Modeling & Control of Hybrid Systems Chapter 4 — Stability

Overview

- 1. Switched systems
- 2. Lyapunov theory for smooth and linear systems
- 3. Stability for any switching signal
- 4. Stability for given switching signal

1. Switched systems

 $\dot{x} = f_{\sigma}(x)$

 $\{f_1(x), f_2(x), \dots, f_N(x)\}$ family of smooth vector fields from \mathbb{R}^n to \mathbb{R}^n

Switching signal $\sigma: [0,\infty) \to \{1,\ldots,N\}$ piecewise constant function

- of time *t*: $\sigma(t)$
- of state x(t): $\sigma(x)$
- of time and state: $\sigma(t,x)$
- or extensions involving memory (like hysteresis)

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1.1 Switched linear systems

Switched linear system: $\dot{x} = A_{\sigma}x$

Special case: Piecewise or multi-modal linear system:

Switching is only state-dependent $\dot{x} = A_i x$ when $x \in \mathscr{X}_i$

Well-posedness: cells form partitioning of the state space \mathbb{R}^n (necessary condition only)

 $\bigcup_{i=1}^{n} \mathscr{X}_{i} = \mathbb{R}^{n} \text{ and interior}(\mathscr{X}_{i}) \cap \text{interior}(\mathscr{X}_{j}) = \varnothing$

Piecewise affine (PWA) systems: \mathscr{X}_i polyhedra

 $\dot{x} = A_i x + a_i$, when $E_i x \ge e_i$, $i \in I := \{1, ..., N\}$

1.2 Problem formulation

Global asymptotic stability (GAS) of a system with state *x*: Something like $\lim_{t\to\infty} x(t) = 0$ for all initial states x_0 .

GUAS: global uniform asymptotic stability: uniform in $\boldsymbol{\sigma}$

- **Problem A** : Find conditions for which the switched system is GAS for *any* switching signal (GUAS)
- **Problem B**: Show that the switched system is GAS for a given switching strategy or a class of switching strategies
- **Problem C**: Construct switching signal that makes the switched system GAS (i.e. stabilization problem)
- \rightarrow Problem C will be treated in Chapter 5

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2. Back to basics: Lyapunov theory for stability of smooth systems

Theorem

Let x = 0 be equilibrium of $\dot{x} = f(x)$ (i.e., f(0) = 0) and let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

- $V(x) \rightarrow \infty$ as $||x|| \rightarrow \infty$ (i.e., V is radially unbounded)
- V(0) = 0 and V(x) > 0, if $x \neq 0$ (i.e., V is positive definite), and

•
$$\dot{V}(x) = L_f V(x) := \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial x} f(x) < 0$$
 for all $x \neq 0$,

then system is GAS for x = 0

Under suitable "technical" conditions (mainly smoothness of f): **Converse theorem**: If x = 0 is GAS equilibrium of $\dot{x} = f(x)$ (i.e., f(0) = 0), then there exists radially unbounded Lyapunov function V(x)

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2.1 Stability of linear systems (continued)

Theorem

The following statements are equivalent:

- $\dot{x} = Ax$ is asymptotically stable;
- there is a *quadratic* Lyapunov function $V(x) = x^T P x$ for some positive definite matrix *P* such that $A^T P + P A < 0$

Moreover, for every asymptotically stable *A* and for any Q > 0 there is a P > 0 such that the following Lyapunov equality holds

 $A^T P + P A = -Q$

Note: system is asymptotically stable if A has only eigenvalues in the open left half-plane

2.1 Stability of linear systems

Consider linear system $\dot{x} = Ax$ and consider quadratic Lyapunov function $V(x) = x^{T}Px$ with *P* symmetric ($P = P^{T}$) and positive definite, i.e.,

- $x^{\mathsf{T}} P x > 0$ for all $x \neq 0$
- (if P symmetric) equivalent: all eigenvalues are positive
- (if *P* symmetric) equivalent: all *leading* principal minors det *P*_{JJ} > 0 for all *J* = {1,...,*j*} for *j* = 1,...,*n*

Note that $\dot{V}(x) = L_{Ax}V(x) = x^T(A^TP + PA)x$

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2.2 Connection of stability of nonlinear system and its linearization

Theorem

Let x = a be equilibrium of $\dot{x} = f(x)$ (i.e., f(a) = 0) with $f : D \to \mathbb{R}^n$ continuously differentiable and D a neighborhood of a. Take

$$\mathbf{A} = \frac{\partial f}{\partial x}(x) \Big|_{x=a}$$

- Equilibrium *a* is locally asymptotically stable, if *A* is asymptotically stable (i.e., all eigenvalues in open left half-plane)
- Equilibrium *a* is unstable (not stable), if there is an eigenvalue of *A* that lies in *open right half-plane*

