Modeling & Control of Hybrid Systems

Chapter 4 — Stability

Overview

1. Switched systems
2. Lyapunov theory for smooth and linear systems
3. Stability for *any* switching signal
4. Stability for *given* switching signal

1. Switched systems

$$\dot{x} = f_\sigma(x)$$

$\{f_1(x), f_2(x), \ldots, f_N(x)\}$ family of smooth vector fields from $\mathbb{R}^n$ to $\mathbb{R}^n$

Switching signal $\sigma : [0, \infty) \to \{1, \ldots, N\}$ piecewise constant function

- of time t: $\sigma(t)$
- of state $x(t)$: $\sigma(x)$
- of time and state: $\sigma(t, x)$
- or extensions involving memory (like hysteresis)

1.1 Switched linear systems

Switched linear system: $\dot{x} = A_\sigma x$

Special case: Piecewise or multi-modal linear system:
Switching is only state-dependent $\dot{x} = A_i x$ when $x \in \mathscr{X}_i$

Well-posedness: cells form partitioning of the state space $\mathbb{R}^n$ (necessary condition only)

$$\bigcup_{i=1}^{n} \mathscr{X}_i = \mathbb{R}^n \text{ and } \mathrm{interior}(\mathscr{X}_i) \cap \mathrm{interior}(\mathscr{X}_j) = \varnothing$$

Piecewise affine (PWA) systems: $\mathscr{X}_i$ polyhedra

$$\dot{x} = A_i x + a_i, \text{ when } E_i x \geqslant e_i, \; i \in I := \{1, \ldots, N\}$$

1.2 Problem formulation

Global asymptotic stability (GAS) of a system with state x:

Something like $\lim_{t \to \infty} x(t) = 0$ for all initial states x_0.

GUAS: global uniform asymptotic stability: uniform in σ

Problem A : Find conditions for which the switched system is GAS for *any* switching signal (GUAS)

Problem B : Show that the switched system is GAS for a given switching strategy or a class of switching strategies

Problem C : Construct switching signal that makes the switched system GAS (i.e. stabilization problem)

$\rightarrow$ Problem C will be treated in Chapter 5

2. Back to basics: Lyapunov theory for stability of smooth systems

Theorem

Let $x = 0$ be equilibrium of $\dot{x} = f(x)$ (i.e., $f(0) = 0$) and let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

- $V(x) \to \infty$ as $\|x\| \to \infty$ (i.e., V is *radially unbounded*)
- $V(0) = 0$ and $V(x) > 0$, if $x \neq 0$ (i.e., V is positive definite), and
- $\dot{V}(x) = L_f V(x) := \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x) < 0$ for all $x \neq 0$,

then system is GAS for $x = 0$

Under suitable "technical" conditions (mainly smoothness of f):

Converse theorem: If $x = 0$ is GAS equilibrium of $\dot{x} = f(x)$ (i.e., $f(0) = 0$), then there exists radially unbounded Lyapunov function $V(x)$

2.1 Stability of *linear* systems

Consider linear system $\dot{x} = Ax$ and consider quadratic Lyapunov function $V(x) = x^{\mathsf{T}} P x$ with P symmetric ($P = P^T$) and positive definite, i.e.,

- $x^{\mathsf{T}} P x > 0$ for all $x \neq 0$
- (if P symmetric) equivalent: all eigenvalues are positive
- (if P symmetric) equivalent: all *leading* principal minors $\det P_{JJ} > 0$ for all $J = \{1, \dots, j\}$ for $j = 1, \dots, n$

Note that $\dot{V}(x) = L_{Ax} V(x) = x^T (A^T P + PA) x$

2.1 Stability of *linear* systems (continued)

Theorem

The following statements are equivalent:

- $\dot{x} = Ax$ is asymptotically stable;
- there is a *quadratic* Lyapunov function $V(x) = x^T P x$ for some positive definite matrix P such that $A^T P + PA < 0$

