Exercises for the course "Optimization in Systems and Control"

Remark:

- Changes made on September 20, 2020 are marked in blue.

1 Exercises for Chapter 1

Exercise 1.1. *Let f be a convex function defined on a set I. If $x_1, x_2, ..., x_n \in I$, and $\lambda_1, \lambda_2, ..., \lambda_n \in [0, 1]$ with $\sum_{i=1}^{n} \lambda_i = 1$, then prove that*

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i) \quad . \tag{1.1}$$

Exercise 1.2. *Use the definition of convex functions to show that the function $f : \mathbb{R}^+ \to \mathbb{R}^+ : x \mapsto \sqrt{x}$ is not convex.*

Exercise 1.3. *Determine for which values $p \geqslant 0$ the function $f : \mathbb{R}_0^+ \to \mathbb{R}^+$ defined by $f(x) = x^p$ is a convex function, with $\mathbb{R}_0^+ = \mathbb{R}^+ \setminus \{0\} = (0, +\infty)$.*
<u>Hint</u>: Use the fact that if the second derivative f'' of the function f with a scalar argument is defined and nonnegative, then f is convex.

Exercise 1.4. *Use previous result and the definition of convex functions to prove that the function f defined by*

$$f(x) = \sum_{i=1}^{n} |x_i|^p, \quad x \in \mathbb{R}^n, p \geq 1$$

is convex.

Exercise 1.5. *Find the Taylor polynomial P_2 of order 2 based at $(0, 0)$ for the function f defined by $f(x, y) = 3xy + 2xy^3$. <u>Note:</u> This Taylor polynomial is defined by:*

$$P_2(x, y) = f(0, 0) + (\nabla f(0, 0))^T \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x, y) H_f(0, 0) \begin{pmatrix} x \\ y \end{pmatrix} \quad .$$

What is an upper bound for $\varepsilon > 0$ so that the error of between $P_2(x, y)$ and $f(x, y)$ is lower than 10^{-6} if $|x|, |y| \leq \varepsilon$? <u>Note:</u> The error is given by:

$$\begin{aligned} R_2(x, y) &= \frac{1}{3!} \sum_{i,j,k=1}^{2} \frac{\partial^3 f(c_1, c_2)}{\partial x_i \partial x_j \partial x_k} h_i h_j h_k \\ &= \frac{1}{3!} \left(\frac{\partial^3 f(c_1, c_2)}{\partial x^3} x^3 + 3 \frac{\partial^3 f(c_1, c_2)}{\partial x^2 y} x^2 y + 3 \frac{\partial^3 f(c_1, c_2)}{\partial x y^2} x y^2 + \frac{\partial^3 f(c_1, c_2)}{\partial y^3} y^3 \right) \quad , \end{aligned}$$

where (c_1, c_2) is any point in the line between $(0, 0)$ and (x, y), and where h_i, h_j, and h_k refer to the x_i, x_j, and x_k component of the vector (x, y).

Exercise 1.6. *Indicate whether or not the following functions $g(x)$ are subgradients of the corresponding functions $f(x)$:*

- $f(x) = |x|$, $x \in \mathbb{R}$: $g(x) = \begin{cases} -1 & if \quad x < 0 \\ 2 & if \quad x = 0 \\ 1 & if \quad x > 0 \end{cases}$
- $f(x) = \max\{f_1(x), f_2(x)\}$, $x \in \mathbb{R}^n$, $f_1(x)$ *and* $f_2(x)$ *convex and continuously differentiable:* $g(x) = \begin{cases} \nabla f_1(x) & if \quad f_1(x) > f_2(x) \\ \nabla f_2(x) & if \quad f_1(x) \leq f_2(x) \end{cases}$

Exercise 1.7. *Let $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$. Show that $\nabla(Ax) = A^T$.*

Exercise 1.8. *Show that the following functions $g(x)$ are subgradients of the corresponding functions $f(x)$:*

- $f(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x)$, $x \in \mathbb{R}^n$, $f_1(x)$ *and* $f_2(x)$ *convex and differentiable:* $g(x) = \alpha_1 \nabla f_1(x) + \alpha_2 \nabla f_2(x)$
- $f(x) = f_1(Ax + b)$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $f_1(x)$ *convex and differentiable:* $g(x) = A^T \nabla f_1(Ax + b)$

Exercise 1.9. *Find the saddle points and local minima and maxima of the following functions:*

- $f_1(x) = 9 - 2x_1 + 4x_2 - x_1^2 - 4x_2^2$
- $f_2(x) = 2x_1^3 + x_1 x_2^2 + 5x_1^2 + x_2^2$

