q, X, f, \mathsf{Init}, \mathsf{Inv}, E, G, R)$ where

- $Q = \{q_1, \ldots, q_N\}$ is finite set of discrete states or *modes*
- $X = \mathbb{R}^n$ is set of continuous states
- $f : Q \times X \to X$ is vector field
- $\mathsf{Init} \subseteq Q \times X$ is set of initial states
- $\mathsf{Inv} : Q \to P(X)$ describes *invariants*
- $E \subseteq Q \times Q$ is set of edges or *transitions*
- $G : E \to P(X)$ is *guard condition*
- $R : E \to P(X \times X)$ is *reset map*

Hybrid automaton $H = (Q, X, f, \mathsf{Init}, \mathsf{Inv}, E, G, R)$

- Hybrid state: (q, x)
- Evolution of continuous state in mode q: $\dot{x} = f(q, x)$
- Invariant Inv: describes conditions that continuous state has to satisfy in given mode
- Guard G: specifies subset of state space where certain transition is enabled
- Reset map R: specifies how new continuous states are related to previous continuous states


$(q_0, x_0) \in \mathsf{Init}$
q_0
$\dot{x} = f(q_0, x)$
$x \in \mathsf{Inv}(q_0)$
q_1
$\dot{x} = f(q_1, x)$
$x \in \mathsf{Inv}(q_1)$
q_2
$\dot{x} = f(q_2, x)$
$x \in \mathsf{Inv}(q_2)$
$G(q_0, q_1)$
$R(q_0, q_1)$
$G(q_1, q_0)$
$R(q_1, q_0)$
$R(q_2, q_0)$
$R(q_2, q_1)$
$G(q_0, q_2)$
$G(q_1, q_2)$
$G(q_2, q_0)$
$G(q_2, q_1)$
$R(q_0, q_2)$
$R(q_1, q_2)$

Evolution of hybrid automaton

- Initial hybrid state $(q_0, x_0) \in \mathsf{Init}$
- Continuous state x evolves according to

$$\dot{x} = f(q_0, x) \quad \text{with } x(0) = x_0$$

  discrete state q remains constant: $q(t) = q_0$

- Continuous evolution can go on as long as $x \in \mathsf{Inv}(q_0)$
- If at some point state x reaches guard $G(q_0, q_1)$, then
  - transition $q_0 \to q_1$ is enabled
  - discrete state *may* change to q_1, continuous state then jumps from current value x^- to new value x^+ with $(x^-, x^+) \in R(q_0, q_1)$
- Next, continuous evolution resumes and whole process is repeated

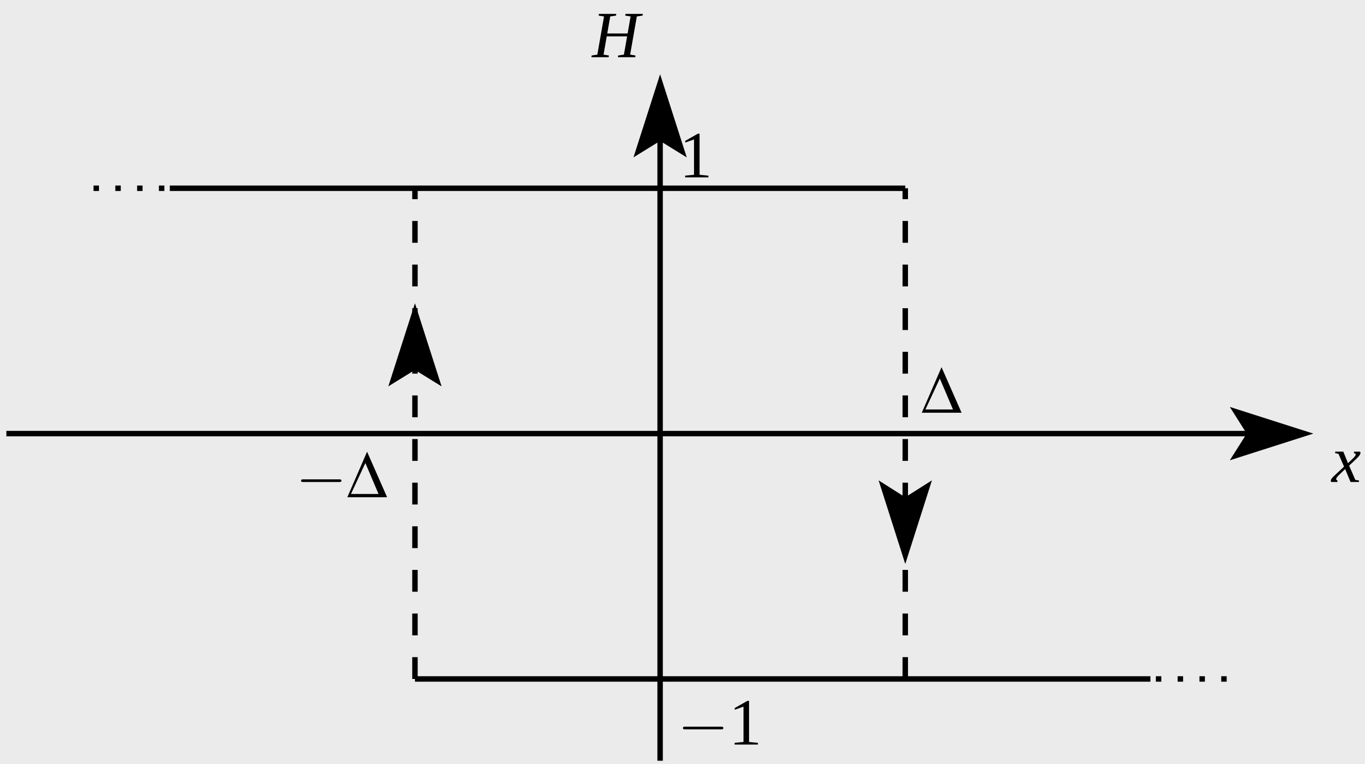
2. Examples of hybrid systems

1. Hysteresis
2. Manual transmission
3. Water-level monitor
4. Supervisor
5. Two-carts system
6. Boost converter

2.1 Hysteresis

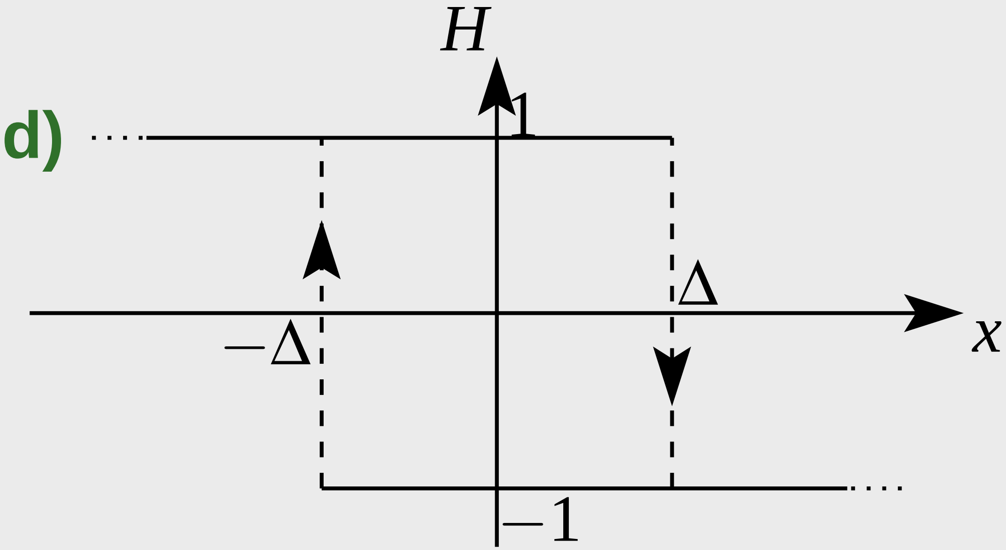
Control system with hysteresis element in the feedback loop :

