Modeling & Control of Hybrid Systems

Chapter 3 — Dynamics & Well-Posedness

Overview

1. Smooth systems: Differential equations
2. Switched systems: Discontinuous differential equations
3. Event times
4. Well-posedness for hybrid automata
5. Well-posedness for complementarity systems

Key issues

- Solution concepts
- Well-posedness: Existence & uniqueness of solutions

1. Smooth systems: differential equations

1.1 Solution concept

Description format / syntax / model

$\downarrow$

solutions / trajectories / executions/ semantics/ behavior

Example: $\dot{x} = f(t,x) \qquad x(t_0) = x_0$

A solution trajectory is a function $x : [t_0,t_1] \to \mathbb{R}^n$ that is continuous, differentiable and satisfies $x(t_0) = x_0$ and

$$\dot{x}(t) = f(t,x(t)) \text{ for all } t \in (t_0,t_1)$$

Issue of **well-posedness:** given initial conditions does there **exist** a solution and is it **unique**?

1.2 Well-posedness

Example: $\dot{x} = 2\sqrt{x}$ with $x(0) = 0$

Two solutions: $x(t) = 0$ and $x(t) = t^2$

Theorem for *local* existence and uniqueness of solutions given initial condition:

Let $f(t,x)$ be piecewise continuous in t and satisfy the following Lipschitz condition: there exist $L > 0$ and $r > 0$ such that

$$\|f(t,x) - f(t,y)\| \leqslant L\|x-y\|$$

for all x and y in neighborhood $B := \{x \in \mathbb{R}^n \mid \|x-x_0\| < r\}$ of x_0 and for all $t \in [t_0,t_1]$.

Then there exists $\delta > 0$ such that unique solution exists on $[t_0,t_0+\delta]$ starting in x_0 at t_0.

1.3 Global well-posedness

Example: $\dot{x} = x^2 + 1$ with $x(0) = 0$ has as solution $x(t) = \tan t$ which is only **locally** defined on $[0, \pi/2)$

Note that we have $\lim_{t \uparrow \pi/2} x(t) = \infty$ $\rightarrow$ Finite escape time!

Theorem (Global Lipschitz condition)

Suppose $f(t,x)$ is piecewise continuous in t and satisfies

$$\|f(t,x) - f(t,y)\| \leqslant L\|x - y\|$$

for all x, y in $\mathbb{R}^n$ and for all $t \in [t_0, t_1]$.

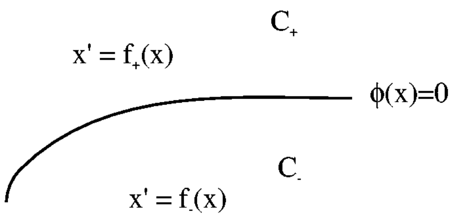
Then unique solution exists on $[t_0, t_1]$ for any initial state x_0 at t_0.

Not necessary: $\dot{x} = -x^3$ is not globally Lipschitz, but has unique global solutions

1.3 Global well-posedness (continued)

These “smooth” phenomena also occur in hybrid systems, but for hybrid systems there is even more awkward stuff due to mode switching (a.o. Zeno)

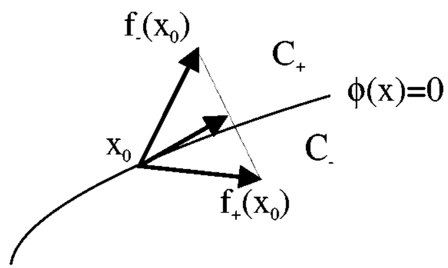
2. Switched systems: Discontinuous differential equations



$$\dot{x} = \begin{cases} f_+(x) & \text{if } x \in C_+ := \{x \mid \phi(x) > 0\} \\ f_-(x) & \text{if } x \in C_- := \{x \mid \phi(x) < 0\} \end{cases}$$