Exercise 1.10. *The optimization problem* $\min f(x_1, x_2) = (x_1 - 3)^4 + (x_1 - 3x_2)^2$ *is solved using the following (gradient-based) algorithm*

$$x_{k+1} = x_k - \lambda_k \frac{\nabla f(x_k)}{\|\nabla f(x_k)\|_2}$$

If the initial point is $x_0 = [0, 0]^T$ and the step $\lambda_k =(0.9)^{k+1}$, use Matlab to indicate which of the following stopping criteria is fulfilled first:

- $\|\nabla f(x_k)\|_2 \leq 3.5$
- $|f(x_k) - f(x_{k-1})| \leq 0.4$
- *Maximum number of iterations* $k_{max} = 10$

Plot the various iteration points and their function values.

Exercise 1.11. *Consider the problem of choosing (x, y) to maximize $f(x, y) = 3x + y$ subject to: $(x + 1)^2 + (y + 1)^2 \leq 5$ and $x \geq 0$, $y \geq 0$.*

- *Suppose that (x^*, y^*) solves this problem. Is there necessarily a value of μ such that (x^*, y^*) satisfies the Kuhn-Tucker conditions?*
- *Now suppose that (x^*, y^*) satisfies the Kuhn-Tucker conditions. Does (x^*, y^*) necessarily solve the problem?*
- *Given the information in your answers to (a) and (b), use the Kuhn-Tucker method to solve the problem.*

2 Exercises for Chapter 2: Linear Programming

Exercise 2.1. *Use the graphical method to solve the following problem:*

$$\min f(x) = x_1 - 2x_2$$

subject to the constraints: $x_1 + x_2 \geq 2$, $-x_1 + x_2 \geq 1$, $x_2 \leq 3$, $x_1, x_2 \geq 0$.
Reformulate the same problem as a linear programming problem in standard form and solve it using the simplex method.

Exercise 2.2. *Two students A and B work at a shop for x and y hours per week, respectively. According to the rules, A can work at most* 8 *hours more than B. But student B can work at most* 6 *hours more than student A. Together they can work at most* 40 *hours per week. Find their maximum combined income per week if student A and student B earn* 15 *and* 17 *euro per hour, respectively.*

3 Exercises for Chapter 3: Quadratic Programming

Exercise 3.1. *Consider the process modeled by the following linear discrete-time system:* $y(n+1) = ay(n) + bu(n) + \frac{1}{1-q^{-1}}e(n)$, *where* $y(n)$ *is the output,* $u(n)$ *the input,* a *and* b *are the model parameters,* $e(n)$ *is white noise of mean value* 0 *and standard deviation* σ. *At time step* n *the output* $y(n)$ *is measured, the output* $y(n-1)$ *and control action* $u(n-1)$ *is also known, and we have to obtain a control action* $u(n)$. *It is easy to show that we can define the 1-step ahead prediction* $\hat{y}(n+1) = (1+a)y(n) - ay(n-1) + b\Delta u(n)$ *with* $\Delta = 1 - q^{-1}$ *with* $\hat{y}(n) = y(n)$.

1. *Obtain the control action* $\Delta u(n)$ *that minimizes* $J = (\hat{y}(n+1) - r)^2 + \lambda\,(\Delta u(n))^2$, *where* λ *is a weighting factor and* r *the output reference.*
2. *Reformulate the following problem as a quadratic programming problem:*
   $\min J_n = \sum_{k=1}^{3}(\hat{y}(n+k) - r)^2 + \lambda \sum_{k=1}^{3} (\Delta u(n+k-1))^2$,
   s.t. $\Delta u_{min} \leq \Delta u(n+k-1) \leq \Delta u_{max}$, $k = 1, 2, 3$.
3. *Reformulate the problem as a quadratic programming problem of Type 1 with as few variables as possible. Assume* $\Delta u_{min} = 0$.

Exercise 3.2. *Solve the following QP problem of type 2:* $\min \frac{1}{2}x^T Hx + c^T x$, *s.t.* $Ax = b$, $x \geq 0$, *where*

$$H = \begin{bmatrix} 1 & -4 & 2 & 1 \\ -4 & 16 & -8 & -4 \\ 2 & -8 & 4 & 2 \\ 1 & -4 & 2 & 1 \end{bmatrix}, \; c = \begin{bmatrix} -1 \\ 0 \\ 7 \\ 4 \end{bmatrix}, \; A = [1, 1, 1, 1], \; b = 4,$$

Exercise 3.3. *Prove that the gradient of* $c^T x$ *is* c *and the Jacobian of* $(Ax - b) = A^T$.