Moreover, for every asymptotically stable A and for any $Q > 0$ there is a $P > 0$ such that the following Lyapunov equality holds

$$A^T P + PA = -Q$$

Note: system is asymptotically stable if A has only eigenvalues in the open left half-plane

2.2 Connection of stability of nonlinear system and its linearization

Theorem

Let $x = a$ be equilibrium of $\dot{x} = f(x)$ (i.e., $f(a) = 0$) with $f : D \to \mathbb{R}^n$ continuously differentiable and D a neighborhood of a. Take

$$A = \frac{\partial f}{\partial x}(x)\Big|_{x=a}$$

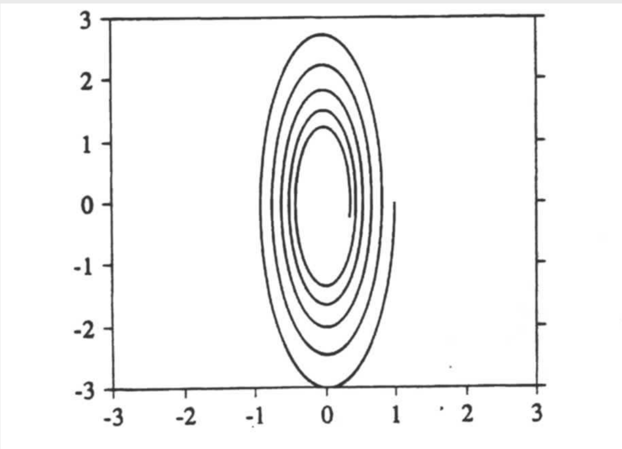
- Equilibrium a is locally asymptotically stable, if A is asymptotically stable (i.e., all eigenvalues in open left half-plane)
- Equilibrium a is unstable (not stable), if there is an eigenvalue of A that lies in *open right half-plane*

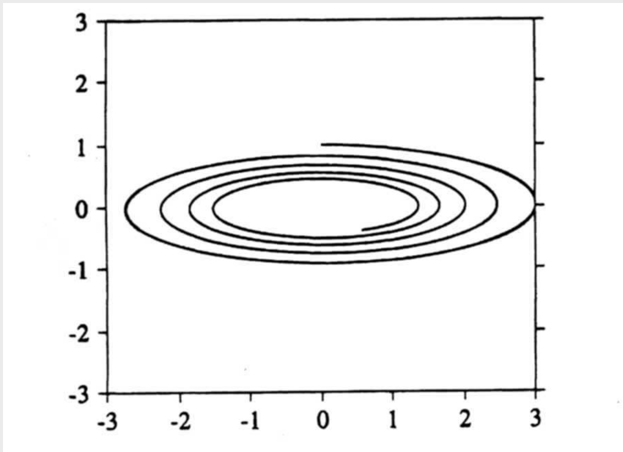
Note: no statements in case all eigenvalues in *closed left half-plane*

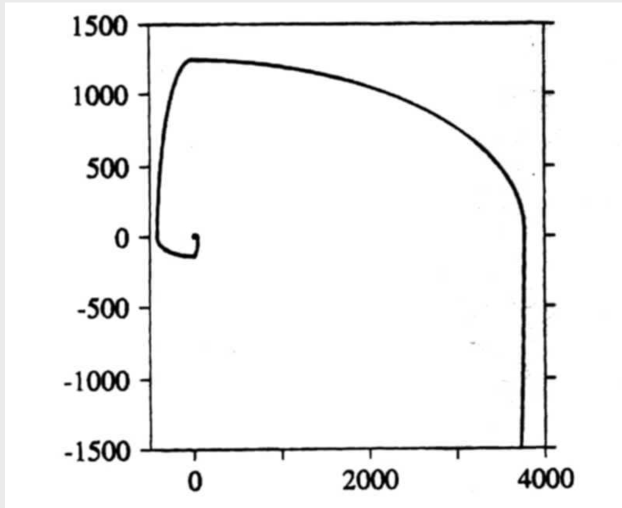
2.3 Combining stable dynamics $\rightarrow$ stable?