- if x in interior of C_- or C_+: just follow!
- if $f_-(x)$ and $f_+(x)$ point in same direction: just follow!
- if $f_+(x)$ points towards C_+ and $f_-(x)$ points towards C_-: At least two trajectories

2.1 Sliding modes



$f_+(x)$ points towards C_- and $f_-(x)$ points towards C_+

$\rightarrow$ no classical solution

- Relaxation: spatial (hysteresis) Δ, time delay τ, smoothing ε
- Chattering / infinitely fast switching (limit case $\Delta, \varepsilon, \tau \downarrow 0$)

Filippov’s convex definition: convex combination of both dynamics

$$\dot{x} = \lambda f_+(x) + (1 - \lambda) f_-(x) \text{ with } 0 \leqslant \lambda \leqslant 1$$

such that x moves (“slides”) along surface $\phi(x) = 0$

2.2 Differential inclusions

$$\dot x = \lambda f_+(x) + (1-\lambda) f_-(x) \text{ with } \begin{cases} \lambda = 1, & \text{if } \phi(x) > 0 \\ 0 \leqslant \lambda \leqslant 1, & \text{if } \phi(x) = 0 \\ \lambda = 0, & \text{if } \phi(x) < 0, \end{cases}$$

i.e., $\lambda \in \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(\phi(x))$ with $\mathrm{sgn}(a) := \begin{cases} \{1\}, & \text{if } a > 0 \\ [-1,1], & \text{if } a = 0 \\ \{-1\}, & \text{if } a < 0 \end{cases}$

Differential inclusion $\dot x \in F(x)$ with set-valued

$$\begin{aligned} F(x) &= \{f_+(x)\} && (\phi(x) > 0) \\ F(x) &= \{f_-(x)\} && (\phi(x) < 0), \\ F(x) &= \{\lambda f_1(x) + (1-\lambda) f_2(x) \mid \lambda \in [0,1]\} && (\phi(x) = 0), \end{aligned}$$

Definition: Function $x : [a,b] \to \mathbb{R}^n$ is *solution* of $\dot x \in F(x)$ if x is absolutely continuous and satisfies $\dot x(t) \in F(x(t))$ for almost all $t \in [a,b]$

Example

$$\phi(x) = x_2,\ f_+(x) = (x_1^2, -x_1 + \tfrac{1}{2}x_1^2)^\mathsf{T},\ \ f_-(x) = (1, x_1^2)^\mathsf{T}$$

Sliding for $x_0 = (1,0)^\mathsf{T}$ as $f_+(x_0) = (1, -\frac{1}{2})^\mathsf{T}$ and $f_-(x_0) = (1,1)^\mathsf{T}$

Sliding behavior: find convex combination such that $\phi(x) = 0$

$$\frac{d\phi}{dt}(x(t)) = \frac{d\phi}{dx}\dot x(t) = \dot x_2(t) = \lambda(-x_1 + \tfrac{1}{2}x_1^2) + (1-\lambda)x_1^2 = 0 \quad \rightarrow$$

$$\lambda(x) = \frac{x_1}{\frac{1}{2}x_1 + 1}$$

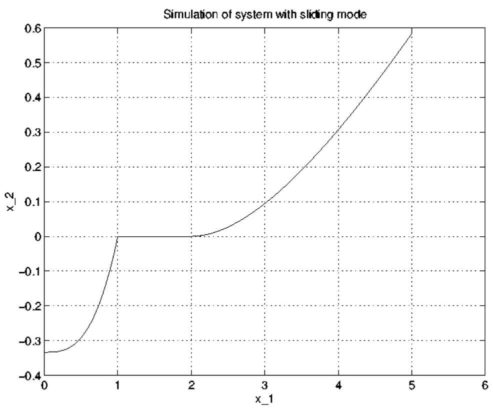
Sliding mode is valid as long as $\lambda(x) \in [0,1]$, “invariant”

$$\dot x_1 = \lambda x_1^2 + (1-\lambda) = \frac{2x_1^3 - x_1 + 2}{x_1 + 2}$$

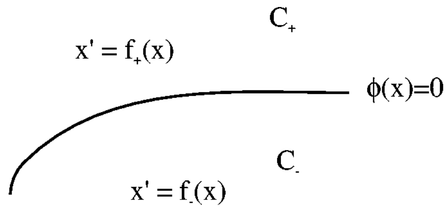
as long as $0 \leqslant x_1 \leqslant 2$

