Exam — November 2015
"Optimization in Systems and Control" (SC4091)

QUESTION 1: Optimization methods I (60 points)

Consider the following optimization problems:

P1. $$\min_{x \in \mathbb{R}^4} |2x_1 + 4x_2 + 3x_3 + 2x_4 + 7| + \exp\big((x_1 + x_2 - 4x_3 - x_4)^4\big)$$
$$\text{s.t. } \|x\|_2 \leq 3 + x_1 - 2x_2 + x_3 - 2x_4$$

P2. $$\max_{x \in \mathbb{R}^5} x_1^4 + x_2^2 + 3x_1x_3 + 2x_4x_5$$
$$\begin{aligned} \text{s.t. } & |x_1 + 2x_2 + 8x_3 - 9x_4 + 8x_5| \leq 7 \\ & -x_1 + 3x_2 - x_3 + 6x_4 + x_5 \geq 2 \\ & \|x\|_\infty \leq 3 \end{aligned}$$

P3. $$\max_{x \in \mathbb{R}^3} \min(4x_1 - 3x_2 + 8x_3 - 5, -2x_1 + 7x_2 - x_3 + 1)$$
$$\text{s.t. } 7|x_1| + 2|x_2| + 5|x_3| \leqslant 10$$

P4. $$\min_{x \in \mathbb{R}^8} \max(F_1(x), 2F_2(x), F_3(x) + 8F_4(x))$$
$$\begin{aligned} \text{s.t. } & \|x\|_\infty^5 \leq 10000 \\ & (x_1^6 + x_2^2 - 3x_5 + 8x_7^2 - x_8)^3 \leqslant 27 \end{aligned}$$

and where the functions F_1, F_2, F_3, and F_4 correspond to the outcomes of physical experiments that take about 45 seconds to carry out and for which it is known that F_1, F_2, F_3, and F_4 are convex in their argument.

P5. $$\min_{x \in \mathbb{R}^3} 9x_1^2 + 4x_2^2 + 2x_1x_2 + 8x_3^2 - 2x_1x_3$$
$$\text{s.t. } \cosh(3x_1^2 + 2x_2^2 + x_3^2) + (4x_1 + 5x_2 + 6x_3)^2 + x_1 - x_3 = 100$$

P6. $$\min_{x \in \mathbb{R}^3} \frac{1}{(x_1 + 3x_2 + 4x_3)^2} + \cosh\big((x_1 + 4x_2 + 5x_3 - 100)^4\big)$$
$$\begin{aligned} \text{s.t. } & \min(x_1, x_2, x_3) \geqslant 1 \\ & \log\big((x_1 - 2)^2 + (x_2 - 3)^2 + (x_3 - 4)^2\big) \geqslant 1000 \end{aligned}$$

P7. $$\max_{x\in\mathbb{R}^3} \; 1-4x_1^2-9x_2^2-8x_3^2+4x_1x_2-2x_1x_3+4x_2x_3$$
$$\text{s.t.} \;\; \|x\|_1 \leqslant 1$$

P8. $$\min_{x\in\mathbb{R}^4} \; x_1+|x_2|+3x_3+4|x_4|$$
$$\begin{aligned}\text{s.t.} \;\; & \exp(x_1)+\exp(2x_2)+\exp(x_3) \leqslant 100 \\ & (x_1^2+x_2^2-x_3-6x_4)^2 \leqslant 64\end{aligned}$$

P9. $$\max_{x\in\mathbb{R}^3} \; \frac{x_1^2+x_2^2+x_3}{1+\sqrt{x_1^4+x_2^6+x_3^2}}$$
$$\text{s.t.} \; x_1^3+\cos(x_2+x_3)+x_2x_3=1$$

P10. $$\max_{x\in\mathbb{R}^4} \; (4x_1+7x_2-4x_3+12x_4)^3$$
$$\begin{aligned}\text{s.t.} \;\; & \exp(x_1+x_2+x_3+x_4)+(x_1-x_2+x_3-8x_4)^2 \leqslant 2 \\ & \frac{1}{1+\|x\|_1} \geqslant 0.2\end{aligned}$$

Tasks:

For each of the problems P1 up to P10:

(a) Determine the best suited optimization method for each of the given problems. Select one of the following methods:

- M1 : Simplex algorithm for linear programming
- M2 : Modified simplex algorithm for quadratic programming
- M3 : Gradient projection method with fixed step size line minimization
- M4 : Gauss-Newton least squares algorithm
- M5 : Steepest descent method
- M6 : Line search method with the Broyden-Fletcher-Goldfarb-Shanno direction and cubic line minimization
- M7 : Lagrange method + Davidon-Fletcher-Powell quasi-Newton algorithm
- M8 : Lagrange method + steepest descent method
- M9 : Penalty function approach + line search method with Powell directions and variable step size line minimization
- M10 : Penalty function approach + steepest descent method
- M11 : Barrier function approach + Nelder-Mead method
- M12 : Genetic algorithm

* If it is necessary to use the selected algorithm in a *multi-start local minimization* procedure, you have to ***indicate*** this clearly in your reply to item (a).

* Give only one solution! If more than one solution is given, the worst one will be taken.

(b) ***Explain*** why the selected algorithm is the *best suited* for the given problem. Note that you may have to perform some extra steps first in order to *simplify* the optimization problem. If you do this, then you have to ***indicate*** it clearly.

