Modeling & Control of Hybrid Systems

Chapter 3 — Dynamics & Well-Posedness

Overview

- 1. Smooth systems: Differential equations
- 2. Switched systems: Discontinuous differential equations
- 3. Event times
- 4. Well-posedness for hybrid automata
- 5. Well-posedness for complementarity systems

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1. Smooth systems: differential equations

1.1 Solution concept

Description format / syntax / model

solutions / trajectories / executions/ semantics/ behavior

Example: $\dot{x} = f(t, x)$ $x(t_0) = x_0$

A solution trajectory is a function $x : [t_0, t_1] \to \mathbb{R}^n$ that is continuous, differentiable and satisfies $x(t_0) = x_0$ and

$$\dot{x}(t) = f(t, x(t))$$
 for all $t \in (t_0, t_1)$

Issue of **well-posedness:** given initial conditions does there **exist** a solution and is it **unique**?

Key issues

- Solution concepts
- Well-posedness: Existence & uniqueness of solutions

1.2 Well-posedness

Example: $\dot{x} = 2\sqrt{x}$ with x(0) = 0

Two solutions: x(t) = 0 and $x(t) = t^2$

Theorem for *local* existence and uniqueness of solutions given initial condition:

Let f(t,x) be piecewise continuous in t and satisfy the following Lipschitz condition: there exist L > 0 and r > 0 such that

 $\|f(t,x) - f(t,y)\| \leq L \|x - y\|$

for all *x* and *y* in neighborhood $B := \{x \in \mathbb{R}^n \mid ||x - x_0|| < r\}$ of x_0 and for all $t \in [t_0, t_1]$.

Then there exists $\delta > 0$ such that unique solution exists on $[t_0, t_0 + \delta]$ starting in x_0 at t_0 .

1.3 Global well-posedness

Example: $\dot{x} = x^2 + 1$ with x(0) = 0 has as solution $x(t) = \tan t$ which is only **locally** defined on $[0, \pi/2)$

Note that we have $\lim_{t\uparrow \pi/2} x(t) = \infty \rightarrow$ Finite escape time!

Theorem (Global Lipschitz condition)

Suppose f(t,x) is piecewise continuous in t and satisfies $||f(t,x) - f(t,y)|| \le L||x-y||$ for all x, y in \mathbb{R}^n and for all $t \in [t_0, t_1]$. Then unique solution exists on $[t_0, t_1]$ for any initial state x_0 at t_0 .

Not necessary: $\dot{x} = -x^3$ is not globally Lipschitz, but has unique global solutions

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2. Switched systems: Discontinuous differential equations

$$\begin{aligned} \mathbf{C}_{+} & \\ \mathbf{x}' = \mathbf{f}_{+}(\mathbf{x}) & \\ & \mathbf{\phi}(\mathbf{x}) = \mathbf{0} & \dot{\mathbf{x}} = \begin{cases} f_{+}(x) & \text{if } x \in C_{+} := \{x \mid \phi(x) > 0\} \\ f_{-}(x) & \text{if } x \in C_{-} := \{x \mid \phi(x) < 0\} \end{cases} \\ & \\ & \mathbf{C}_{-} & \\ & \mathbf{x}' = \mathbf{f}_{-}(\mathbf{x}) & \end{cases} \end{aligned}$$

- if x in interior of C_- or C_+ : just follow!
- if $f_{-}(x)$ and $f_{+}(x)$ point in same direction: just follow!
- if $f_+(x)$ points towards C_+ and $f_-(x)$ points towards C_- : At least two trajectories

1.3 Global well-posedness (continued)

These "smooth" phenomena also occur in hybrid systems, but for hybrid systems there is even more awkward stuff due to mode switching (a.o. Zeno)

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- $f_+(x)$ points towards C_- and $f_-(x)$ points towards C_+
- \rightarrow no classical solution
- Relaxation: spatial (hysteresis) Δ , time delay au, smoothing arepsilon
- \bullet Chattering / infinitely fast switching (limit case $\Delta, \varepsilon, \tau \downarrow 0)$

Filippov's convex definition: convex combination of both dynamics

$$\dot{x} = \lambda f_+(x) + (1 - \lambda) f_-(x)$$
 with $0 \leq \lambda \leq 1$

such that *x* moves ("slides") along surface $\phi(x) = 0$

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 $\begin{aligned} \textbf{2.2 Differential inclusions} \\ \dot{x} &= \lambda f_+(x) + (1-\lambda) f_-(x) \text{ with } \begin{cases} \lambda = 1, & \text{ if } \phi(x) > 0 \\ 0 \leqslant \lambda \leqslant 1, & \text{ if } \phi(x) = 0 \\ \lambda = 0, & \text{ if } \phi(x) < 0, \end{cases} \\ \textbf{i.e., } \lambda &\in \frac{1}{2} + \frac{1}{2} \text{sgn}(\phi(x)) \text{ with } \text{sgn}(a) &:= \begin{cases} \{1\}, & \text{ if } a > 0 \\ [-1,1], & \text{ if } a = 0 \\ \{-1\}, & \text{ if } a < 0 \end{cases} \end{aligned}$