Example (continued)

Initial state $(0, -\frac{1}{3})$, at time $t = 1$ the surface $x_2 = 0$ is hit in $(1,0)^\mathsf{T}$, trajectory slides along surface till time $t_{\text{leave}} \approx 1.531$ ($x_1(t_{\text{leave}}) = 2$), and then leaves surface again to C_+



2.3 A well-posedness result for sliding modes



Theorem

Assume

- f_- and f_+ are continuously differentiable (C^1)
- ϕ is C^2, discontinuity vector $h(x) := f_+(x) - f_-(x)$ is C^1

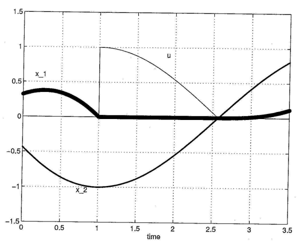
If for each x with $\phi(x) = 0$ at least one of the conditions

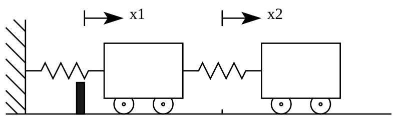
- $f_+(x)$ points towards C_- or
- $f_-(x)$ points towards C_+

holds (where for different points x a different condition may hold), then the Filippov solutions exist and are unique

3. Event times

3.1 Admissible event times





event times set $\mathscr{E}$ is $\{0, 1, 1+\frac{\pi}{2}\}$

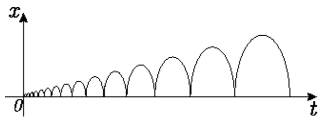
Definition: Set $\mathscr{E} \subset \mathbb{R}_+$ is *admissible event times set*, if it is closed and countable, and $0 \in \mathscr{E}$ (0: initial time)

3.2 Accumulation points

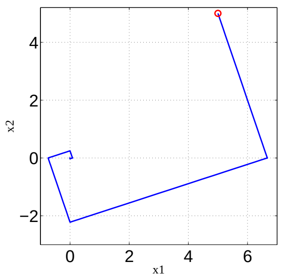
- $t \in \mathscr{E}$ is said to be *left accumulation point* of $\mathscr{E}$, if for all $t' > t$ $(t, t') \cap \mathscr{E}$ is not empty
- t is called *right accumulation point*, if for all $t' < t$ $(t', t) \cap \mathscr{E}$ is not empty

Definition Admissible event times set $\mathscr{E}$ (or the corresponding solution) is said to be *left (right) Zeno free*, if it does not contain any left (right) accumulation points

- Bouncing ball $\longrightarrow$ right accumulation point
- Time-reversed bouncing ball $\longrightarrow$ left accumulation point



3.3 Effects of choice of solution concept



Filippov's example (*reverse* of Ch. 1):

$\dot{x}_1 = \text{sgn}(x_1) - 2\text{sgn}(x_2)$
$\dot{x}_2 = 2\text{sgn}(x_1) + \text{sgn}(x_2),$

Left accumulation point $\longrightarrow$ $\mathscr{E}$ is not left Zeno free!

