

# Modeling & Control of Hybrid Systems

## Chapter 4 — Stability

### Overview

1. Switched systems
2. Lyapunov theory for smooth and linear systems
3. Stability for *any* switching signal
4. Stability for *given* switching signal

hs\_stab.1

### 1.1 Switched linear systems

**Switched linear system:**  $\dot{x} = A_{\sigma}x$

**Special case: Piecewise or multi-modal linear system:**

Switching is only state-dependent  $\dot{x} = A_i x$  when  $x \in \mathcal{X}_i$

Well-posedness: cells form partitioning of the state space  $\mathbb{R}^n$  (necessary condition only)

$$\bigcup_{i=1}^n \mathcal{X}_i = \mathbb{R}^n \text{ and } \text{interior}(\mathcal{X}_i) \cap \text{interior}(\mathcal{X}_j) = \emptyset$$

Piecewise affine (PWA) systems:  $\mathcal{X}_i$  polyhedra

$$\dot{x} = A_i x + a_i, \text{ when } E_i x \geq e_i, \quad i \in I := \{1, \dots, N\}$$

hs\_stab.3

### 1. Switched systems

$$\dot{x} = f_{\sigma}(x)$$

$\{f_1(x), f_2(x), \dots, f_N(x)\}$  family of smooth vector fields from  $\mathbb{R}^n$  to  $\mathbb{R}^n$

Switching signal  $\sigma : [0, \infty) \rightarrow \{1, \dots, N\}$  piecewise constant function

- of time  $t$ :  $\sigma(t)$
- of state  $x(t)$ :  $\sigma(x)$
- of time and state:  $\sigma(t, x)$
- or extensions involving memory (like hysteresis)

hs\_stab.2

### 1.2 Problem formulation

Global asymptotic stability (GAS) of a system with state  $x$ :

Something like  $\lim_{t \rightarrow \infty} x(t) = 0$  for all initial states  $x_0$ .

GUAS: global uniform asymptotic stability: uniform in  $\sigma$

**Problem A :** Find conditions for which the switched system is GAS for *any* switching signal (GUAS)

**Problem B :** Show that the switched system is GAS for a given switching strategy or a class of switching strategies

**Problem C :** Construct switching signal that makes the switched system GAS (i.e. stabilization problem)

→ Problem C will be treated in Chapter 5

hs\_stab.4

## 2. Back to basics: Lyapunov theory for stability of smooth systems

### Theorem

Let  $x = 0$  be equilibrium of  $\dot{x} = f(x)$  (i.e.,  $f(0) = 0$ ) and let  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuously differentiable function such that

- $V(x) \rightarrow \infty$  as  $\|x\| \rightarrow \infty$  (i.e.,  $V$  is *radially unbounded*)
- $V(0) = 0$  and  $V(x) > 0$ , if  $x \neq 0$  (i.e.,  $V$  is positive definite), and
- $\dot{V}(x) = L_f V(x) := \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial x} f(x) < 0$  for all  $x \neq 0$ ,

then system is GAS for  $x = 0$

Under suitable “technical” conditions (mainly smoothness of  $f$ ):

**Converse theorem:** If  $x = 0$  is GAS equilibrium of  $\dot{x} = f(x)$  (i.e.,  $f(0) = 0$ ), then there exists radially unbounded Lyapunov function  $V(x)$

hs\_stab.5

## 2.1 Stability of linear systems (continued)

### Theorem

The following statements are equivalent:

- $\dot{x} = Ax$  is asymptotically stable;
- there is a *quadratic* Lyapunov function  $V(x) = x^T P x$  for some positive definite matrix  $P$  such that  $A^T P + PA < 0$

Moreover, for every asymptotically stable  $A$  and for any  $Q > 0$  there is a  $P > 0$  such that the following Lyapunov equality holds

$$A^T P + PA = -Q$$

Note: system is asymptotically stable if  $A$  has only eigenvalues in the open left half-plane

hs\_stab.7

## 2.1 Stability of linear systems

Consider linear system  $\dot{x} = Ax$  and consider quadratic Lyapunov function  $V(x) = x^T P x$  with  $P$  symmetric ( $P = P^T$ ) and positive definite, i.e.,

- $x^T P x > 0$  for all  $x \neq 0$
- (if  $P$  symmetric) equivalent: all eigenvalues are positive
- (if  $P$  symmetric) equivalent: all *leading* principal minors  $\det P_{JJ} > 0$  for all  $J = \{1, \dots, j\}$  for  $j = 1, \dots, n$