Exercise 3.4. *Solve the following optimization problem:*

$$\min f(x) = -8x_1 - 16x_2 + x_1^2 + 4x_2^2$$

subject to: $x_1 + x_2 \leq 5$, $x_1 \leq 3$, $x_1 \geq 0$, $x_2 \geq 0$

4 Exercises for Chapter 4: Nonlinear optimization without constraints

Exercise 4.1. *Perform three iterations to find the minimum of* $f(x_1, x_2) = (x_1-3)^4+(x_1-3x_2)^2$ *using:*

- *Newton's method (use* $x_0 = [0, 0]^T$*).*
- *Levenberg-Marquardt's method (use* $x_0 = [0, 0]^T$*, and* λ*=1.1).*
- *Broyden-Fletcher-Goldfarb-Shanno's method (use* $x_0 = [0, 0]^T$*).*
- *Davidon-Fletcher-Powell's method (use* $x_0 = [0, 0]^T$*).*

Exercise 4.2. *Use the golden section method to find the value of* x *that minimizes the function*

$$f(x) = -\min\left\{\frac{x}{2}, 2-(x-3)^2, 2-\frac{x}{2}\right\}.$$

Use the fact that the function is strictly unimodal on $[0, 8]$*. Perform five iterations. Compare the results with those obtained with the Fibonacci method and with a fixed-step method (take a step length* $\Delta s = 2$*).*

Exercise 4.3. *Answer the following questions:*

- *Why is Newton's method for minimizing multivariate functions not a descent method and how should it be modified to become a descent method?*
  Note: An optimization method is called a descent method if $f(x_{k+1}) \leq f(x_k)$ *for all* k*, where* f *is the objective function and* x_k *is the* k*th iteration point.*
- *Is the steepest-descent algorithm is a descent method?*
- *Are the steps in the steepest-descent algorithm orthogonal?*

Exercise 4.4. *Using a steepest-descent method with update formula*

$$x_{k+1} = x_k - \frac{\nabla f(x_k)}{\|\nabla f(x_k)\|_2} \ ,$$

find the minimum of the quadratic function

$$f(x_1, x_2) = (x_1 - 2)^2 + 4(x_2 - 3)^2,$$

starting from $x_0 = [0, 0]^T$*. Using Matlab, plot the algorithmic moves (*x_k *as function of* k*) and verify the zigzag property of the algorithm.*

Exercise 4.5. *Show that the choice of* λ *in the golden section method indeed results in reuse of points from one iteration to the next.*

Exercise 4.6. *Show that the choice of* λ *in the Fibonacci method indeed results in reuse of points from one iteration to the next.*

Exercise 4.7. *Prove the expressions on page 36 for gradient of* $f(x)$ *and the Hessian (4.1)*

5 Exercises for Chapter 5: Constraints in nonlinear optimization

Exercise 5.1. *Consider the constrained minimization problem:*

$$\begin{aligned}
&\min f(x_1, x_2, x_3) = x_1^2 + 8x_2^2 + 3x_1x_3 \\
&\textit{subject to} \\
&x_1 - x_2 + x_3 = 1 \\
&x_1 + x_2 = 2
\end{aligned}$$

Solve this problem using the method of elimination of constraints.

Exercise 5.2. *Using Matlab, apply sequential quadratic programming to solve the problem:*

$$\begin{aligned}
&\min f(x_1, x_2) = (x_1 - \tfrac{9}{4})^2 + (x_2 - 2)^2 \\
&\textit{subject to} \\
&x_1^2 - x_2 \leq 0 \\
&x_1 + x_2 \leq 6 \\
&x_1, x_2 \geq 0
\end{aligned}$$

Starting with the point $x_0 = [0, 0]^T$, show the evolution of the search direction d_k (Step 2 of the algorithm as listed in the lecture notes), step length in the line optimization s_k of (Step 3), and the optimization variables $(x_k)_1$, $(x_k)_2$ as function of the iteration step k.

Exercise 5.3. *Solve the problem:*

$$\max_{x,y,z}(x + y)$$

subject to $x^2 + 2y^2 + z^2 = 1$ and $x + y + z = 1$.

6 Exercises for Chapter 6: Convex optimization

Exercise 6.1. *Perform two iterations of the ellipsoid algorithm to solve the program:*

$$\begin{aligned}
&min f(x_1, x_2) = 4(x_1 - 10)^2 + (x_2 - 4)^2 \\
&\textit{subject to} \\
&x_1 - x_2 \leq 10 \\
&x_1 - x_2 \geq 3 \\
&x_1 \geq 0
\end{aligned}$$

Plot the feasible region and the algorithmic steps. Take $[0, 0]^T$ as starting point.

