Modeling & Control of Hybrid Systems

Chapter 5 – Switched Control

Overview

1. Introduction & motivation for hybrid control

Several "classical" control methods for continuous-time systems are hybrid:

- variable structure control
- sliding mode control
- relay control
- gain scheduling
- bang-bang time-optimal control
- fuzzy control

$\longrightarrow$ common characteristic: **switching**

1.1 Motivation for switched controllers

Theorem (Brockett's necessary condition)

Consider system

$$\dot{x} = f(x,u) \qquad \text{with } x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ f(0,0) = 0$$

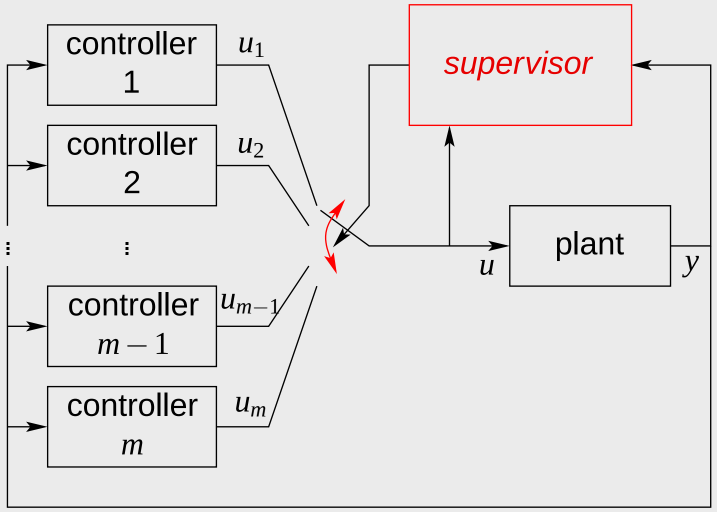
where f is smooth function.
If system is asymptotically stabilizable (around $x = 0$) using *continuous* feedback law $u = \alpha(x)$,
then image of every open neighborhood of $(x,u) = (0,0)$ under f contains open neighborhood of $x = 0$

1.1 Motivation for switched controllers (continued)

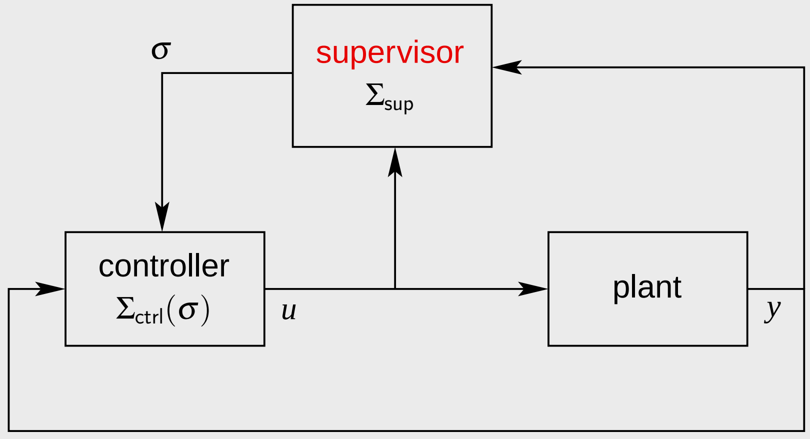
- For non-holonomic integrator:
$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= xv - yu \end{aligned}$$
- Is asymptotically stabilizable (see later)
- Satisfies Brockett's necessary condition?
  - if $f_1 = f_2 = 0$ then $f_3 = 0$
  - hence, $(0,0,\varepsilon)$ cannot belong to image of f for any $\varepsilon \neq 0$
    $\longrightarrow$ image of open neighborhood of $(x,u) = (0,0)$ under f does *not* contain open neighborhood of $(x,y,z) = (0,0,0)$
  - so non-holonomic integrator cannot be stabilized by continuous feedback

$\longrightarrow$ hybrid control schemes necessary to stabilize it!