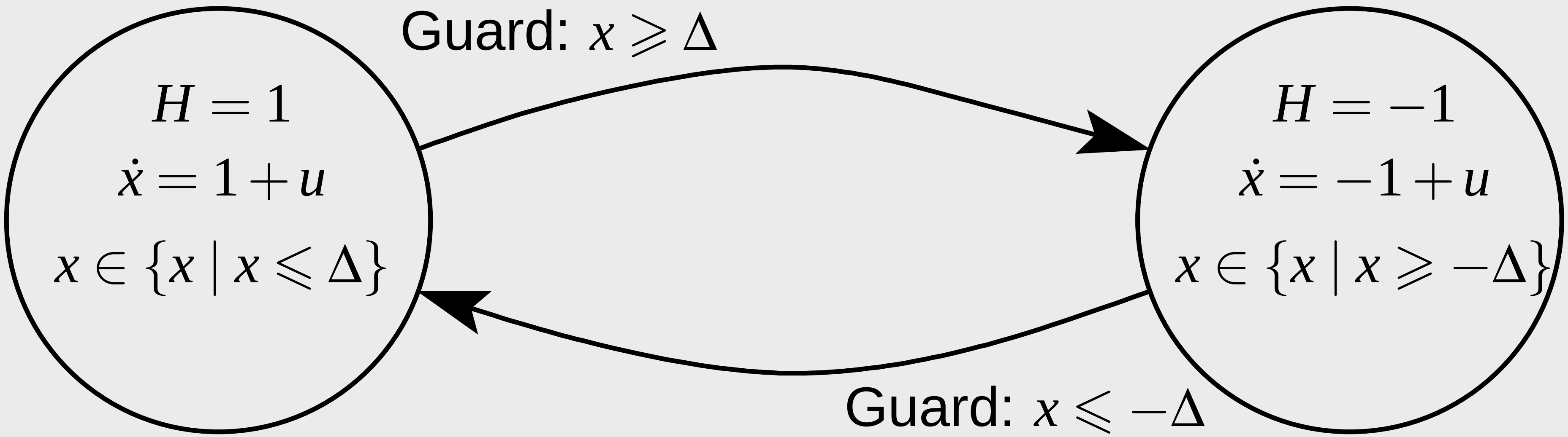
$$\dot{x} = H(x) + u$$



2.1 Hysteresis (continued)

$$\dot{x} = H(x) + u$$





2.2 Manual transmission

Simple model of manual transmission

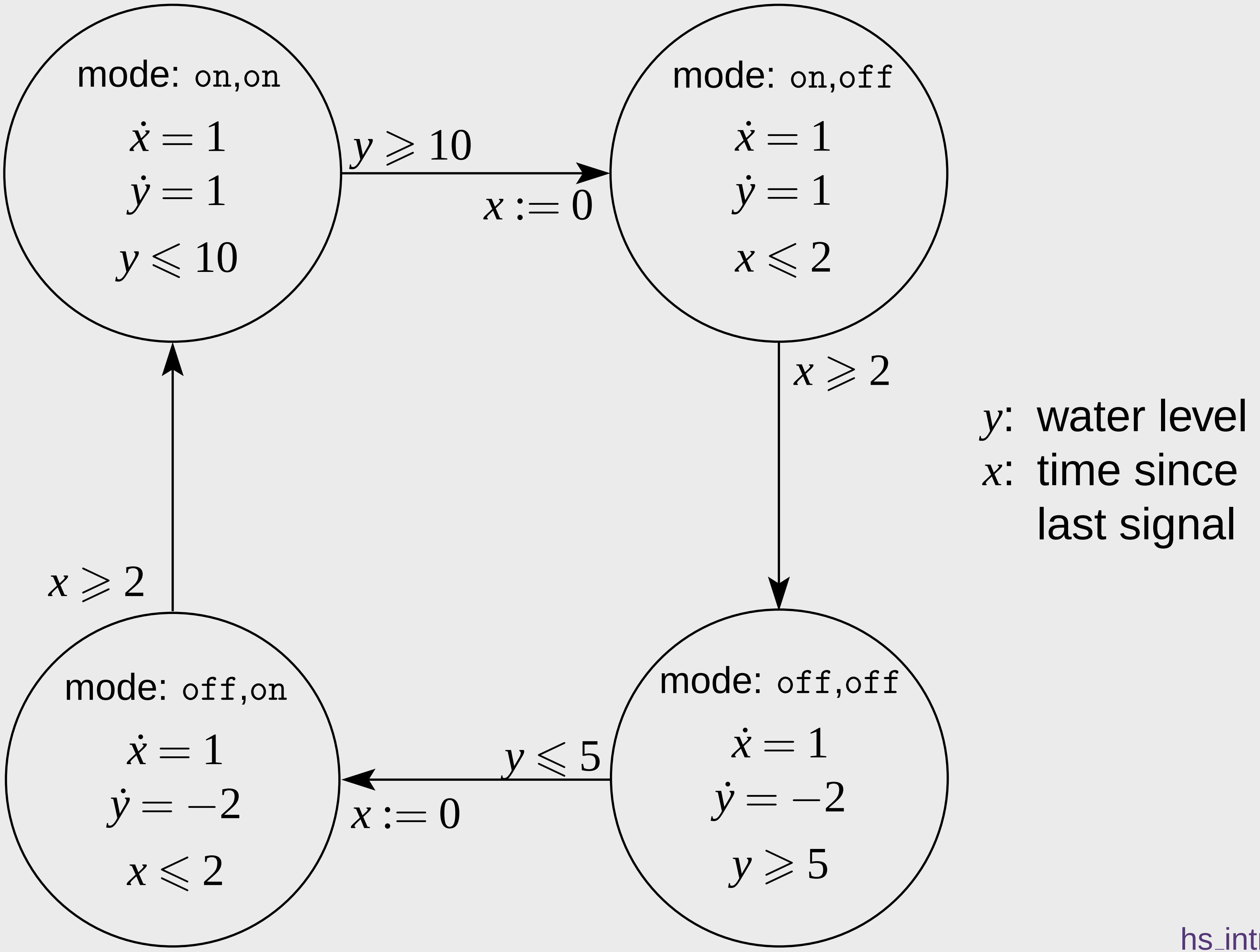
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{-a x_2 + u}{1 + v}$$

with v: gear shift position $v \in \{1, 2, 3, 4\}$
u: acceleration
a: parameter

$\rightarrow$ hybrid system with four modes, 2-dimensional continuous state, controlled transitions (switchings), and no resets

