Exam — November 2016
"Optimization in Systems and Control" (SC42055)

QUESTION 1: Optimization methods I (60 points)

Consider the following optimization problems:

P1. $$\begin{aligned}\min_{x\in\mathbb{Z}^3}\ & \big(\max(4x_1-3x_2+8x_3-5,\,-2x_1+7x_2-x_3+1)\big)^3\\ \text{s.t. }\ & 1\leqslant 0.7|x_1|+0.2|x_2|+0.5|x_3|\leqslant 80\end{aligned}$$

P2. $$\begin{aligned}\min_{x\in\mathbb{R}^4}\ & 2x_1^2+2x_1x_2+x_2^2+4x_3^2+4x_3x_4+7x_4^2-3x_1-4x_2+8x_3+1\\ \text{s.t. }\ & \exp(x_1)+\exp(x_2)+\exp(x_3)+\exp(x_4)\leq 216\\ & \min(x_1+2x_3+x_4,\,4x_1-x_2+6x_3)\geq 2\\ & 1\leq x_1^2+x_2^2+x_3^2+x_4^2\leq 15\end{aligned}$$

P3. $$\begin{aligned}\min_{x\in\mathbb{R}^5}\ & 3x_1^4+x_2^2+2x_3^2+2x_1x_3+2x_4x_5\\ \text{s.t. }\ & (x_1+2x_2+8x_3-9x_4+8x_5)^4\leq 16\\ & \sqrt{(-x_1+3x_2-x_3+6x_4+x_5)^2}\geq 2\\ & \|x\|_1\leq 3\end{aligned}$$

P4. $$\begin{aligned}\min_{x\in\mathbb{R}^3}\ & \arctan\big(4x_1^2+2x_1x_2+2x_1x_3+4x_2^2+2x_2x_3+4x_3^2-x_1+x_2-x_3\big)\\ \text{s.t. }\ & (3x_1-x_2+6x_3)^4\leqslant 256\\ & \sqrt{|x_1+x_2+x_3|}\leqslant 4\end{aligned}$$

P5. $$\begin{aligned}\min_{x\in\mathbb{R}^4}\ & \sqrt[3]{(x_1+6x_2+8x_3-9x_4-10)^2}\\ \text{s.t. }\ & x_1^3+x_3^2+3x_2^2+2x_2^2x_3^2+4x_3^2x_4^2+8x_4^4=2\end{aligned}$$

P6. $$\begin{aligned}\min_{x_1,x_2,x_3\in\{0,1,\ldots,30\}}\ & x_1^2+2x_1x_2+3x_2^2+4x_3^2-x_1-5x_2+x_3+1\\ \text{s.t. }\ & 1+(x_1^2+x_2^2)^2+x_3^4\leq 20^4\\ & |x_1+x_2-x_3|\leq 3\end{aligned}$$

P7. $\max_{v\in\mathbb{R}^{10}} \|G\|_\infty$

where the (stable) transfer function G depends on v and is defined by

$$G(z,v) = \frac{z^2 + \dfrac{v_1v_2 - v_3v_5}{1+v_1^2+v_2^2+v_6^2}z - \dfrac{v_4v_9 - v_7v_{10}}{1+v_7^2+v_8^2}}{\left(z - \dfrac{1}{2+v_2^2+2v_5^2+3v_9^2}\right)\left(z - \dfrac{1}{3+2v_4^2+8v_{10}^2}\right)\left(z + \dfrac{1}{4+v_7^2}\right)}$$

with z the z-transform variable.

P8. $\max_{x\in\mathbb{R}^3} -\exp(x_1^2 + 2x_1x_2 + x_2^2 + x_3^2 - x_1 - 3x_2 + 4x_3)$

s.t. $\|0.2x\|_2 \leq 1$

$\log_2(4 + x_1 + x_2 - x_3) \leq 1$

P9. $\min_{x\in\mathbb{R}^4} x_1^4 + 7x_2^2 - 8x_3x_4 + 2x_3 - 9$

s.t. $\exp(1 + x_1^4x_2^2) + 3x_2x_3^2 + 7x_1^2x_4\sin(x_1 + x_3 - x_4) = 1$

P10. $\min_{x\in\mathbb{R}^3} \dfrac{1}{(x_1+3x_2+4x_3)^6} + 0.0001\cosh(x_1 + 4x_2 + 5x_3 - 100)^4$

s.t. $x_1 + 3x_2 + 4x_3 \geqslant 1$

$\log\left((x_1-2)^2 + (x_2-3)^2 + (x_3-4)^2\right) \leqslant 2$

Tasks:

For each of the problems P1 up to P10:

(a) Give the *most simple type* of optimization problem that the given problem can be reduced to, *irrespective* of the optimization algorithm used later on[1]. Select one of the following types:

- LP: (mixed-integer) linear programming problem
- QP: convex quadratic programming problem
- CP: convex optimization problem
- NCU: nonconvex unconstrained optimization problem
- NCC: nonconvex constrained optimization problem

Note that you may have to perform some extra steps first in order to *simplify* the optimization problem. If you do this, then you have to ***indicate*** it clearly in your reply to item (d).