Exercise 6.2. *Use the interior-point algorithm to solve the program:*

$$\begin{aligned}
&min f(x_1, x_2) = -x_1x_2 \\
&\textit{subject to} \\
&1 - x_1^2 - x_2^2 \geq 0
\end{aligned}$$

Plot the feasible region and the algorithmic steps. Use first $[0.1, 0.1]^T$ and next $[-0.1, -0.1]^T$ as starting points.

Exercise 6.3. *Are the following functions convex or not? Why?*

1. $f : \mathbb{R} \to \mathbb{R} : x \mapsto (x^2 + 1)^2$
2. $f : \mathbb{R} \to \mathbb{R} : x \mapsto (x^2 - 3x)^2$
3. $f : \mathbb{R} \to \mathbb{R} : x \mapsto 2^x$
4. $f : \mathbb{R} \to \mathbb{R} : x \mapsto \left(\frac{1}{2}\right)^x$
5. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} : x \mapsto \frac{1}{x}$
6. $f : [1, +\infty) \to \mathbb{R} : x \mapsto \frac{1}{x}$
7. $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto \cosh(x^2 + y^2)$

Exercise 6.4. *On page 60 it is stated that if P is symmetric then the conditions $P > 0$ and $A^T P + PA < 0$ can be recast as an LMI. Prove this statement.*
Hint: Write P as a linear combination of symmetric basis matrices, each having only one (diagonal) entry or two (off-diagonal) entries equal to 1, the other entries being equal to 0.

Exercise 6.5. *If the function f is convex, is f^2 then always convex?*
If the function f is convex and nonnegative, is f^2 then always convex?

Exercise 6.6.

1. *Prove that the sum of a linear function and a convex function is convex.*
2. *Prove that the sum of a linear function and a nonconvex function is nonconvex.*
3. *Provide examples to show that the sum of a convex function and a nonconvex one, can be either convex or nonconvex.*
4. *Provide examples to show that the sum of two nonconvex functions can be either convex or nonconvex.*

7 Exercises for Chapter 7: Global optimization

Exercise 7.1. *Using the routine simulannealbnd of Matlab, minimize the following function,*

$$f(x) = -e^{-2\ln(2)(\frac{x-0.008}{0.854})} \sin^6(5\pi(x^{0.75} - 0.05)), \quad x \in [0, 1].$$

Plot the current iteration point, the function value, and the temperature function.

Exercise 7.2. *The Himmelblau function has four peaks in the points $(3; 2)$, $(-3.799; -3.283)$, $(-2.805; 3.131)$, and $(3.584; -1.848)$, and it is defined by*

$$f(x_1, x_2) = \frac{2186 - (x_1^2 + x_2 - 11)^2 - (x_1 + x_2^2 - 7)^2}{2186}, \quad x_1, x_2 \in [-6, 6].$$

Using the routine ga of Matlab, generate an optimizer capable to detect the four optimal solutions.

Exercise 7.3.

Discuss the main differences between multi-start local optimization methods, simulated annealing, and genetic algorithms.

8 Exercises for Chapter 11: Integer optimization

Exercise 8.1. *Consider the process modeled by the following linear discrete-time system:* $y(n+1) = ay(n) + bu(n) + e(n)$*, where* $y(n)$ *is the output,* $u(n) \in \{0,1\}$ *the input (binary input),* $a = 0.9$ *and* $b = 0.1$ *are the model parameters, and* $e(n)$ *is white noise of mean value* 0 *and standard deviation* σ*. At instant time* n *the output* $y(n) = 0.5$ *is measured and we have to obtain a control action* $u(n) \in \{0,1\}$*. Let us define the prediction* $\hat{y}(n+1) = ay(n) + bu(n)$*, and* $\hat{y}(n+k) = a\hat{y}(n+k-1) + bu(n+k-1)$ *for* $k \in \{2,3,4,5\}$*.*

- *Obtain the control action* $u(n) \in \{0,1\}$ *that minimizes* $J = (\hat{y}(n+1) - r)^2 + \lambda u(n)^2$*, where* $\lambda = 0.01$ *is a weighting factor and* $r = 1$ *the output reference.*

- *Using branch-and-bound, obtain the control sequence* $U = [u(n), u(n+1), u(n+2)]$*, that minimize* $\min J_n^{n+2} = \sum_{k=1}^{3}(\hat{y}(n+k) - r)^2 + \lambda \sum_{k=1}^{3} u(n+k-1)^2$*.*