2.3 Water-level monitor

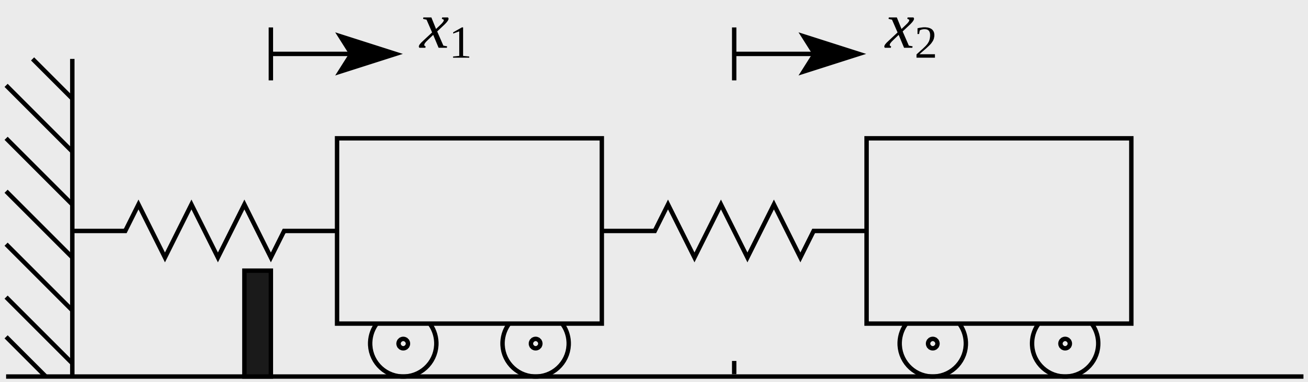
- variables:
  - $y(t)$: water level, continuous
  - $x(t)$: time elapsed since last signal was sent by monitor, continuous
  - $P(t)$: status of pump, $\in \{\mathtt{on}, \mathtt{off}\}$
  - $S(t)$: nature of signal last sent by monitor, $\in \{\mathtt{on}, \mathtt{off}\}$
- dynamics of system:
  - water level rises 1 unit per second when pump is on and falls 2 units per second when pump is off
  - when water level rises to 10 units, monitor sends switch-off signal; after delay of 2 seconds pump turns off
  - when water level falls to 5 units, monitor sends switch-on signal; after delay of 2 seconds pump switches on



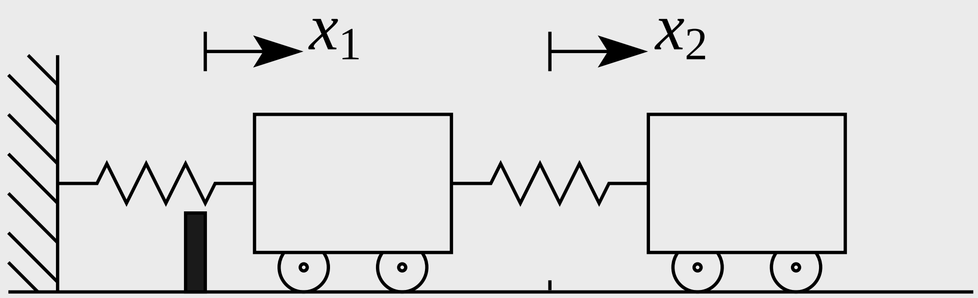
mode: on,on
$\dot{x} = 1$
$\dot{y} = 1$
$y \leqslant 10$
$y \geqslant 10$
$x := 0$
mode: on,off
$\dot{x} = 1$
$\dot{y} = 1$
$x \leqslant 2$
$x \geqslant 2$
mode: off,off
$\dot{x} = 1$
$\dot{y} = -2$
$y \geqslant 5$
$y \leqslant 5$
$x := 0$
mode: off,on
$\dot{x} = 1$
$\dot{y} = -2$
$x \leqslant 2$
$x \geqslant 2$
y: water level
x: time since last signal

2.4 Two-carts system

- Two carts connected by spring
- Left cart attached to wall by spring;
  motion constrained by completely inelastic stop
  Stop is placed at equilibrium position of left cart
- Masses of carts and spring constants = 1



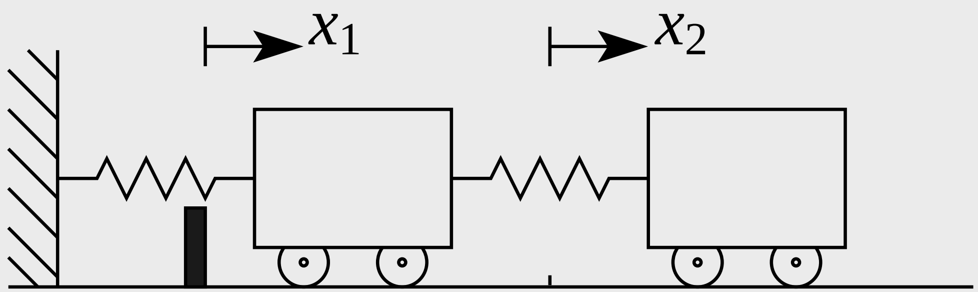
2.4 Two-carts system (continued)



- x_1, x_2: deviations of left and right cart from equilibrium position
- x_3, x_4: velocities of left and right cart
- z: reaction force exerted by stop
- Evolution:

$$\begin{aligned}
\dot{x}_1(t) &= x_3(t)\\
\dot{x}_2(t) &= x_4(t)\\
\dot{x}_3(t) &= -2x_1(t) + x_2(t) + z(t)\\
\dot{x}_4(t) &= x_1(t) - x_2(t)
\end{aligned}$$

2.4 Two-carts system (continued)



To model stop:

- define $w(t) = x_1(t)$
- $w(t) \geqslant 0$ (since w is position of left cart w.r.t. stop)
- force exerted by stop can act only in positive direction $\rightarrow z(t) \geqslant 0$
- if left cart not at stop $(w(t) > 0)$, reaction force vanishes: $z(t) = 0$
- if $z(t) > 0$ then cart must necessarily be at the stop: $w(t) = 0$

$$0 \leqslant w(t) \perp z(t) \geqslant 0$$

2.4 Two-carts system (continued)

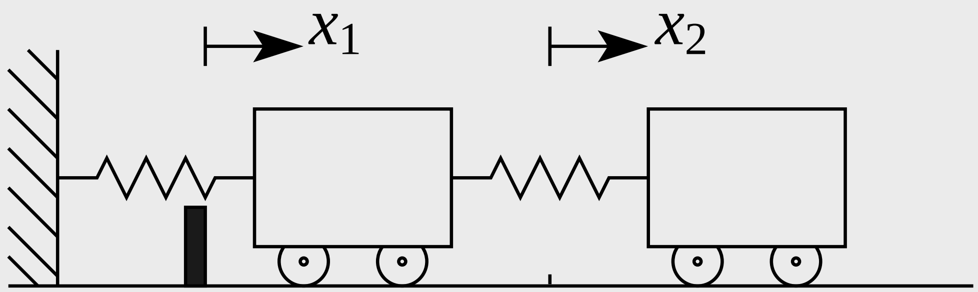
System can be represented by two modes (stop active or not)

$\boxed{z=0}$ unconstrained	constrained $\boxed{w=0}$
$\dot{x}_1(t) = x_3(t)$	$\dot{x}_1(t) = x_3(t)$
$\dot{x}_2(t) = x_4(t)$	$\dot{x}_2(t) = x_4(t)$
$\dot{x}_3(t) = -2x_1(t) + x_2(t)$	$\dot{x}_3(t) = -2x_1(t) + x_2(t) + z(t)$
$\dot{x}_4(t) = x_1(t) - x_2(t)$	$\dot{x}_4(t) = x_1(t) - x_2(t)$
$z(t) = 0$	$w(t) = x_1(t) = 0$
ODE (in state)	DAE (as z is not explicit)