1.2 Switching control/logic



1.2 Switching control/logic (continued)



$\rightarrow$ shared controller state variables

1.2 Switching control/logic (continued)

Main problem: **Chattering** (i.e., very fast switching)

1. **Hysteresis switching logic**
   - Let $h > 0$, let π_σ be a performance criterion (to be minimized)
   - If supervisor changes value of σ to q, then σ is held *fixed* at q until $\pi_p + h < \pi_q$ for some p
     $\rightarrow$ σ is set equal to p
     $\Rightarrow$ threshold parameter $h > 0$ prevents infinitely fast switching
   - Similar idea: boundary layer around switching surface in sliding mode control
2. **Dwell-time switching logic**
   once symbol σ is chosen by supervisor it remains constant for at least $\tau > 0$ time units (τ: "dwell time")

2. Stabilization of switched linear systems via suitable switching (Problem C)

$$\dot{x} = A_i x, \quad i \in I := \{1, 2, \ldots, N\}$$

Find switching rule σ as function of time/state such that closed-loop system is asymptotically stable

2.1 Quadratic stabilization via *single* Lyapunov function

Select $\sigma(x) : \mathbb{R}^n \rightarrow I := \{1, 2, \ldots, N\}$ such that closed-loop system has single quadratic Lyapunov function $x^T P x$

One solution: if some convex combination of A_i is stable:

$$A := \sum \alpha_i A_i \quad (\alpha_i \geqslant 0, \ \sum \alpha_i = 1) \text{ is stable}$$

Select $Q > 0$ and let $P > 0$ be solution of $A^T P + PA = -Q$

Quadratic stabilization (continued)

- From $x^T(A^TP+PA)x = -x^TQx < 0$ it follows that

$$\sum_i \alpha_i [x^T(A_i^TP+PA_i)x] \ < \ 0$$

- For each x there is at least one mode with $x^T(A_i^TP+PA_i)x < 0$ or stronger

$$\bigcup_{i\in I} \{x \mid x^T(A_i^TP+PA_i)x \leqslant -\frac{1}{N}x^TQx\} \ = \ \mathbb{R}^n$$

- Switching rule:

$$i(x) \ := \ \arg\min x^T(A_i^TP+PA_i)x$$

- Leads possibly to sliding modes. Alternative?

Alternative switching rule for quadratic stabilization

- Modified switching rule (based on hysteresis switching logic):
  - stay in mode i as long as $x^T(A_i^TP+PA_i)x \leqslant -\frac{1}{2N}x^TQx$
  - when bound reached, switch to a new mode j that satisfies

$$x^T(A_j^TP+PA_j)x \leqslant -\frac{1}{N}x^TQx$$

- There is a lower bound on the duration in each mode!
- No conservatism for 2 modes (necessary & sufficient for this case):

  Theorem: If there exists a quadratically stabilizing state-dependent switching law for the switched linear system with $N=2$, then matrices A_1 and A_2 have a stable convex combination

2.2 Stabilization via *multiple* Lyapunov functions (Problem C)

Main idea: Find function $V_i(x) = x^TP_ix$ that decreases for $\dot{x} = A_ix$ in some region

Define $\mathscr{X}_i := \{x \mid x^T[A_i^TP_i+P_iA_i]x < 0\}$

If $\mathscr{X}_1 \cup \mathscr{X}_2 = \mathbb{R}^n$, try to switch to satisfy multiple Lyapunov criterion to guarantee asymptotic stability.

Find P_1 and P_2 such that they satisfy the coupled conditions:

$$x^T(P_1A_1+A_1^TP_1)x < 0 \text{ when } x^T(P_1-P_2)x \geqslant 0,\ x \neq 0$$

and

$$x^T(P_2A_2+A_2^TP_2)x < 0 \text{ when } x^T(P_2-P_1)x \geqslant 0,\ x \neq 0$$

Then $\sigma(t) =$ arg max$\{V_i(x(t)) \mid i=1,2\}$ is stabilizing (V_σ will be continuous)

2.3 S-procedure

S-procedure If there exist β_1, $\beta_2 \geqslant 0$ such that

$$\begin{aligned} -P_1A_1 - A_1^TP_1 + \beta_1(P_2-P_1) > 0 \\ -P_2A_2 - A_2^TP_2 + \beta_2(P_1-P_2) > 0 \end{aligned}$$

then $\sigma(t) =$ arg max$\{V_i(x(t)) \mid i=1,2\}$

$\rightarrow$ only finds switching sequence (discrete inputs)!
What if also continuous inputs are present?

2.4 Stabilization of switched linear systems with continuous inputs

Switched linear system with inputs:

$$\dot{x} = A_i x + B_i u, \; i \in I = \{1, \ldots, N\}$$

Now both $\sigma : [0, \infty) \to I$ and feedback controllers $u = K_i x$ are to determined

Case 1: Determine K_i such that closed loop stable under arbitrary switching (assuming we **know** mode)!