Differential inclusion $\dot{x} \in F(x)$ with set-valued

$$F(x) = \{f_{+}(x)\} \qquad (\phi(x) > 0)$$

$$F(x) = \{f_{-}(x)\} \qquad (\phi(x) < 0),$$

$$F(x) = \{\lambda f_{1}(x) + (1 - \lambda)f_{2}(x) \mid \lambda \in [0, 1]\} \qquad (\phi(x) = 0)$$

Definition: Function $x : [a,b] \to \mathbb{R}^n$ is *solution* of $\dot{x} \in F(x)$ if x is absolutely continuous and satisfies $\dot{x}(t) \in F(x(t))$ for almost all $t \in [a,b]$

Example (continued)

Initial state $(0, -\frac{1}{3})$, at time t = 1 the surface $x_2 = 0$ is hit in $(1, 0)^{\mathsf{T}}$, trajectory slides along surface till time $t_{\mathsf{leave}} \approx 1.531$ ($x_1(t_{\mathsf{leave}}) = 2$), and then leaves surface again to C_+



Example

$$\phi(x) = x_2, f_+(x) = (x_1^2, -x_1 + \frac{1}{2}x_1^2)^{\mathsf{T}}, \ f_-(x) = (1, x_1^2)^{\mathsf{T}}$$

Sliding for $x_0 = (1,0)^T$ as $f_+(x_0) = (1,-\frac{1}{2})^T$ and $f_-(x_0) = (1,1)^T$

Sliding behavior: find convex combination such that $\phi(x) = 0$

$$\frac{d\phi}{dt}(x(t)) = \frac{d\phi}{dx}\dot{x}(t) = \dot{x}_2(t) = \lambda(-x_1 + \frac{1}{2}x_1^2) + (1-\lambda)x_1^2 = 0 \quad \to \\ \lambda(x) = \frac{x_1}{\frac{1}{2}x_1 + 1}$$

Sliding mode is valid as long as $\lambda(x) \in [0,1]$, "invariant"

$$\dot{x}_1 = \lambda x_1^2 + (1 - \lambda) = \frac{2x_1^3 - x_1 + 2}{x_1 + 2}$$

as long as $0 \leq x_1 \leq 2$

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Assume

- f_- and f_+ are continuously differentiable (C^1)
- ϕ is C^2 , discontinuity vector $h(x) := f_+(x) f_-(x)$ is C^1

If for each *x* with $\phi(x) = 0$ at least one of the conditions

- $f_+(x)$ points towards C_- or
- $f_{-}(x)$ points towards C_{+}

holds (where for different points *x* a different condition may hold), then the Filippov solutions exist and are unique

3. Event times

3.1 Admissible event times





event times set \mathscr{E} is $\{0, 1, 1+\frac{\pi}{2}\}$

Definition: Set $\mathscr{E} \subset \mathbb{R}_+$ is *admissible event times set*, if it is closed and countable, and $0 \in \mathscr{E}$ (0: initial time)

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Filippov's example (reverse of Ch. 1):

$$\dot{x}_1 = \operatorname{sgn}(x_1) - 2\operatorname{sgn}(x_2)$$

$$\dot{x}_2 = 2\operatorname{sgn}(x_1) + \operatorname{sgn}(x_2),$$

Left accumulation point $\rightarrow \mathscr{E}$ is not left Zeno free!

Well-posedness:

- If solution concept left Zeno free: only one solution from origin (Filippov's example)
- If solution concept right Zeno free: only local existence (bouncing ball)
- If solution concept allows Zeno, then multiple solutions from origin (Filippov's example) and global solutions for bouncing ball



3.2 Accumulation points

- $t \in \mathscr{E}$ is said to be *left accumulation point* of \mathscr{E} , if for all t' > t $(t,t') \cap \mathscr{E}$ is not empty
- *t* is called *right accumulation point*, if for all $t' < t(t',t) \cap \mathscr{E}$ is not empty

Definition Admissible event times set \mathscr{E} (or the corresponding solution) is said to be *left (right) Zeno free*, if it does not contain any left (right) accumulation points

- Bouncing ball \rightarrow right accumulation point
- \bullet Time-reversed bouncing ball \rightarrow left accumulation point



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4. Well-posedness for hybrid automata

Hybrid automaton H = (Q, X, f, lnit, lnv, E, G, R)

- Hybrid state: (q, x)
- Evolution of continuous state in mode q: $\dot{x} = f(q, x)$
- Invariant Inv: describes conditions that continuous state has to satisfy at given mode
- Guard *G*: specifies subset of state space where certain transition is enabled
- Reset map *R*: specifies how new continuous states are related to previous continuous states

 $(q_0, x_0) \in \mathsf{Init}$



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4.1 Hybrid time trajectory

Definition: Hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$ is finite $(N < \infty)$ or infinite $(N = \infty)$ sequence of intervals of real line, such that

- $I_i = [\tau_i, \tau'_i]$ with $\tau_i \leq \tau'_i = \tau_{i+1}$ for $0 \leq i < N$;
- if $N < \infty$, either $I_N = [\tau_N, \tau'_N]$ with $\tau_N \leqslant \tau'_N \neq \infty$ or $I_N = [\tau_N, \tau'_N)$ with $\tau_N \leqslant \tau'_N \leqslant \infty$.