System stays in mode as long as

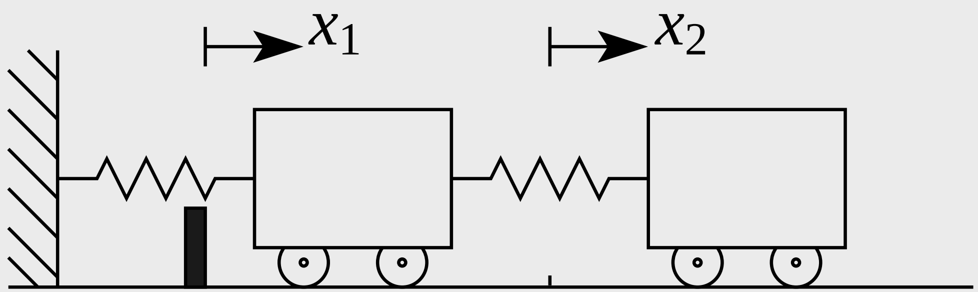
unconstrained	constrained
$z(t) = 0,\ w(t) > 0$	$w(t) = 0,\ z(t) > 0$

Mode transitions for two-carts system



- **Unconstrained → constrained**
  Suppose $x(\tau) = (0^+, -1, -1, 0)^\mathsf{T} \rightarrow w(t) > 0$ tends to be violated
  Left cart hits stop and stays there. Velocity of left cart is reduced to zero instantaneously (purely inelastic collision)

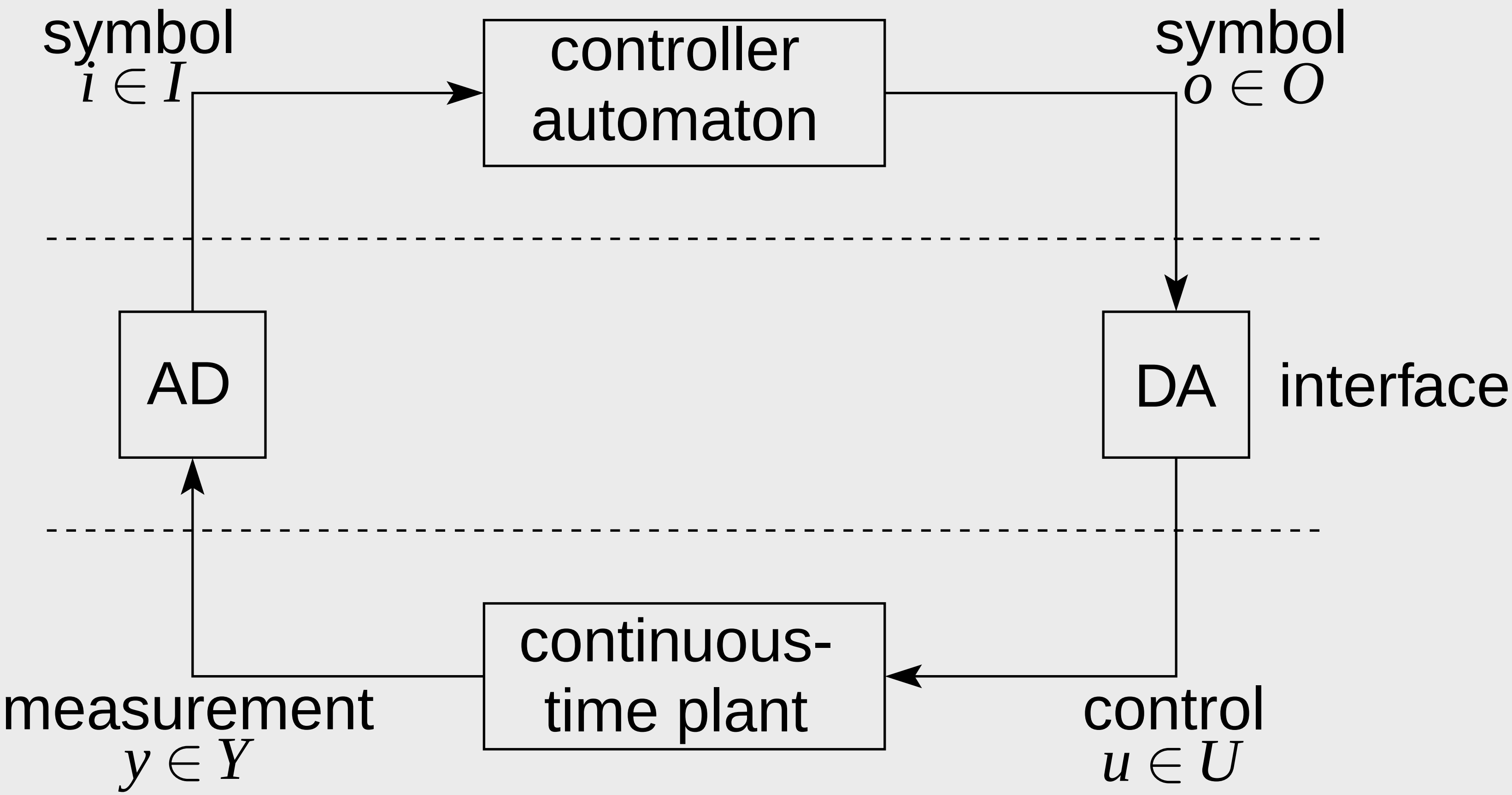
- **Constrained → unconstrained**
  Suppose $x(\tau) = (0, 0, 0, 1)^\mathsf{T} \rightarrow z(t) > 0$ tends to be violated
  Right cart is moving to right of its equilibrium position, so spring between carts pulls left cart away from stop



- **Unconstrained → unconstrained with re-initialization according to constrained mode**
  Consider $x(\tau) = (0^+, 1, -1, 0)^\mathsf{T} \to w(t) > 0$ tends to be violated
  At impact, velocity of left cart is reduced to 0, i.e., state reset to $(0, 1, 0, 0)^\mathsf{T}$
  Right cart is at right of its equilibrium position, pulls left cart away from stop → smooth continuation in unconstrained mode

  So: After the reset, no smooth continuation is possible in constrained mode → second mode change, back to unconstrained mode