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 x_2 < 0 \\ A_2 x, & \text{if } x_1 x_2 > 0 \end{cases}$$

$$A_1 = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix};\ A_2 = \begin{pmatrix} -1 & 100 \\ -10 & -1 \end{pmatrix} \text{ Eigenvalues } = -1 \pm 31.6j$$







$\rightarrow$ combined system unstable!

3. Global asymptotic stability for *any* switching signal ($\rightarrow$ GUAS)?

Also for constant switching signals $\sigma(t) = i$ for all t

$$\Downarrow$$

$\dot{x} = f_i(x)$ should be globally asymptotically stable

$$\Downarrow$$

There is a radially unbounded Lyapunov function for each i!

3.1 Common Lyapunov function approach

$\rightarrow$ Try to find one shared Lyapunov function that decreases along any of the submodels

A C^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is called *common Lyapunov function* for $\dot{x} = f_\sigma(x)$ with $\sigma \in \{1, \ldots, N\}$ if

$$\dot{V}(x) = L_{f_i}V(x) = \frac{\partial V}{\partial x} f_i(x) < 0, \text{ when } x \neq 0 \text{ and for all } i = 1, \ldots, N$$

Theorem

If all smooth submodels share positive definite radially unbounded common Lyapunov function, then switched system is globally uniformly asymptotically stable (GUAS)

Converse theorem

Necessary and sufficient condition:

Theorem

If switched system is GUAS, then all f_i share positive definite radially unbounded common Lyapunov function.

Hence, no conservatism in result!

3.2 Switched linear systems: Common *quadratic* Lyapunov function approach

Stability of switched linear systems of the form

$$\dot{x} = A_\sigma x, \quad \sigma \in \{1, \ldots, N\}$$

Common Lyapunov function of quadratic type $V(x) = x^T P x$ for positive definite P?

$$\dot{V}(x) = L_{f_i} V(x) := \frac{\partial V}{\partial x} f_i(x) = x^\top [P A_i + A_i^T P] x < 0 \text{ for all } x \neq 0 \text{ and } i$$

Hence, we obtain **linear matrix inequalities** (LMIs)

$$A_i^T P + P A_i < 0 \text{ for all } i = 1, \ldots, N \text{ and } P > 0$$

Quadratic stability: there exists a quadratic Lyapunov function $V(x) = x^T P x$ with $\dot{V}(x) \leqslant -\varepsilon \|x\|^2$ for some $\varepsilon > 0$

Infeasibility test for common quadratic Lyapunov function

$$A_i^T P + P A_i < 0 \quad \text{for all } i = 1, \ldots, N \text{ and } P > 0$$

Dual theorem:
The set of LMIs is infeasible (i.e., no quadratic stability) if and only if there exist positive definite matrices R_i, $i = 1, \ldots, N$ such that

$$\sum_{i=1}^{N} (A_i^T R_i + R_i A_i) > 0$$

Converse *quadratic* Lyapunov function theorem?

Asymptotic stability of switched linear system $\dot{x} = A_\sigma x \Rightarrow$ existence of common quadratic Lyapunov function???

Answer is negative

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

is GUAS, but no common quadratic Lyapunov function by infeasibility condition

$$R_1 = \begin{pmatrix} 0.2996 & 0.7048 \\ 0.7048 & 2.4704 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0.2123 & -0.5532 \\ -0.5532 & 1.9719 \end{pmatrix}$$

However, there is common Lyapunov function of form
$V(x) = \max_{i=1,2,\ldots,k} (l_i^T x)^2$

Conditions for existence of common *quadratic* Lyapunov function

Theorem

If matrices $\{A_1,\ldots,A_N\}$ commute pairwise (i.e., $A_iA_j = A_jA_i$) for all i,j and are all stable, then there exists common quadratic Lyapunov function $P = P_N$, that can be found from solving following set of Lyapunov equalities successively:

$$
\begin{aligned}
A_1^T P_1 + P_1 A_1 &= -I \\
A_2^T P_2 + P_2 A_2 &= -P_1 \\
A_3^T P_3 + P_3 A_3 &= -P_2 \\
&\vdots \\
A_N^T P_N + P_N A_N &= -P_{N-1}
\end{aligned}
$$

More involved conditions exist (cf. references in lecture notes!)