• For instance,

$$\begin{split} \tau &= \{[0,2],[2,3],\{3\},\{3\},[3,4.5],\{4.5\},[4.5,6]\} \\ \tau &= \{[0,2],[2,3],[3,4.5],\{4.5\},[4.5,6],[6,\infty)\} \\ I_i &= [1-\frac{1}{2^i},1-\frac{1}{2^{i+1}}] \end{split}$$

& e = {τ₀, τ₁, τ₂,...}
Note: No left accumulations of event times!

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4.2 Well-posedness for hybrid automata

- Initial well-posedness: if hybrid automaton is non-blocking + deterministic, i.e., if there is no
 - dead-lock: no smooth continuation and no jump possible
 - splitting of trajectories (non-determinism)
- \rightarrow there exist theoretical conditions, but not easy to check

4.2 Well-posedness for hybrid automata (continued)

- However, no statements by hybrid automata theory on existence, absence, or continuation
 - beyond **live-lock**: an infinite number of jumps at one time instant, so no solution on $[0, \varepsilon)$ for some $\varepsilon > 0$
 - for **left accumulations** of event times \rightarrow prevent uniqueness:



– for right~accumulations of event times \rightarrow prevent global existence

4.3 Obstruction — local existence

 \rightarrow Live-lock: Infinitely many jumps at one time instant

$$v_{1}(0) = 1 \quad v_{2}(0) = 0 \quad v_{3}(0) = 0$$

$$v_{1} \colon 1 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{3}{8} \quad \frac{3}{8} \quad \frac{11}{32} \quad \dots \quad \frac{1}{3}$$

$$v_{2} \colon 0 \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{3}{8} \quad \frac{5}{16} \quad \frac{11}{32} \quad \dots \quad \frac{1}{3}$$

$$v_{3} \colon 0 \quad 0 \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{5}{16} \quad \frac{5}{16} \quad \dots \quad \frac{1}{3}$$

smooth continuation possible with constant velocity after infinite number of events

 \rightarrow exclude live-lock or show convergence of state *x* for local existence

Discrete mode is a function of continuous state here!

 \Rightarrow not for general hybrid automata!!!

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5. Well-posedness for complementarity systems (+ MLD, PWA, MMPS,...)

5.1 Discrete-time LCS

$$x(k+1) = Ax(k) + Bz(k) + Eu(k)$$
$$w(k) = Cx(k) + Dz(k) + Fu(k)$$
$$0 \le w(k) \perp z(k) \ge 0$$

Well-posedness:

Given x(k), $u(k) \rightarrow x(k+1)$, z(k), w(k) uniquely determined

4.4 Obstruction — global existence: Zenoness

 \rightarrow **Right accumulation** of event times

Reversed Filippov's example:

$$\begin{aligned} \dot{x}_1 &= -\operatorname{sgn}(x_1) + 2\operatorname{sgn}(x_2) \\ \dot{x}_2 &= -2\operatorname{sgn}(x_1) - \operatorname{sgn}(x_2) \end{aligned}$$



 \rightarrow show the existence of the left limit $\lim_{t\uparrow \tau^*} x(t)$ for global existence

Discrete mode is a function of continuous state!

 \Rightarrow not for general hybrid automata!!!

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5.1 Discrete-time LCS (continued) Algebraic problem: **Linear complementarity problem** LCP(q, M): Given vector $q \in \mathbb{R}^m$ and matrix $M \in \mathbb{R}^{m \times m}$ find $z \in \mathbb{R}^m$ such that $0 \leq q + Mz \perp z \geq 0$

 $M \in \mathbb{R}^{m \times m}$ is *P*-matrix, if det $M_{II} > 0$ for all $I \subseteq \{1, \dots, m\}$

Theorem

Discrete-time LCS is well-posed if *D* is a P-matrix

Necessary in case $\operatorname{im}[C F] = \mathbb{R}^n$

5.2 Initial well-posedness for continuous-time LCS Consider LCS:

 $\dot{x}(t) = Ax(t) + Bz(t), \quad w(t) = Cx(t) + Dz(t), \quad 0 \le z(t) \perp w(t) \ge 0$

Define $G(s) := C(s\mathscr{I} - A)^{-1}B + D$ $Q(s) = C(s\mathscr{I} - A)^{-1}$

LCS is initially well-posed if and only if for all x_0 LCP $(Q(\sigma)x_0, G(\sigma))$ is uniquely solvable for sufficiently large $\sigma \in \mathbb{R}$

- *dynamical* properties can now be linked to *static* results on LCPs which are abundant in literature!
- $G(\sigma)$ being P-matrix for sufficiently large σ is sufficient condition for initial well-posedness

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