Note: no statements in case all eigenvalues in *closed left half-plane*

2.3 Combining stable dynamics \rightarrow stable?

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 x_2 < 0\\ A_2 x, & \text{if } x_1 x_2 > 0 \end{cases}$$

$$A_1 = \begin{pmatrix} -1 & 10\\ -100 & -1 \end{pmatrix}; A_2 = \begin{pmatrix} -1 & 100\\ -10 & -1 \end{pmatrix} \text{ Eigenvalues } = -1 \pm 31.6j$$

3. Global asymptotic stability for *any* switching signal (\rightarrow GUAS)?

Also for constant switching signals $\sigma(t) = i$ for all t

 $\dot{x} = f_i(x)$ should be globally asymptotically stable

There is a radially unbounded Lyapunov function for each *i*!

∜

 \rightarrow combined system unstable!

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4000

2000

-500 -1000 -1500

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3.1 Common Lyapunov function approach

 \rightarrow Try to find one shared Lyapunov function that decreases along any of the submodels

A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is called *common Lyapunov function* for $\dot{x} = f_{\sigma}(x)$ with $\sigma \in \{1, \dots, N\}$ if

$$\dot{V}(x) = L_{f_i}V(x) = \frac{\partial V}{\partial x}f_i(x) < 0$$
, when $x \neq 0$ and for all $i = 1, \dots, N$

Theorem

If all smooth submodels share positive definite radially unbounded common Lyapunov function, then switched system is globally uniformly asymptotically stable (GUAS)

Converse theorem

Necessary and sufficient condition:

Theorem

If switched system is GUAS, then all f_i share positive definite radially unbounded common Lyapunov function.

Hence, no conservatism in result!

3.2 Switched linear systems: Common *quadratic* Lyapunov function approach

Stability of switched linear systems of the form

 $\dot{x} = A_{\sigma}x, \ \sigma \in \{1, \dots, N\}$

Common Lyapunov function of quadratic type $V(x) = x^T P x$ for positive definite *P*?

$$\dot{V}(x) = L_{f_i}V(x) := \frac{\partial V}{\partial x}f_i(x) = x^{\mathsf{T}}[PA_i + A_i^TP]x < 0 \text{ for all } x \neq 0 \text{ and } i$$

Hence, we obtain linear matrix inequalities (LMIs)

$$A_i^T P + PA_i < 0$$
 for all $i = 1, \dots, N$ and $P > 0$

Quadratic stability: there exists a quadratic Lyapunov function $V(x) = x^T P x$ with $\dot{V}(x) \leq -\varepsilon ||x||^2$ for some $\varepsilon > 0$

Converse quadratic Lyapunov function theorem?

Asymptotic stability of switched linear system $\dot{x} = A_{\sigma}x \Rightarrow$ existence of common quadratic Lyapunov function???

Answer is negative

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

is GUAS, but no common quadratic Lyapunov function by infeasibility condition

$$R_1 = \begin{pmatrix} 0.2996 & 0.7048 \\ 0.7048 & 2.4704 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0.2123 & -0.5532 \\ -0.5532 & 1.9719 \end{pmatrix}$$

However, there is common Lyapunov function of form $V(x) = \max_{i=1,2,...,k} (l_i^T x)^2$

Infeasibility test for common quadratic Lyapunov function

$$A_i^T P + P A_i < 0$$
 for all $i = 1, \dots, N$ and $P > 0$

Dual theorem:

The set of LMIs is infeasible (i.e., no quadratic stability) if and only if there exist positive definite matrices R_i , i = 1, ..., N such that

$$\sum_{i=1}^{N} (A_i^T R_i + R_i A_i) > 0$$

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Conditions for existence of common *quadratic* Lyapunov function

Theorem

If matrices $\{A_1, ..., A_N\}$ commute pairwise (i.e., $A_iA_j = A_jA_i$) for all i, j and are all stable, then there exists common quadratic Lyapunov function $P = P_N$, that can be found from solving following set of Lyapunov equalities successively:

$$A_{1}^{T}P_{1} + P_{1}A_{1} = -I$$

$$A_{2}^{T}P_{2} + P_{2}A_{2} = -P_{1}$$

$$A_{3}^{T}P_{3} + P_{3}A_{3} = -P_{2}$$

$$\vdots$$

$$A_{N}^{T}P_{N} + P_{N}A_{N} = -P_{N-1}$$

More involved conditions exist (cf. references in lecture notes!)