4. Global asymptotic stability for *given* switching strategy?

4.1 Multiple Lyapunov approach

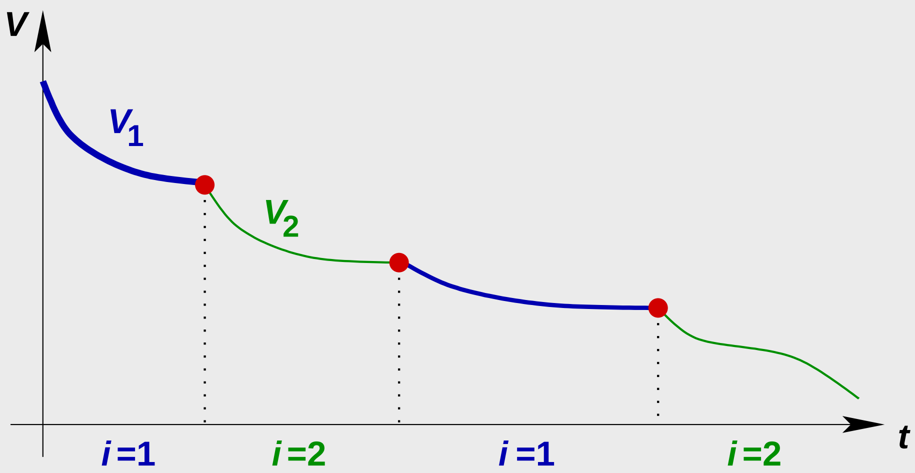
Switched system with $\dot{x} = f_i(x)$, $i = 1, 2$ are GAS with Lyapunov function $V_i(x)$

Assumption: no common Lyapunov function $\rightarrow$ not GUAS

Let switching times be given by t_k, $k = 0, 1, 2, \ldots$ and suppose that

$$V_{\sigma(t_{k-1})}(x(t_k)) = V_{\sigma(t_k)}(x(t_k)) \text{ for all } k = 1, 2, \ldots$$

V_σ is now continuous Lyapunov function $\Rightarrow$ switched system is GAS



4.2 Most general theorem

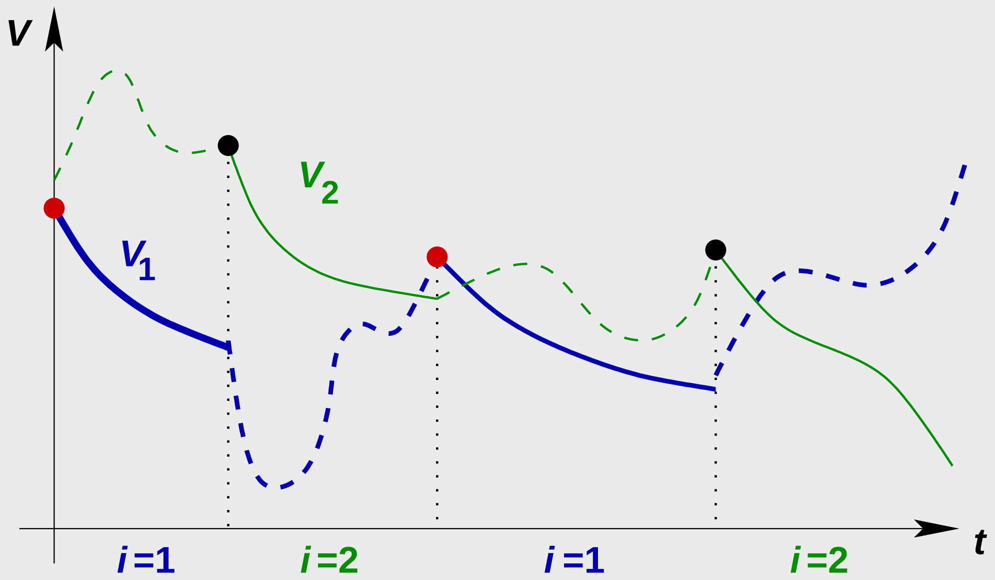
Theorem

Consider switched system with all submodels $\dot{x} = f_i(x)$ GAS with corresponding Lyapunov function V_i