Case 2: Determine both $\sigma : [0, \infty) \to I$ and K_i

Case 3 (for PWL systems): Given σ as function of state, determine K_i

Case 1: Stabilization of switched linear system under arbitrary switching

$$\dot{x} = A_i x + B_i u, \; i \in I = \{1, \ldots, N\}$$

Sufficient condition: find common *quadratic* Lyapunov function $V(x) = x^T P x$ for some positive definite matrix P and $K_1, \ldots, K_N$

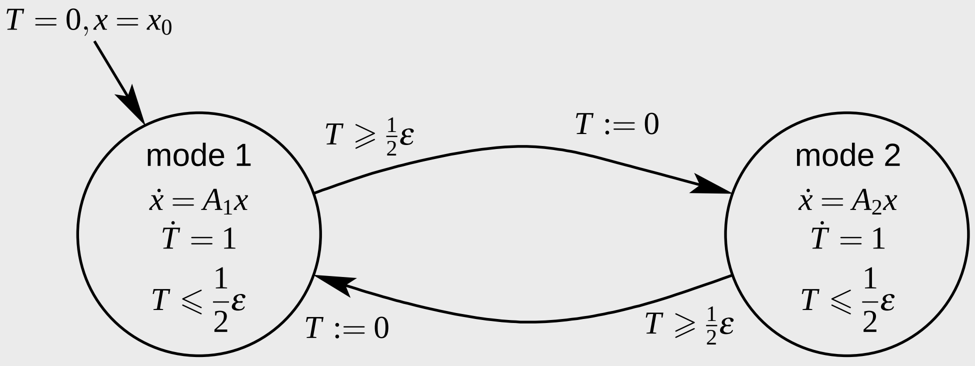
$$(A_i + B_i K_i)^T P + P(A_i + B_i K_i) < 0 \text{ for all } i = 1, \ldots, N \text{ and } P > 0$$

$\to$ LMIs (also for Cases 2 and 3)

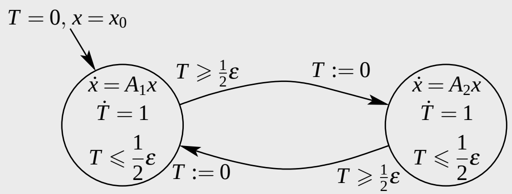
$\longrightarrow$ state-based switching in this section, ... next ...

3. Time-controlled switching & pulse width modulation

If dynamical system switches between several subsystems
$\to$ stability properties of total system may be quite different from those of subsystems



3.1 Time-controlled switching



- $x(t_0 + \frac{1}{2}\varepsilon) = \exp(\frac{1}{2}\varepsilon A_1) x_0 \; = \; x_0 + \frac{\varepsilon}{2} A_1 x_0 + \frac{\varepsilon^2}{8} A_1^2 x_0 + \cdots$

  $$\begin{aligned} x(t_0 + \varepsilon) &= (I + \tfrac{\varepsilon}{2} A_2 + \tfrac{\varepsilon^2}{8} A_2^2 + \cdots)(I + \tfrac{\varepsilon}{2} A_1 + \tfrac{\varepsilon^2}{8} A_1^2 + \cdots) x_0 \\ &= (I + \varepsilon[\tfrac{1}{2} A_1 + \tfrac{1}{2} A_2] + \tfrac{\varepsilon^2}{8}[A_1^2 + A_2^2 + 2 A_2 A_1] + \cdots) x_0. \end{aligned}$$

- Compare with

  $$\exp[\varepsilon(\tfrac{1}{2} A_1 + \tfrac{1}{2} A_2)] \; = \; I + \varepsilon[\tfrac{1}{2} A_1 + \tfrac{1}{2} A_2] + \tfrac{\varepsilon^2}{8}[A_1^2 + A_2^2 + A_1 A_2 + A_2 A_1] + \cdots$$

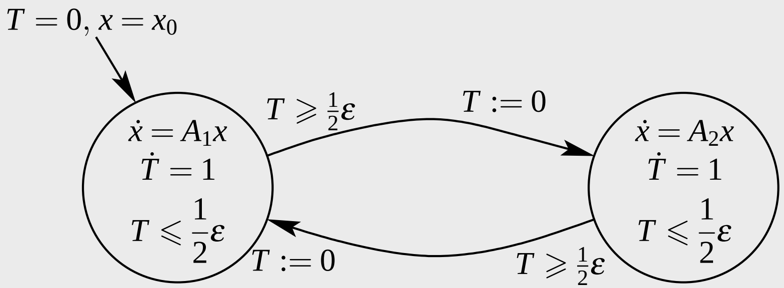
  $\to$ same for $\varepsilon \approx 0$