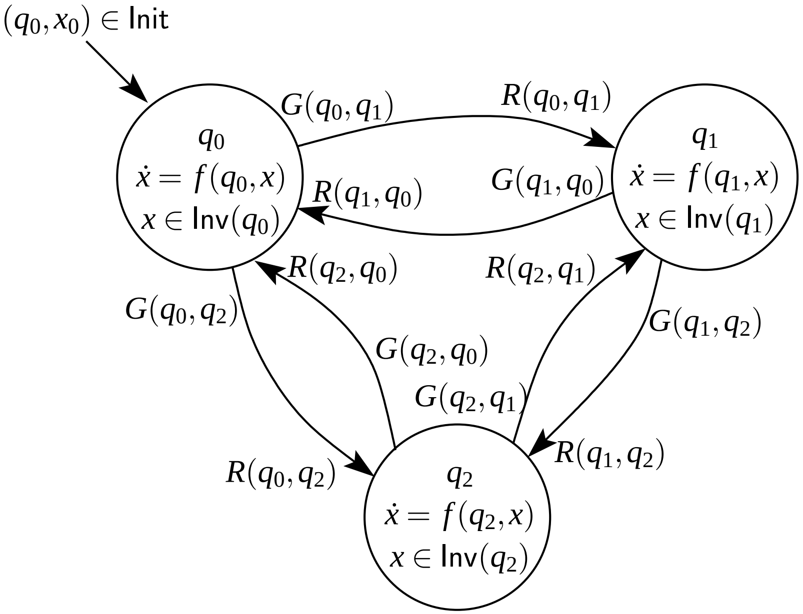
Well-posedness:

- If solution concept left Zeno free: only one solution from origin (Filippov's example)
- If solution concept right Zeno free: only local existence (bouncing ball)
- If solution concept allows Zeno, then multiple solutions from origin (Filippov's example) and global solutions for bouncing ball

4. Well-posedness for hybrid automata

Hybrid automaton $H = (Q, X, f, \text{Init}, \text{Inv}, E, G, R)$

- Hybrid state: (q, x)
- Evolution of continuous state in mode q: $\dot{x} = f(q, x)$
- Invariant Inv: describes conditions that continuous state has to satisfy at given mode
- Guard G: specifies subset of state space where certain transition is enabled
- Reset map R: specifies how new continuous states are related to previous continuous states



4.1 Hybrid time trajectory

Definition: A hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$ is finite ($N < \infty$) or infinite ($N = \infty$) sequence of intervals of real line, such that

- $I_i = [\tau_i, \tau_i']$ with $\tau_i \leqslant \tau_i' = \tau_{i+1}$ for $0 \leqslant i < N$;
- if $N < \infty$, either $I_N = [\tau_N, \tau_N']$ or $I_N = [\tau_N, \tau_N')$ with $\tau_N \leqslant \tau_N' \leqslant \infty$.

- For instance,

$$\tau = \{[0,2], [2,3], \{3\}, \{3\}, [3,4.5], \{4.5\}, [4.5,6]\}$$

$$\tau = \{[0,2], [2,3], [3,4.5], \{4.5\}, [4.5,6], [6,\infty)\}$$

$$I_i = [1-2^i, 1-2^{i+1}]$$

- $\mathscr{E} = \{\tau_0, \tau_1, \tau_2, \ldots\}$
- Note: No left accumulations of event times!

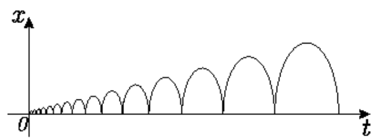
4.2 Well-posedness for hybrid automata

- **Initial well-posedness**: if hybrid automaton is non-blocking + deterministic, i.e., if there is no
  - **dead-lock**: no smooth continuation and no jump possible
  - splitting of trajectories (non-determinism)

$\rightarrow$ there exist theoretical conditions, but not easy to check

4.2 Well-posedness for hybrid automata (continued)

- However, no statements by hybrid automata theory on existence, absence, or continuation
  - beyond **live-lock**: an infinite number of jumps at one time instant, so no solution on $[0, \varepsilon)$ for some $\varepsilon > 0$
  - for **left accumulations** of event times $\rightarrow$ prevent uniqueness:

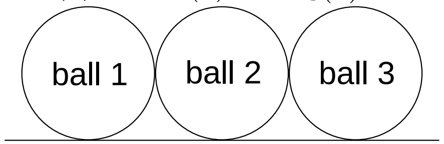


  - for **right-accumulations** of event times $\rightarrow$ prevent global existence

4.3 Obstruction — local existence

$\rightarrow$ **Live-lock:** Infinitely many jumps at one time instant

$v_1(0)=1 \quad v_2(0)=0 \quad v_3(0)=0$



$$\begin{array}{lllllllll} v_1: & 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{11}{32} & \cdots & \frac{1}{3} \\ v_2: & 0 & \frac{1}{2} & \frac{1}{4} & \frac{3}{8} & \frac{5}{16} & \frac{11}{32} & \cdots & \frac{1}{3} \\ v_3: & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{5}{16} & \frac{5}{16} & \cdots & \frac{1}{3} \end{array}$$

smooth continuation possible with constant velocity after infinite number of events

$\rightarrow$ exclude live-lock or show convergence of state x for local existence

Discrete mode is a function of continuous state here!

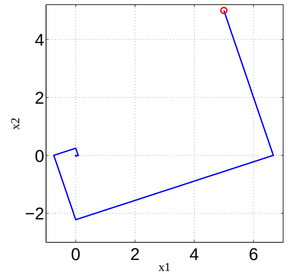
$\Rightarrow$ not for general hybrid automata!!!

4.4 Obstruction — global existence: Zenoness

$\rightarrow$ **Right-accumulation** of event times

Reversed Filippov's example:

$$\dot{x}_1 = -\ \text{sgn}(x_1) + 2\ \text{sgn}(x_2)$$
$$\dot{x}_2 = -2\ \text{sgn}(x_1) -\ \text{sgn}(x_2)$$



$\rightarrow$ show the existence of the left limit $\lim_{t \uparrow \tau^*} x(t)$ for global existence

Discrete mode is a function of continuous state!

$\Rightarrow$ not for general hybrid automata!!!

5. Well-posedness for complementarity systems (+ MLD, PWA, MMPS,. . .)

5.1 Discrete-time LCS

$$\begin{aligned} x(k+1) &= Ax(k) + Bz(k) + Eu(k) \\ w(k) &= Cx(k) + Dz(k) + Fu(k) \\ 0 \leqslant w(k) &\perp z(k) \geqslant 0 \end{aligned}$$

Well-posedness:

Given $x(k)$, $u(k)$ $\rightarrow$ $x(k+1)$, $z(k)$, $w(k)$ uniquely determined

5.1 Discrete-time LCS (continued)

Algebraic problem:

Linear complementarity problem LCP(q, M):

Given vector $q \in \mathbb{R}^m$ and matrix $M \in \mathbb{R}^{m \times m}$ find $z \in \mathbb{R}^m$ such that

$$0 \leqslant q + Mz \perp z \geqslant 0$$

$M \in \mathbb{R}^{m \times m}$ is *P-matrix*, if $\det M_{II} > 0$ for all $I \subseteq \{1, \ldots, m\}$

Theorem

Discrete-time LCS is well-posed if D is a P-matrix

Necessary in case $\text{im}[C\ F] = \mathbb{R}^n$

5.2 Initial well-posedness for continuous-time LCS

Consider LCS:

$$\dot{x}(t) = Ax(t) + Bz(t), \qquad w(t) = Cx(t) + Dz(t), \;\; 0 \leqslant z(t) \perp w(t) \geqslant 0$$

Define $G(s) := C(s\mathscr{I} - A)^{-1}B + D \qquad Q(s) = C(s\mathscr{I} - A)^{-1}$

LCS is initially well-posed if and only if for all x_0 LCP$(Q(\sigma)x_0, G(\sigma))$ is uniquely solvable for sufficiently large $\sigma \in \mathbb{R}$

- *dynamical* properties can now be linked to *static* results on LCPs which are abundant in literature!
- $G(\sigma)$ being P-matrix for sufficiently large σ is sufficient condition for initial well-posedness