Suppose that for every pair of switching times (t_k, t_l), $k < l$ with $\sigma(t_k) = \sigma(t_l) = i$ and $\sigma(t_m) \neq i$ for $t_k < t_m < t_l$, we have

$$V_i(x(t_l)) - V_i(x(t_k)) \leqslant -\rho(\|x(t_k)\|) < 0,$$

then switched system is GAS



4.3 State-dependent switchings: Single Lyapunov function

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 x_2 \leqslant 0 \\ A_2 x, & \text{if } x_1 x_2 > 0 \end{cases} \quad \text{with } A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix};\ A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

• No common quadratic Lyapunov function
• However, for $V(x) = x_1^2 + x_2^2$ it holds that $\dot{V} < 0$ along the nonzero solutions of the switched system, which implies GAS

Relaxation w.r.t. common Lyapunov function approach: Indeed, we only need

$$L_{A_1 x} V(x) < 0 \text{ if } x_1 x_2 \leqslant 0 \text{ and } L_{A_2 x} V(x) < 0 \text{ if } x_1 x_2 > 0$$

Hence, general set-up:

Find V such that $L_{f_i} V(x)$ is only negative where $\dot{x} = f_i(x)$ can be active

4.4 State-dependent switchings: Multiple Lyapunov function

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x_1 \leqslant 0 \\ A_2 x, & \text{if } x_1 > 0, \end{cases} \quad \text{where } A_1 = \begin{pmatrix} -5 & -4 \\ -1 & -2 \end{pmatrix};\ A_2 = \begin{pmatrix} -2 & -4 \\ 20 & -2 \end{pmatrix}$$

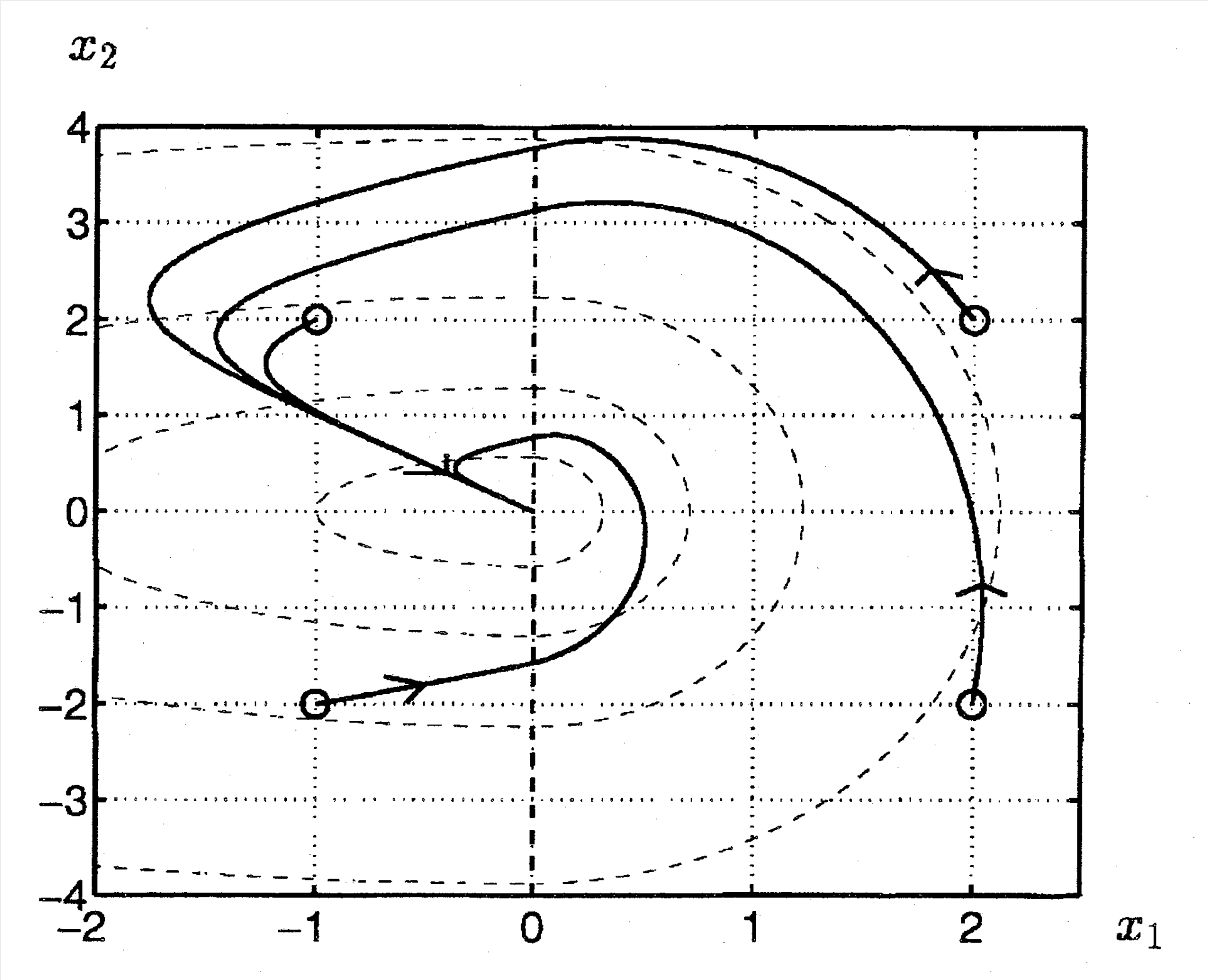
No common Lyapunov function and no quadratic function as in previous example