- So for $\varepsilon \to 0$ solution of switched system tends to solution of

  $$\dot{x} \; = \; (\tfrac{1}{2} A_1 + \tfrac{1}{2} A_2) x \qquad \text{(“averaged” system)}$$

- Possible that A_1 and A_2 are stable, whereas matrix $\frac{1}{2} A_1 + \frac{1}{2} A_2$ is unstable, or vice versa

Example

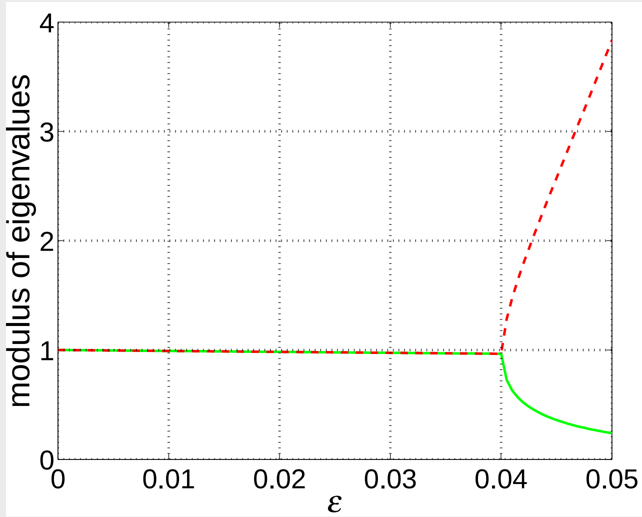


- Consider

$$A_1 = \begin{bmatrix} -0.5 & 1 \\ 100 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -100 \\ -0.5 & -1 \end{bmatrix}$$

- A_1, A_2 unstable, but matrix $\frac{1}{2}(A_1 + A_2)$ is stable
  $\rightarrow$ switched system should be stable if frequency of switching is sufficiently high
- Minimal switching frequency found by computing eigenvalues of the mapping $\exp(\frac{1}{2}\varepsilon A_1)\exp(\frac{1}{2}\varepsilon A_2)$ (Why?)

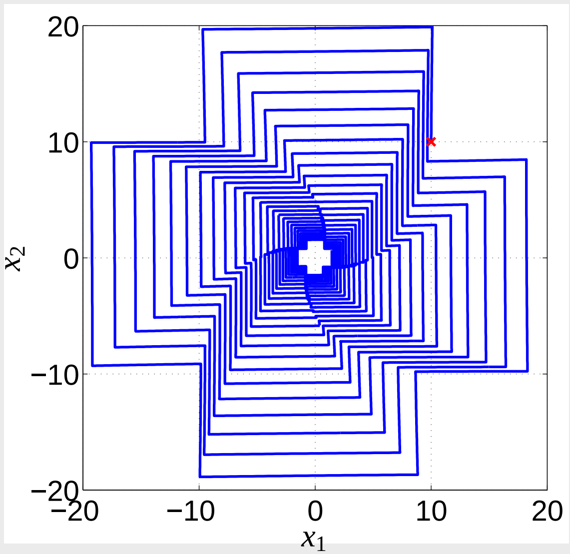
Example (continued)



$\rightarrow$ maximal value of ε: 0.04 (=50 Hz)

Example (continued)

for ε =0.02



3.2 Pulse width modulation

- Assume mode 1 followed during $h\varepsilon$, and mode 2 during $(1-h)\varepsilon$
  $\rightarrow$ behavior of system is well approximated by system

$$\dot{x} = \big(hA_1 + (1-h)A_2\big)x$$

- Parameter h might be considered as control input
- If h varies, should be on time scale that is much slower than the time scale of switching
- If mode 1 is "power on" and mode 2 is "power off", then h is known as *duty ratio*
- Power electronics: fast switching theoretically provides possibility to regulate power without loss of energy
  $\rightarrow$ used in power converters (e.g., Boost converter)

3.2 Pulse width modulation (continued)

- System: $\dot{x} = f(x,u), \quad u \in \{0,1\}$
- Duty cycle: Δ (fixed)
- u is switched exactly one time from 1 to 0 in each cycle
- Duty ration α: fraction of duty cycle for which $u = 1$

$$\begin{array}{ll} u(\tau) = 1 & \text{for } t \leqslant \tau < t + \alpha\Delta \\ u(\tau) = 0 & \text{for } t + \alpha\Delta \leqslant \tau < t + \Delta \end{array}$$