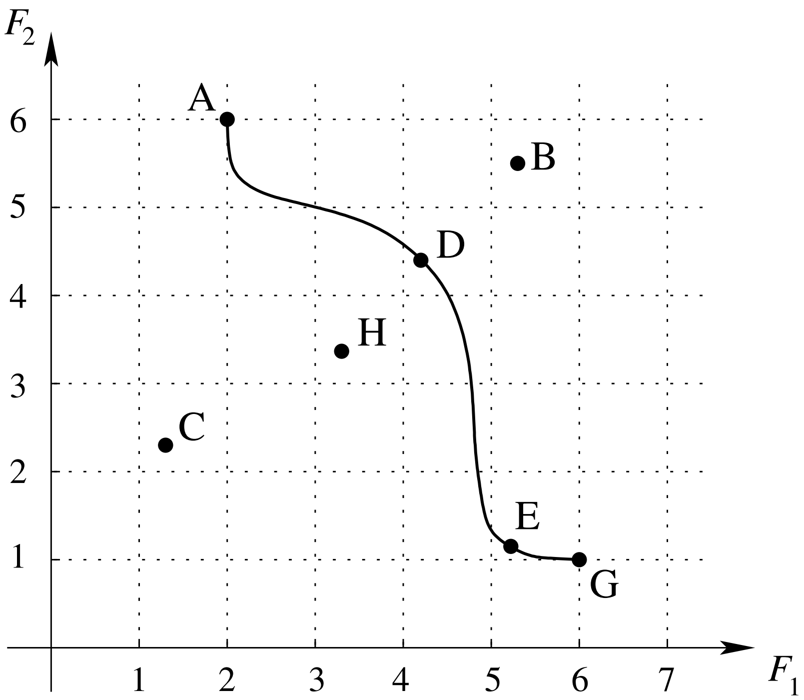
(c) Give the most appropriate stopping criterion for the selected optimization method. Do *not* reply by referring to your answer to a previous or subsequebnt problem.

QUESTION 2: Optimization methods II (25 points)

1. The topic of this question is Pareto optimality:

   (a) Briefly define and explain the concept of Pareto optimality. Illustrate your explanation with one or more pictures.

   (b) Consider the following plot with the set of all Pareto-optimal points (indicated by the curve A-D-E-G) of a multi-objective minimization problem with the vector of objectives $F = [F_1 \ \ F_2]^T$:



      i. Which of the points A, B, C, D, E, G, H can possibly be obtained as the result of a heuristic optimization method (e.g., random search)?

      ii. Which of the points A, B, C, D, E, G, H can possibly be obtained as the result of a weighted-sum optimization method?

      Explain and motivate your answers.

   (c) If we now apply the ε-constraint method with F_2 as primary objective subject to the constraint $F_1 \leqslant \varepsilon$, what is the result that is obtained for the optimal values F_1^* and F_2^* of respectively F_1 and F_2 if

      i. $\varepsilon = 1$

      ii. $\varepsilon = 3$

      iii. $\varepsilon = 7$

      Explain and motivate your answers.

2. Now consider the following optimization problem:

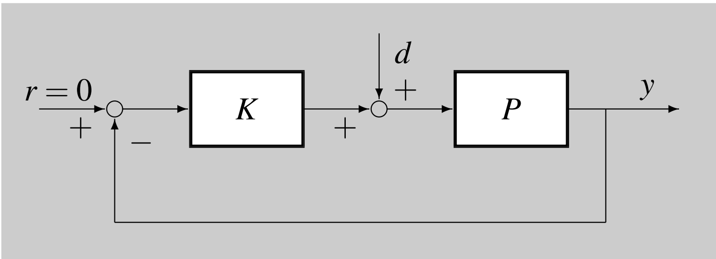
$$\begin{aligned} \min_{x \in S} \, & f(x) \\ \text{s.t.} \; & g(x) \geqslant 0 \\ & h(x) = 0 \;\; , \end{aligned}$$

with $S \subseteq \mathbb{R}^n$, f a scalar-valued function with domain $\mathbb{R}^n$, and g and h vector-valued functions with domain $\mathbb{R}^n$.

Under which conditions on S, f, g, and h is the above optimization problem convex?
Note that we want you to give the *least restrictive conditions*!
Motivate or prove your answer.

QUESTION 3: Controller design (15 points)

Consider the following closed-loop system consisting of the stable system with transfer function P and controller with transfer function $K(\cdot,\theta)$ with θ a real-valued parameter. Let $r(k)=0$ for all k and let $\mathrm{RMS}^2(d)=\sigma_d^2$ denote the variance of the zero-mean white noise sequence d.



Let the closed-loop configuration be given by the state-space realization

$$\begin{aligned} x(k+1) &= A_{\mathrm{cl}}x(k)+B_{\mathrm{cl}}d(k) \\ y(k) &= C_{\mathrm{cl}}x(k)+D_{\mathrm{cl}}d(k) \end{aligned}$$

where A_{cl}, B_{cl}, C_{cl}, and D_{cl} are the closed-loop system matrices, with A_{cl} and B_{cl} constant, and C_{cl} and D_{cl} affine in the parameter θ. Consider the objective function $f=\mathrm{RMS}^2(y)$.

Tasks:

a) Let $X=E[x(k+1)x^T(k+1)]$. Show that X is the solution of the discrete-time Lyapunov equation

$$X = A_{\mathrm{cl}}XA_{\mathrm{cl}}^T + B_{\mathrm{cl}}B_{\mathrm{cl}}^T\sigma_d^2$$

b) Now derive an expression for f as a function of $C_{\mathrm{cl}}(\theta)$ and $D_{\mathrm{cl}}(\theta)$.

c) Show that f is convex in θ.

d) Argue that if A_{cl} and B_{cl} would also be affine in the parameter θ, the question whether the function f is convex in θ is much harder to answer.

End of the exam