(b) Now determine the *best suited optimization method* for the given problem. Select one of the following methods:

M1 : Simplex algorithm for linear programming
M2 : Gradient projection method with fixed step size line minimization
M3 : Ellipsoid algorithm
M4 : Gauss-Newton least squares algorithm
M5 : Levenberg-Marquardt algorithm
M6 : Steepest descent method
M7 : Line search method with the Broyden-Fletcher-Goldfarb-Shanno direction and cubic line minimization
M8 : Lagrange method + Davidon-Fletcher-Powell quasi-Newton algorithm
M9 : Lagrange method + steepest descent method
M10 : Sequential quadratic programming
M11 : Simulated annealing
M12 : Branch-and-bound method for mixed-integer linear programming

* If it is necessary to use the selected algorithm in a *multi-start optimization* procedure, you have to ***indicate*** this clearly in your reply to item (b) by putting a checkmark in the Multi-start box.

* Give only one solution! If more than one solution is given, the worst one will be taken.

(c) Give the most appropriate stopping criterion for the selected optimization method.
Do *not* reply by referring to your answer to a previous or subsequent problem. Such a statement will considered to be a wrong answer.

(d) ***Explain and motivate*** your reply to items (a) and (b).

[1]So if you have a nonconvex constrained problem that you will solve using, e.g., a barrier or penalty function method in item (b), you should still mark NCC here.

QUESTION 2: Optimization methods II (20 points)

<u>Tasks:</u>

1. Explain how the branch-and-bound method for mixed-integer linear programming works. Will such an algorithm always yield the globally optimal solution in a finite amount of time? If not, when not? Explain your answer.

2. Consider the following optimization problem:

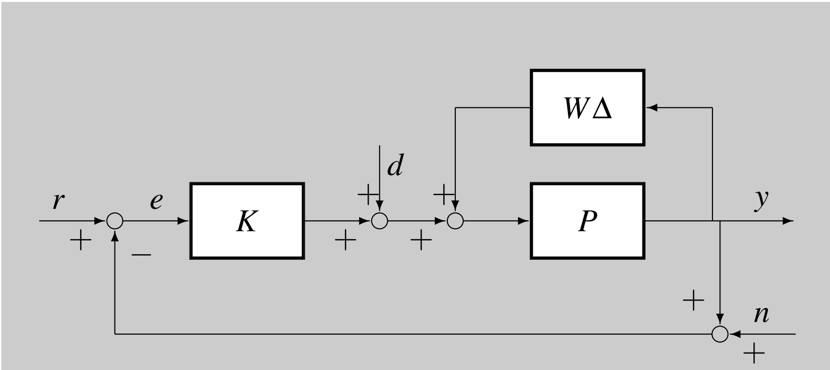
$$\min_{(x,y)\in\mathbb{R}^2} x^2 + 2y^2 - xy - 7y + 13 \quad . \tag{1}$$

   a) Is the objective function of this optimization problem convex or not? Why?

   b) Compute all the points (x^*, y^*) in which a local optimum of the given optimization problem is reached. Characterize these points (are they maximima or minima?).

QUESTION 3: Controller design (20 points)

Consider the following closed-loop system consisting of the stable system P and a controller K with a transfer function that depends on a parameter θ:



The signal r is the reference signal, and d and n are noise signals. A reverse additive model error is given by $W\Delta$ where W is a known filter and Δ is the model error satisfying $\|\Delta\|_\infty < 1$.
Let the plant P be given by the transfer function

$$P(q) = \frac{1}{1 - 0.5q^{-1}} \quad ,$$

where q denotes the forward shift operator.
The transfer function of the controller has the form

$$K(\theta, q) = Q(\theta, q)(1 - P(q)Q(\theta, q))^{-1} \quad ,$$

where Q is parameterized as

$$Q(\theta, q) = \theta_0 + \theta_1 q^{-1} + \theta_2 q^{-2} \quad .$$

Tasks:

a) Assume $\Delta = 0$. Determine the transfer function matrix M in terms of P and Q such that

$$\begin{bmatrix} y(k) \\ e(k) \end{bmatrix} = M(\theta, q) \begin{bmatrix} d(k) \\ r(k) \\ n(k) \end{bmatrix} \quad .$$

b) Assume $\Delta = 0$. Furthermore, let r be a unit step function (i.e., $r(k) = 1$ for $k \geq 0$), and let $d = n = 0$. Suppose we want to design a stabilizing controller with the output y asymptotically tracking the reference r. Derive a condition on the parameters θ_0, θ_1, θ_2 that guarantees this.

c) Let $r = d = 0$, and let $\|n\|_2 \leq 1$. Further assume $\Delta = 0$. Consider the objective function

$$J = \min_K \max_n \|e\|_2$$

Rewrite this optimization problem as a problem in terms of the parameter θ.

d) Compute the parameter θ that optimizes the objective function J defined in Task c) above.

e) Consider the model error Δ with $\|\Delta\|_\infty < 1$. Derive a constraint on the transfer function Q to guarantee robust stability.

$\boxed{\text{End of the exam}}$