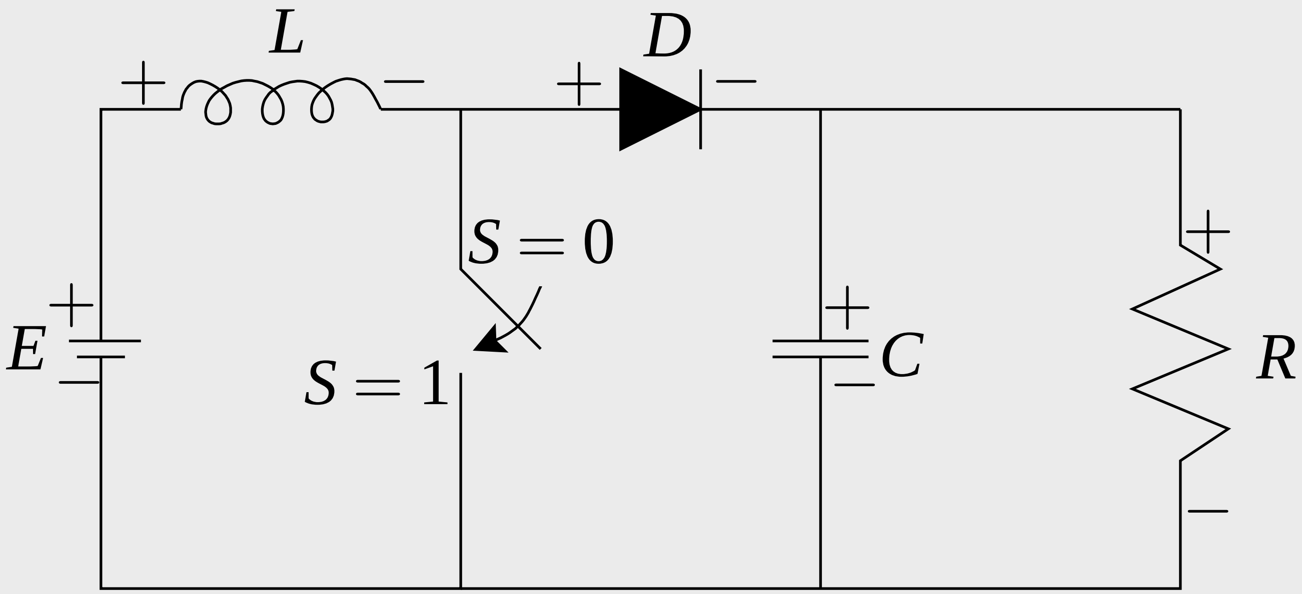
2.5 Supervisor model

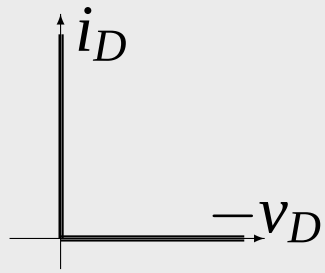


Controller is input-output automaton:

$$\begin{aligned} q^{\#} &= \nu(q,i) \\ o &= \eta(q,i) \end{aligned}$$

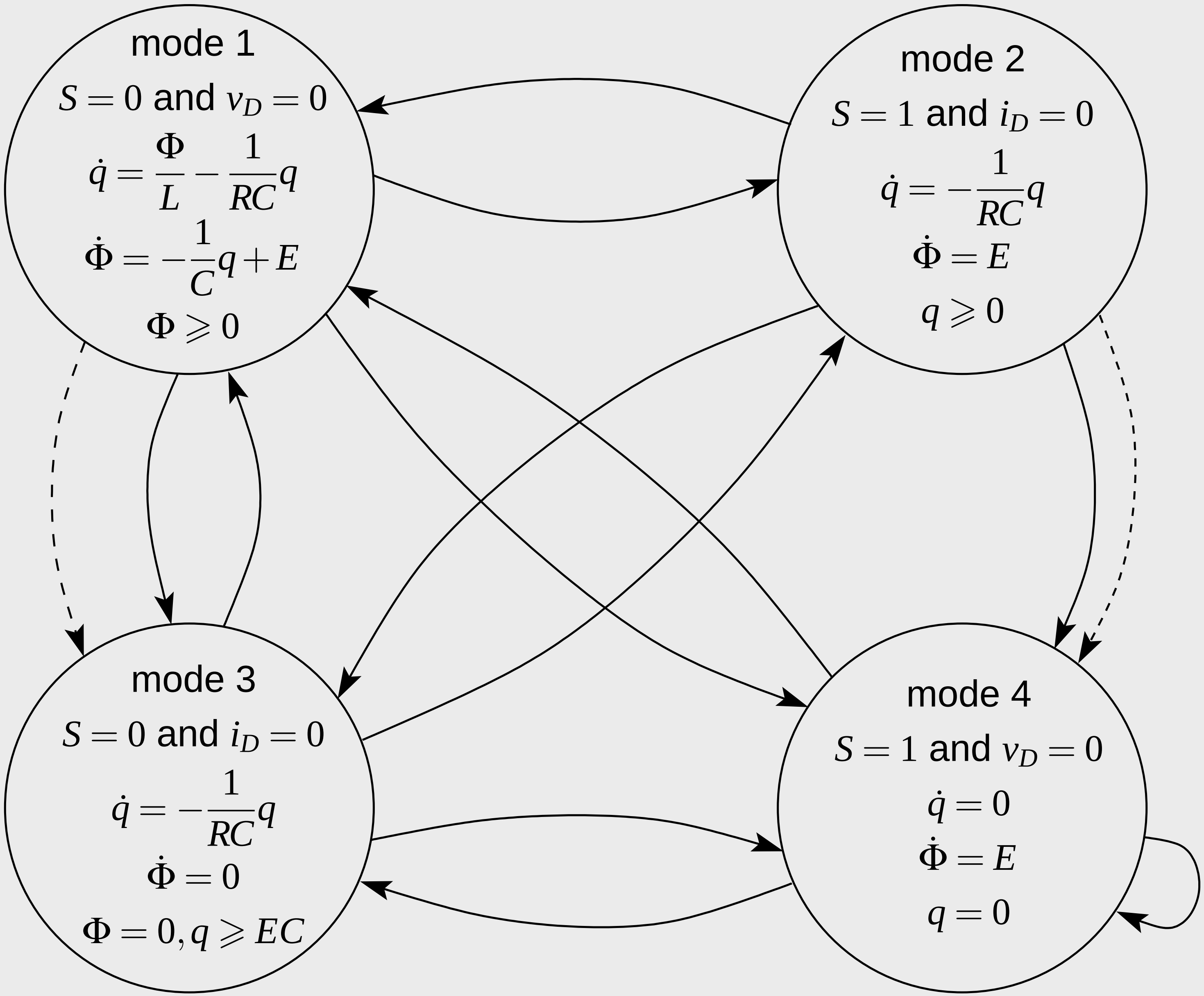
2.6 Boost converter





- presence of switch and diode introduces hybrid dynamics
- 4 modes:
  $(v_S = 0, v_D = 0)$, $(v_S = 0, i_D = 0)$, $(i_S = 0, v_D = 0)$, $(i_S = 0, i_D = 0)$

transition	guard	reset
mode 1$\rightarrow$mode 2	$S=1$ and $q \geqslant 0$	
mode 1$\rightarrow$mode 3	$\phi = 0$ and $q > CE$	
mode 1$\dashrightarrow$mode 3	$\phi < 0$	$\phi^+ = 0$
mode 1$\rightarrow$mode 4	$S=1$ and $q \leqslant 0$	$q^+ = 0$
mode 2$\rightarrow$mode 1	$S=0$ and $\phi \geqslant 0$	
mode 2$\rightarrow$mode 3	$S=0$ and $\phi \leqslant 0$	$\phi^+ = 0$
mode 2$\rightarrow$mode 4	$q=0$	
mode 2$\dashrightarrow$mode 4	$q<0$	$q^+ = 0$
mode 3$\rightarrow$mode 1	$q=CE$	
mode 3$\rightarrow$mode 2	$S=1$ and $q \geqslant 0$	
mode 3$\rightarrow$mode 4	$S=1$ and $q \leqslant 0$	$q^+ = 0$
mode 4$\rightarrow$mode 1	$S=0$ and $\phi \geqslant 0$	
mode 4$\rightarrow$mode 3	$S=0$ and $\phi \leqslant 0$	$\phi^+ = 0$
mode 4$\rightarrow$mode 4	$q<0$	$q^+ = 0$



mode 1
$S = 0$ and $v_D = 0$
$\dot{q} = \frac{\Phi}{L} - \frac{1}{RC} q$
$\dot{\Phi} = -\frac{1}{C} q + E$
$\Phi \geqslant 0$
mode 2
$S = 1$ and $i_D = 0$
$\dot{q} = -\frac{1}{RC} q$
$\dot{\Phi} = E$
$q \geqslant 0$
mode 3
$S = 0$ and $i_D = 0$
$\dot{q} = -\frac{1}{RC} q$
$\dot{\Phi} = 0$
$\Phi = 0, q \geqslant EC$
mode 4
$S = 1$ and $v_D = 0$
$\dot{q} = 0$
$\dot{\Phi} = E$
$q = 0$