However, consider 2 quadratic Lyapunov functions $V_i(x) = x^\top P_i x$ with

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}$$

V_i is Lyapunov function for $\dot{x} = A_i x$

V_σ (with $\sigma = 1$ if $x_1 \leqslant 0$ and $\sigma = 2$ when $x_1 > 0$) is continuous and strictly decreasing


x_2
4
3
2
1
0
−1
−2
−3
−4
−2
−1
0
1
2
x_1

4.5 More general set-up for piecewise linear systems

$$\dot{x} = A_i x \text{ if } x \in \mathscr{X}_i$$

Several relaxations possible w.r.t. common *quadratic* Lyapunov function:

- One can require that derivative $L_{f_i(x)}V(x)$ of $V(x) = x^T P x$ is only negative in region where subsystem is active
- One can use multiple Lyapunov functions, say $V_i(x) = x^T P_i x$, for each submodel and "connect them" in a suitable way
- One can require that the Lyapunov function $V_i(x) = x^T P_i x$ is only positive definite in its active region

4.6 Relaxation: S-procedure

Aim: $V(x) = x^T P x$, $P > 0$ such that $x^T[A_i^T P + PA_i]x < 0$ for $0 \neq x \in \mathscr{X}_i$

Find: $S_i(x)$ based on $\mathscr{X}_i$ with $S_i(x) \geqslant 0$ when $x \in \mathscr{X}_i$

Next: search for $\beta \geqslant 0$ satisfying

$$x^T A_i^T P x + x^T P A_i x + \beta S_i(x) < 0 \text{ for all } x$$

Result: Since $S_i(x)$ might be negative outside $\mathscr{X}_i$, so less conservative than $A_i^T P + PA_i < 0$ (i.e., $x^T A_i^T P x + x^T P A_i x < 0$ for all x)

Computationally interesting: $S_i(x) = x^T S_i x$, then LMI:

$$\text{Find } \beta_i \geqslant 0 \text{ and } P > 0 \text{ such that } A_i^T P + PA_i + \beta_i S_i < 0$$

+ other relaxations (cf. lecture notes)