Note that  $\dot{V}(x) = L_{Ax} V(x) = x^T (A^T P + PA)x$

hs\_stab.6

## 2.2 Connection of stability of nonlinear system and its linearization

### Theorem

Let  $x = a$  be equilibrium of  $\dot{x} = f(x)$  (i.e.,  $f(a) = 0$ ) with  $f : D \rightarrow \mathbb{R}^n$  continuously differentiable and  $D$  a neighborhood of  $a$ . Take

$$A = \left. \frac{\partial f}{\partial x}(x) \right|_{x=a}$$

- Equilibrium  $a$  is locally asymptotically stable, if  $A$  is asymptotically stable (i.e., all eigenvalues in open left half-plane)
- Equilibrium  $a$  is unstable (not stable), if there is an eigenvalue of  $A$  that lies in *open right half-plane*

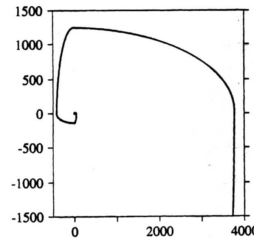
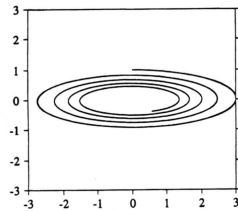
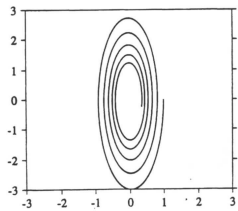
Note: no statements in case all eigenvalues in *closed left half-plane*

hs\_stab.8

## 2.3 Combining stable dynamics → stable?

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 x_2 < 0 \\ A_2 x, & \text{if } x_1 x_2 > 0 \end{cases}$$

$$A_1 = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix}; A_2 = \begin{pmatrix} -1 & 100 \\ -10 & -1 \end{pmatrix} \text{ Eigenvalues } = -1 \pm 31.6j$$



→ combined system unstable!

hs\_stab.9

## 3.1 Common Lyapunov function approach

→ Try to find one shared Lyapunov function that decreases along any of the submodels

A  $C^1$  function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is called *common Lyapunov function* for  $\dot{x} = f_\sigma(x)$  with  $\sigma \in \{1, \dots, N\}$  if

$$\dot{V}(x) = L_{f_i} V(x) = \frac{\partial V}{\partial x} f_i(x) < 0, \text{ when } x \neq 0 \text{ and for all } i = 1, \dots, N$$

### Theorem

If all smooth submodels share positive definite radially unbounded common Lyapunov function, then switched system is globally uniformly asymptotically stable (GUAS)

hs\_stab.11

## 3. Global asymptotic stability for *any* switching signal (→ GUAS)?

Also for constant switching signals  $\sigma(t) = i$  for all  $t$

↓

$\dot{x} = f_i(x)$  should be globally asymptotically stable

↓

There is a radially unbounded Lyapunov function for each  $i$ !

hs\_stab.10

## Converse theorem

Necessary and sufficient condition:

### Theorem

If switched system is GUAS, then all  $f_i$  share positive definite radially unbounded common Lyapunov function.

Hence, no conservatism in result!

hs\_stab.12

### 3.2 Switched linear systems: Common quadratic Lyapunov function approach

Stability of switched linear systems of the form

$$\dot{x} = A_{\sigma}x, \quad \sigma \in \{1, \dots, N\}$$

Common Lyapunov function of quadratic type  $V(x) = x^T P x$  for positive definite  $P$ ?

$$\dot{V}(x) = L_{f_i} V(x) := \frac{\partial V}{\partial x} f_i(x) = x^T [P A_i + A_i^T P] x < 0 \text{ for all } x \neq 0 \text{ and } i$$

Hence, we obtain **linear matrix inequalities** (LMIs)

$$A_i^T P + P A_i < 0 \text{ for all } i = 1, \dots, N \text{ and } P > 0$$

**Quadratic stability:** there exists a quadratic Lyapunov function  $V(x) = x^T P x$  with  $\dot{V}(x) \leq -\varepsilon \|x\|^2$  for some  $\varepsilon > 0$

hs\_stab.13

### Converse quadratic Lyapunov function theorem?

Asymptotic stability of switched linear system  $\dot{x} = A_{\sigma}x \Rightarrow$  existence of common quadratic Lyapunov function???