2.6 Boost converter (continued)

Hybrid automaton model is very involved

Alternatively, one may use the more compact model

$$\begin{aligned}
\dot{q} &= -\frac{1}{RC}q + i_D \\
\dot{\phi} &= v_S + E \\
-v_D &= \frac{1}{C}q + v_S \\
i_S &= \tfrac{1}{L}\phi - i_D \\
0 \leqslant i_D &\perp -v_D \geqslant 0 \\
v_S &\perp i_S
\end{aligned}$$

$\longrightarrow$ also complementarity relation (as in two-carts system)

3. Examples with Zeno behavior

- *Zeno behavior* : infinitely many mode switches in finite time interval
- Examples

1. bouncing ball
2. reversed Filippov’s system
3. two-tank system
4. three-balls example

3.1 Bouncing ball

- Dynamics: $\ddot{x} = -g$ subject to $x \geqslant 0$ ($x(t)$: height)
- Newton's restitution rule ($0 < e < 1$):

$$\dot{x}(\tau+) = -e\dot{x}(\tau-) \quad \text{when } x(\tau-) = 0,\ \dot{x}(\tau-) < 0$$

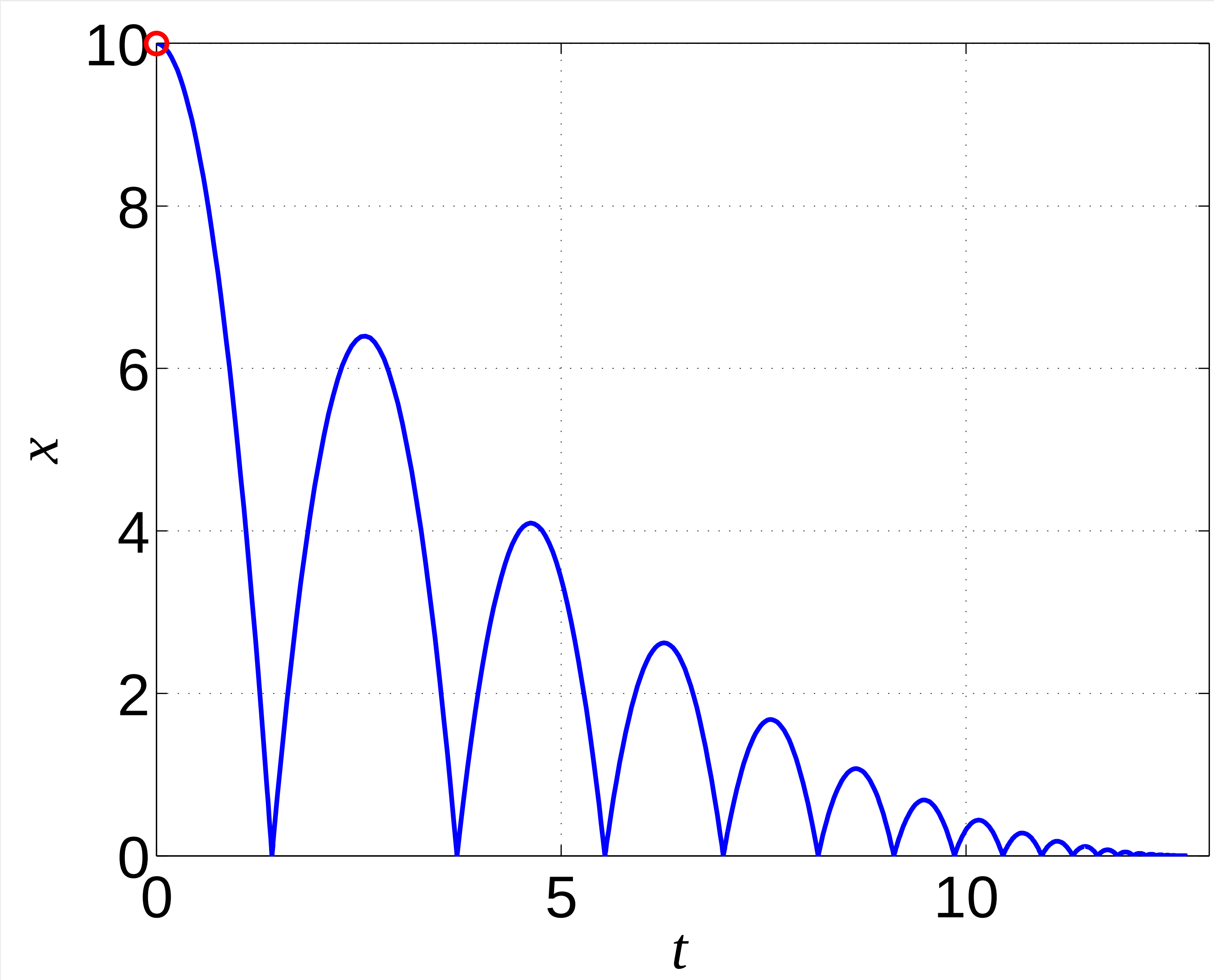
- Assuming $x(0) = 0$, $\dot{x}(0) > 0$, event times are related through

$$\tau_{i+1} = \tau_i + \frac{2e^i\dot{x}(0)}{g}$$

- Sequence has finite limit $\tau^* = \frac{2\dot{x}(0)}{g-ge} < \infty$ (geometric series)
- Physical interpretation: ball is at rest within finite time span, but after infinitely many bounces → Zeno behavior

  In this case: infinite number of state re-initializations, set of event times contains *right-accumulation point*

3.1 Bouncing ball (continued)



3.2 Reversed Filippov's example

- Dynamics:

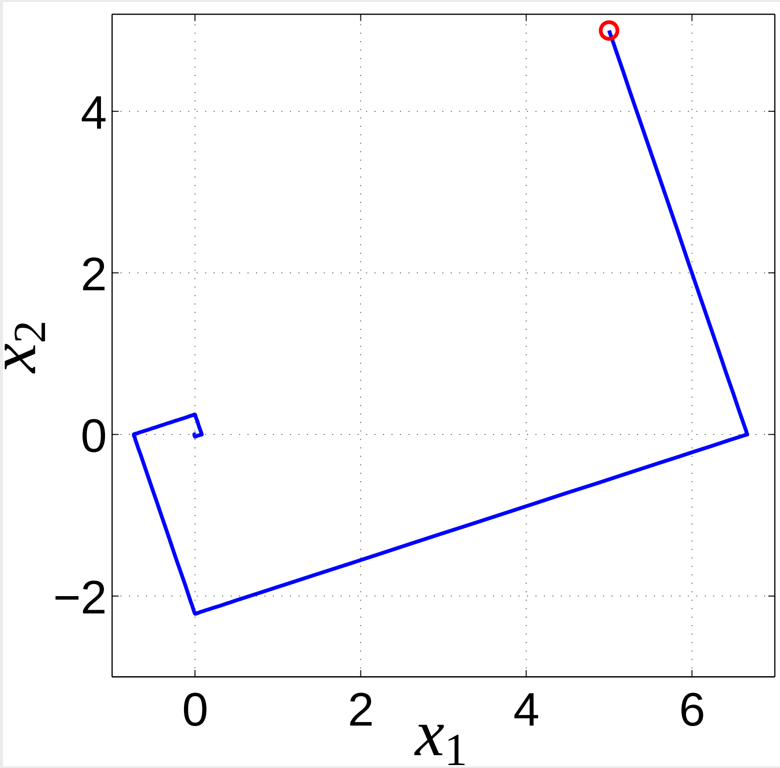
$$\begin{aligned} \dot{x}_1 &= -\mathrm{sgn}(x_1) + 2\mathrm{sgn}(x_2) \\ \dot{x}_2 &= -2\mathrm{sgn}(x_1) - \mathrm{sgn}(x_2), \end{aligned}$$

  with

$$\begin{cases} \mathrm{sgn}(x) = 1 & \text{if } x > 0 \\ \mathrm{sgn}(x) = -1 & \text{if } x < 0 \\ \mathrm{sgn}(x) \in [-1, 1] & \text{when } x = 0 \end{cases}$$