- Hence, $x(t+\Delta) = x(t) + \int_t^{t+\alpha\Delta} f(x(\tau),1)d\tau + \int_{t+\alpha\Delta}^{t+\Delta} f(x(\tau),0)d\tau$
- Ideal averaged model ($\Delta \to 0$):

$$\dot{x}(t) = \lim_{\Delta\to 0} \frac{x(t+\Delta) - x(t)}{\Delta} = \alpha f(x(t),1) + (1-\alpha) f(x(t),0)$$

4. Sliding mode control

- Consider $\dot{x}(t) = f(x(t),u(t))$ with u scalar
- Suppose switching feedback control scheme:

$$u(t) = \begin{cases} \phi_+(x(t)) & \text{if } h(x(t)) > 0 \\ \phi_-(x(t)) & \text{if } h(x(t)) < 0 \end{cases}$$

- Surface $\{x \mid h(x) = 0\}$ is called *switching surface*
- Let $f_+(x) = f(x,\phi_+(x))$ and $f_-(x) = f(x,\phi_-(x))$, then

$$\dot{x} = \frac{1}{2}(1+v)f_+(x) + \frac{1}{2}(1-v)f_-(x), \qquad v = \mathrm{sgn}(h(x))$$

- Use solutions in Filippov's sense if "chattering"

4. Sliding mode control (continued)

- Assume "desired behavior" whenever constraint $s(x) = 0$ is satisfied
- Set $\{x \mid s(x) = 0\}$ is called *sliding surface*
- Find control law u such that

$$\frac{1}{2}\frac{d}{dt}s^2 \leqslant -\alpha|s|$$

  where $\alpha > 0$
  $\to$ squared "distance" to sliding surface decreases along all system trajectories

Properties of sliding mode control

- Quick succession of switches may occur
  $\to$ increased wear, high-frequency vibrations
  $\Rightarrow$ - embed sliding surface in thin *boundary layer*
  - smoothen discontinuity by replacing sgn by steep sigmoid function
  - Note: modifications may deteriorate performance of closed-loop system
- Main advantages of sliding mode control:
  - conceptually simple
  - robustness w.r.t. uncertainty in system data
- Possible disadvantage:
  - excitation of unmodeled high-frequency modes

5. Stabilization by switching control

- For multi-model *linear* systems
  $\rightarrow$ use techniques for quadratic stabilization using single or multiple Lyapunov function
- Stabilization of non-holonomic systems using hybrid feedback control (e.g., non-holonomic integrator)
  $\rightarrow$ rather ad hoc
  not structured
  complicated analysis and proofs

Stabilization of non-holonomic integrator

- System: $\dot{x} = u, \quad \dot{y} = v, \quad \dot{z} = xv - yu$
- Sliding mode control: $u = -x + y\,\mathrm{sgn}(z)$
  $v = -y - x\,\mathrm{sgn}(z)$
- Lyapunov function for (x,y) subspace: $V(x,y) = \frac{1}{2}(x^2+y^2)$
  $\Rightarrow \dot{V} = -x^2 + xy\,\mathrm{sgn}(z) - y^2 - xy\,\mathrm{sgn}(z) = -(x^2+y^2) = -2V$
  $\Rightarrow x, y \rightarrow 0$
- $\dot{z} = xv - yu = -(x^2+y^2)\,\mathrm{sgn}(z) = -2V\,\mathrm{sgn}(z)$
  So $|z|$ will decrease and reach 0 provided that

$$2\int_0^\infty V(\tau)d\tau > |z(0)|$$

$\rightarrow$ z will reach 0 in finite time

Stabilization of non-holonomic integrator (continued)

- Since $V(t) = V(0)e^{-2t} = \frac{1}{2}(x^2(0)+y^2(0))e^{-2t}$
  condition for system to be asymptotically stable is

$$\frac{1}{2}(x^2(0)+y^2(0)) \geqslant |z(0)|$$

$\rightarrow$ defines parabolic region $\mathscr{P} = \{(x,y,z) \mid 0.5(x^2+y^2) \leqslant |z|\}$

- If initial conditions do *not* belong to $\mathscr{P}$ then sliding mode control asymptotically stabilizes system
- If initial state is inside $\mathscr{P}$:
  - first use control law (e.g., nonzero constant control) to steer system outside $\mathscr{P}$
  - then use sliding mode control

$\rightarrow$ *hybrid control scheme*