4. Global asymptotic stability for given switching strategy?

4.1 Multiple Lyapunov approach

Switched system with $\dot{x} = f_i(x)$, i = 1, 2 are GAS with Lyapunov function $V_i(x)$

Assumption: no common Lyapunov function $\ \rightarrow \ not \ GUAS$

Let switching times be given by t_k , k = 0, 1, 2, ... and suppose that

$$V_{\sigma(t_{k-1})}(x(t_k)) = V_{\sigma(t_k)}(x(t_k))$$
 for all $k = 1, 2, ...$

 V_{σ} is now continuous Lyapunov function \Rightarrow switched system is GAS



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4.3 State-dependent switchings: Single Lyapunov function

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 x_2 \leq 0\\ A_2 x, & \text{if } x_1 x_2 > 0 \end{cases} \text{ with } A_1 = \begin{pmatrix} -1 & -1\\ 1 & -1 \end{pmatrix}; A_2 = \begin{pmatrix} -1 & -10\\ 0.1 & -1 \end{pmatrix}$$

• No common quadratic Lyapunov function

• However, for $V(x) = x_1^2 + x_2^2$ it holds that $\dot{V} < 0$ along the nonzero solutions of the switched system, which implies GAS

Relaxation w.r.t. common Lyapunov function approach: Indeed, we only need

$$L_{A_1x}V(x) < 0$$
 if $x_1x_2 \leqslant 0$ and $L_{A_2x}V(x) < 0$ if $x_1x_2 > 0$

Hence, general set-up:

Find V such that $L_{f_i}V(x)$ is only negative where $\dot{x} = f_i(x)$ can be active hs_stab.19

4.2 Most general theorem

Theorem

Consider switched system with all submodels $\dot{x} = f_i(x)$ GAS with corresponding Lyapunov function V_i

Suppose that for every pair of switching times (t_k, t_l) , k < l with $\sigma(t_k) = \sigma(t_l) = i$ and $\sigma(t_m) \neq i$ for $t_k < t_m < t_l$, we have

$$V_i(x(t_l)) - V_i(x(t_k)) \leq -\rho(||x(t_k)||) < 0,$$

then switched system is GAS



4.4 State-dependent switchings: Multiple Lyapunov function

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 \leq 0\\ A_2 x, & \text{if } x_1 > 0, \end{cases} \text{ where } A_1 = \begin{pmatrix} -5 & -4\\ -1 & -2 \end{pmatrix}; A_2 = \begin{pmatrix} -2 & -4\\ 20 & -2 \end{pmatrix}$$

No common Lyapunov function and no quadratic function as in previous example

However, consider 2 quadratic Lyapunov functions $V_i(x) = x^T P_i x$ with

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}$$

 V_i is Lyapunov function for $\dot{x} = A_i x$

 V_{σ} (with $\sigma=1$ if $\mathit{x}_1\leqslant 0$ and $\sigma=2$ when $\mathit{x}_1>0$) is continuous and strictly decreasing

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4.5 More general set-up for piecewise linear systems

$$\dot{x} = A_i x \text{ if } x \in \mathscr{X}_i$$

Several relaxations possible w.r.t. common *quadratic* Lyapunov function:

- One can require that derivative $L_{f_i(x)}V(x)$ of $V(x) = x^T P x$ is only negative in region where subsystem is active
- One can use multiple Lyapunov functions, say $V_i(x) = x^T P_i x$, for each submodel and "connect them" in a suitable way
- One can require that the Lyapunov function $V_i(x) = x^T P_i x$ is only positive definite in its active region

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4.6 Relaxation: S-procedure

Aim: $V(x) = x^T P x$, P > 0 such that $x^T [A_i^T P + P A_i] x < 0$ for $0 \neq x \in \mathscr{X}_i$

Find: $S_i(x)$ based on \mathscr{X}_i with $S_i(x) \ge 0$ when $x \in \mathscr{X}_i$

Next: search for $\beta \ge 0$ satisfying

 $x^{T}A_{i}^{T}Px + x^{T}PA_{i}x + \beta S_{i}(x) < 0$ for all x

Result: Since $S_i(x)$ might be negative outside \mathscr{X}_i , so less conservative than $A_i^T P + PA_i < 0$ (i.e., $x^T A_i^T P x + x^T PA_i x < 0$ for all x)

Computationally interesting: $S_i(x) = x^T S_i x$, then LMI:

Find $\beta_i \ge 0$ and P > 0 such that $A_i^T P + PA_i + \beta_i S_i < 0$

+ other relaxations (cf. lecture notes)