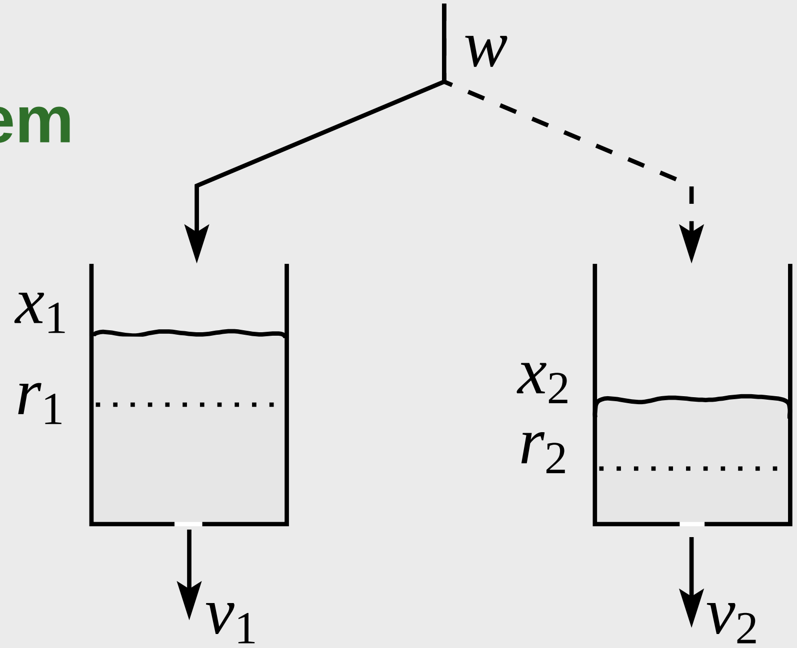
- Solutions system are spiraling towards origin, which is an equilibrium

3.2 Reversed Filippov's example (continued)



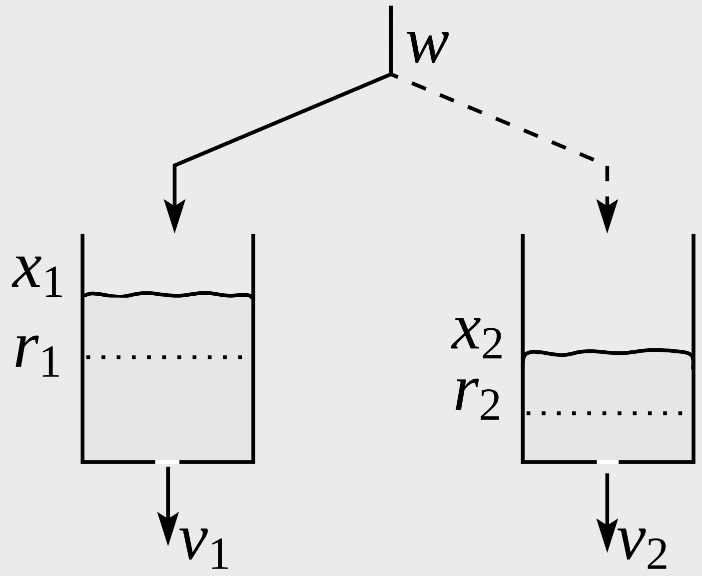
- Since $\frac{\mathrm{d}}{\mathrm{d}t}(|x_1(t)|+|x_2(t)|)=-2$, solutions reach origin in finite time
- Solutions go through infinite number of mode transitions (relay switches) $\rightarrow$ Zeno behavior

3.3 Two-tank system



- Two tanks (x_i: volume of water in tank)
- Tanks are leaking at constant rate $v_i > 0$
- Water is added at constant rate w through hose, which at any point in time is dedicated to either one tank or the other
- Objective: keep water volumes above r_1 and r_2
- Controller that switches inflow to tank 1 whenever $x_1 \leqslant r_1$ and to tank 2 whenever $x_2 \leqslant r_2$

Description of two-tank system as hybrid automaton

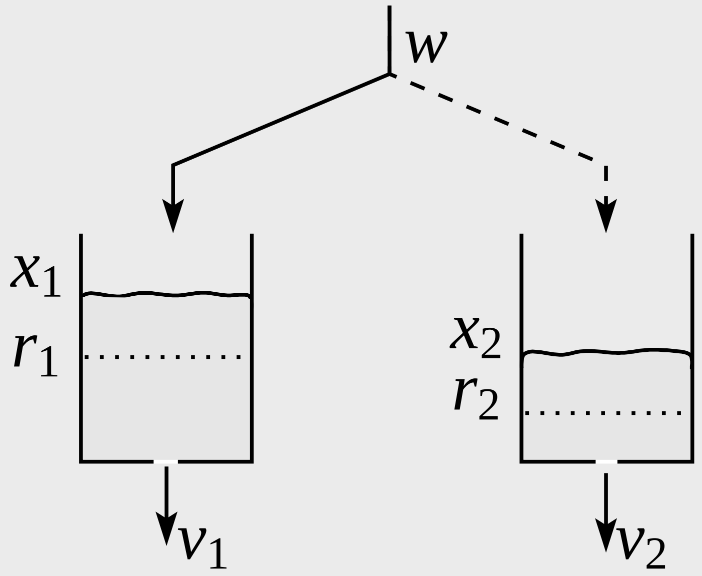


- Two modes: filling tank 1 (mode q_1) or tank 2 (mode q_2)
- Evolution of continuous state:

$$\begin{cases} \dot{x}_1 = w - v_1 \\ \dot{x}_2 = -v_2 \end{cases} \text{in mode } q_1 \qquad \begin{cases} \dot{x}_1 = -v_1 \\ \dot{x}_2 = w - v_2 \end{cases} \text{in mode } q_2$$

- $\text{Init} = \{q_1, q_2\} \times \{(x_1, x_2) \mid x_1 \geqslant r_1 \text{ and } x_2 \geqslant r_2\}$

Description of two-tank system as hybrid automaton (cont.)