Answer is negative

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

is GUAS, but no common quadratic Lyapunov function by infeasibility condition

$$R_1 = \begin{pmatrix} 0.2996 & 0.7048 \\ 0.7048 & 2.4704 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0.2123 & -0.5532 \\ -0.5532 & 1.9719 \end{pmatrix}$$

However, there is common Lyapunov function of form

$$V(x) = \max_{i=1,2,\dots,k} (l_i^T x)^2$$

hs\_stab.15

### Infeasibility test for common quadratic Lyapunov function

$$A_i^T P + P A_i < 0 \quad \text{for all } i = 1, \dots, N \text{ and } P > 0$$

#### Dual theorem:

The set of LMIs is infeasible (i.e., no quadratic stability) if and only if there exist positive definite matrices  $R_i$ ,  $i = 1, \dots, N$  such that

$$\sum_{i=1}^N (A_i^T R_i + R_i A_i) > 0$$

hs\_stab.14

### Conditions for existence of common quadratic Lyapunov function

#### Theorem

If matrices  $\{A_1, \dots, A_N\}$  commute pairwise (i.e.,  $A_i A_j = A_j A_i$ ) for all  $i, j$  and are all stable, then there exists common quadratic Lyapunov function  $P = P_N$ , that can be found from solving following set of Lyapunov equalities successively:

$$\begin{aligned} A_1^T P_1 + P_1 A_1 &= -I \\ A_2^T P_2 + P_2 A_2 &= -P_1 \\ A_3^T P_3 + P_3 A_3 &= -P_2 \\ &\vdots \\ A_N^T P_N + P_N A_N &= -P_{N-1} \end{aligned}$$

More involved conditions exist (cf. references in lecture notes!)

hs\_stab.16

## 4. Global asymptotic stability for given switching strategy?

### 4.1 Multiple Lyapunov approach

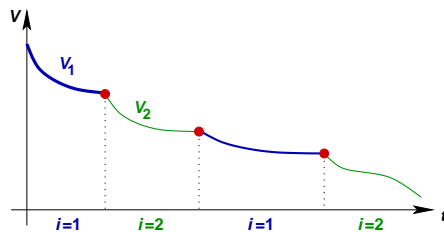
Switched system with  $\dot{x} = f_i(x)$ ,  $i = 1, 2$  are GAS with Lyapunov function  $V_i(x)$

Assumption: no common Lyapunov function  $\rightarrow$  not GUAS

Let switching times be given by  $t_k$ ,  $k = 0, 1, 2, \dots$  and suppose that

$$V_{\sigma(t_{k-1})}(x(t_k)) = V_{\sigma(t_k)}(x(t_k)) \text{ for all } k = 1, 2, \dots$$

$V_\sigma$  is now continuous Lyapunov function  $\Rightarrow$  switched system is GAS



hs\_stab.17

### 4.3 State-dependent switchings: Single Lyapunov function

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 x_2 \leq 0 \\ A_2 x, & \text{if } x_1 x_2 > 0 \end{cases} \text{ with } A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}; A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

- No common quadratic Lyapunov function
- However, for  $V(x) = x_1^2 + x_2^2$  it holds that  $\dot{V} < 0$  along the nonzero solutions of the switched system, which implies GAS

**Relaxation w.r.t. common Lyapunov function approach:** Indeed, we only need

$$L_{A_1 x} V(x) < 0 \text{ if } x_1 x_2 \leq 0 \text{ and } L_{A_2 x} V(x) < 0 \text{ if } x_1 x_2 > 0$$

Hence, general set-up:

Find  $V$  such that  $L_{f_i} V(x)$  is only negative where  $\dot{x} = f_i(x)$  can be active

hs\_stab.19

## 4.2 Most general theorem

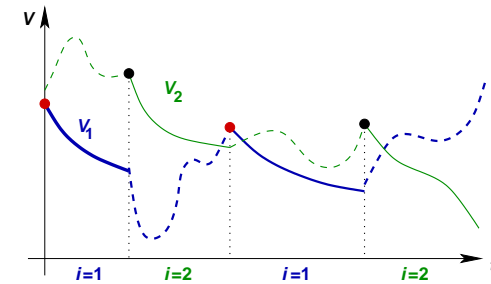
### Theorem

Consider switched system with all submodels  $\dot{x} = f_i(x)$  GAS with corresponding Lyapunov function  $V_i$