- Invariants: $\mathsf{Inv}(q_1) = \{x \in \mathbb{R}^2 \mid x_2 \geqslant r_2\}$

  $\mathsf{Inv}(q_2) = \{x \in \mathbb{R}^2 \mid x_1 \geqslant r_1\}$

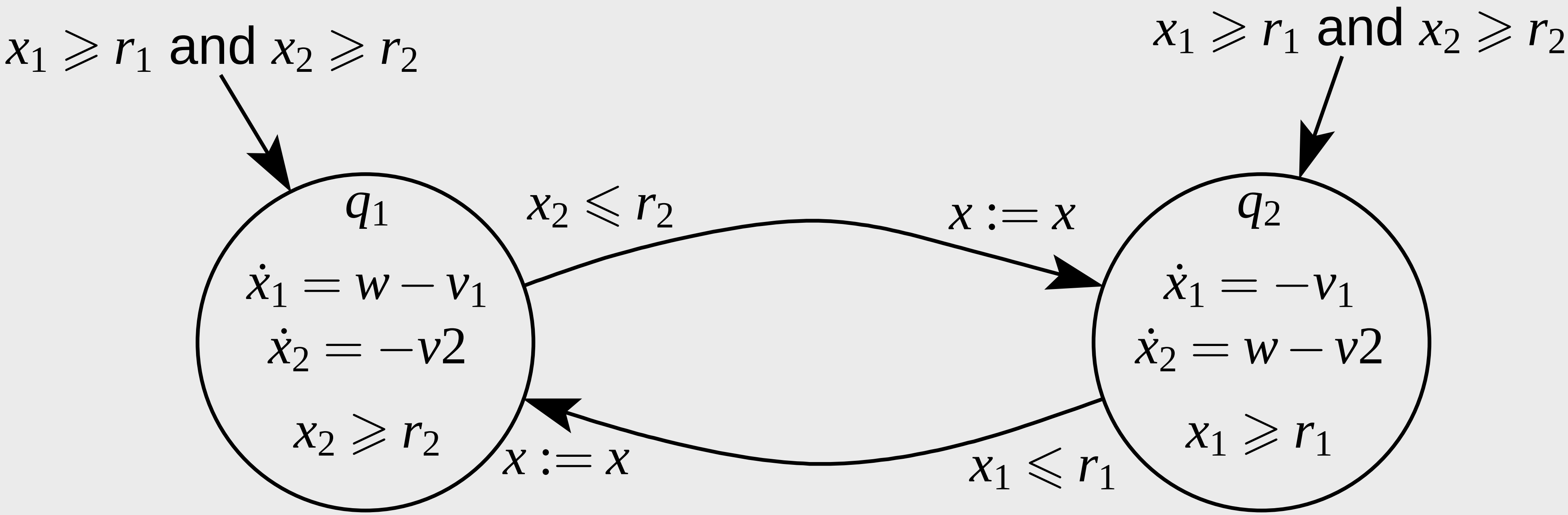
- Guards: $G(q_1, q2) = \{x \in \mathbb{R}^2 \mid x_2 \leqslant r_2\}$

  $G(q_2, q1) = \{x \in \mathbb{R}^2 \mid x_1 \leqslant r_1\}$

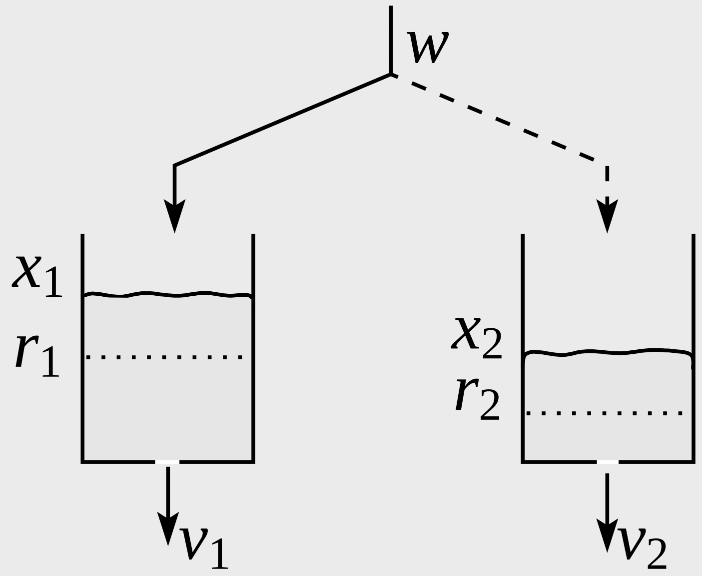
- No resets:

$$R(q_1, q_2) = R(q_2, q_1) = \{(x^-, x^+) \mid x^-, x^+ \in \mathbb{R}^2 \text{ and } x^- = x^+\}$$

Description of two-tank system as hybrid automaton (cont.)

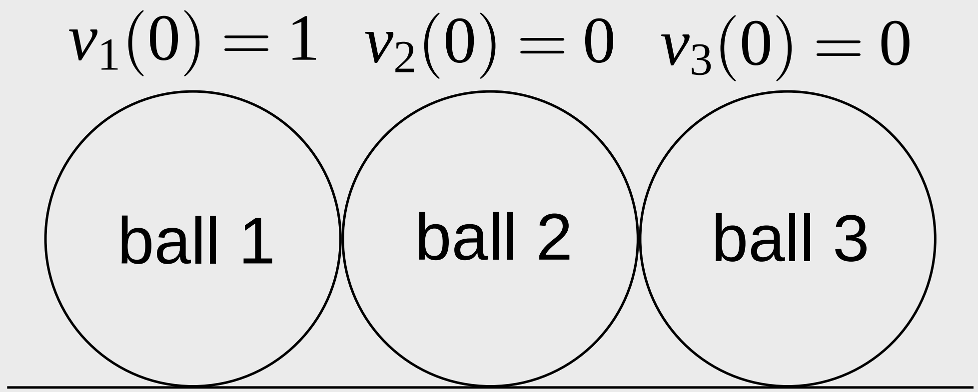


Two-tank system and Zeno behavior



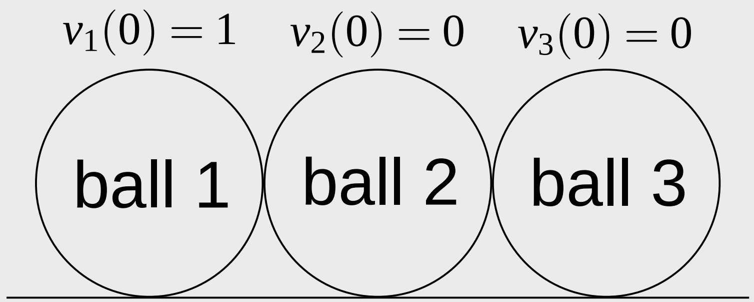
- Assume total outflow $v_1 + v_2 > w$
- Control objective cannot be met and tanks will empty in finite time
- Infinitely many switchings in finite time $\rightarrow$ Zeno behavior

3.4 Three-balls example



- System consisting of three balls
- Inelastic impacts modeled by successions of simple impacts
- Suppose unit masses, touching at time 0, and
  $v_1(0) = 1$, $v_2(0) = v_3(0) = 0$
- We model all impacts separately $\rightarrow$
  - first, inelastic collision between balls 1 and 2, resulting in
    $v_1(0+) = v_2(0+) = 0.5$, $v_3(0+) = 0$

3.4 Three-balls example (continued)



- next, ball 2 hits ball 3, resulting in $v_1(0{+}{+}) = \frac{1}{2}$, $v_2(0{+}{+}) = v_3(0{+}{+}) = \frac{1}{4}$
- next, ball 1 hits ball 2 again, etc.

  $\longrightarrow$ sequence of resets:

$$\begin{array}{llllllll} v_1: & 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{11}{32} & \dots \\ v_2: & 0 & \frac{1}{2} & \frac{1}{4} & \frac{3}{8} & \frac{5}{16} & \frac{11}{32} & \dots \\ v_3: & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{5}{16} & \frac{5}{16} & \dots \end{array}$$

  converges to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^{\mathrm{T}}$

- Afterwards, smooth continuation is possible with constant and equal velocity for all balls
- Infinite number of events (resets) at one time instant, sometimes called *live-lock* $\rightarrow$ another special case of Zeno behavior