Suppose that for every pair of switching times  $(t_k, t_l)$ ,  $k < l$  with  $\sigma(t_k) = \sigma(t_l) = i$  and  $\sigma(t_m) \neq i$  for  $t_k < t_m < t_l$ , we have

$$V_i(x(t_l)) - V_i(x(t_k)) \leq -\rho(\|x(t_k)\|) < 0,$$

then switched system is GAS



hs\_stab.18

### 4.4 State-dependent switchings: Multiple Lyapunov function

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 \leq 0 \\ A_2 x, & \text{if } x_1 > 0, \end{cases} \text{ where } A_1 = \begin{pmatrix} -5 & -4 \\ -1 & -2 \end{pmatrix}; A_2 = \begin{pmatrix} -2 & -4 \\ 20 & -2 \end{pmatrix}$$

No common Lyapunov function and no quadratic function as in previous example

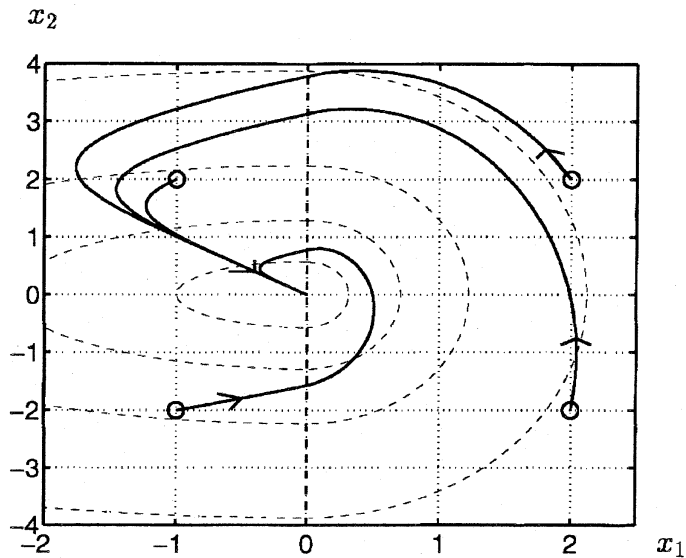
However, consider 2 quadratic Lyapunov functions  $V_i(x) = x^T P_i x$  with

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}$$

$V_i$  is Lyapunov function for  $\dot{x} = A_i x$

$V_\sigma$  (with  $\sigma = 1$  if  $x_1 \leq 0$  and  $\sigma = 2$  when  $x_1 > 0$ ) is continuous and strictly decreasing

hs\_stab.20



hs\_stab.21

## 4.5 More general set-up for piecewise linear systems

$$\dot{x} = A_i x \text{ if } x \in \mathcal{X}_i$$

Several relaxations possible w.r.t. common *quadratic* Lyapunov function:

- One can require that derivative  $L_{f_i(x)}V(x)$  of  $V(x) = x^T P x$  is only negative in region where subsystem is active
- One can use multiple Lyapunov functions, say  $V_i(x) = x^T P_i x$ , for each submodel and “connect them” in a suitable way
- One can require that the Lyapunov function  $V_i(x) = x^T P_i x$  is only positive definite in its active region

hs\_stab.22

## 4.6 Relaxation: S-procedure

**Aim:**  $V(x) = x^T P x$ ,  $P > 0$  such that  $x^T [A_i^T P + P A_i] x < 0$  for  $0 \neq x \in \mathcal{X}_i$

**Find:**  $S_i(x)$  based on  $\mathcal{X}_i$  with  $S_i(x) \geq 0$  when  $x \in \mathcal{X}_i$

**Next:** search for  $\beta \geq 0$  satisfying

$$x^T A_i^T P x + x^T P A_i x + \beta S_i(x) < 0 \text{ for all } x$$

**Result:** Since  $S_i(x)$  might be negative outside  $\mathcal{X}_i$ , so less conservative than  $A_i^T P + P A_i < 0$  (i.e.,  $x^T A_i^T P x + x^T P A_i x < 0$  for all  $x$ )

**Computationally interesting:**  $S_i(x) = x^T S_i x$ , then LMI:

$$\text{Find } \beta_i \geq 0 \text{ and } P > 0 \text{ such that } A_i^T P + P A_i + \beta_i S_i < 0$$

+ other relaxations (cf. lecture notes)

hs\_stab.23