Exam — November 2017
"Optimization in Systems and Control" (SC42055)

QUESTION 1: Optimization methods I (60 points)

Consider the following optimization problems:

P1. $$\begin{aligned} \min_{x\in\mathbb{R}^5} \ & 3x_1^4 + x_2^2 + 2x_3^2 + 2x_1x_3 + 2x_4x_5 \\ \text{s.t.} \ & 2^{x_1+2x_2+8x_3-9x_4+8x_5} \leq 2048 \\ & \sqrt{(-x_1+3x_2-x_3+6x_4+x_5)^2} \geq 2 \\ & \|x\|_1 \leq 5 \end{aligned}$$

P2. $$\begin{aligned} \min_{x\in\mathbb{Z}^3} \ & x_1^4 + 2x_2^2x_3^2 - 3x_1 + 4x_2 - 9x_3 - 5 \\ \text{s.t.} \ & \|x\|_2 \leq 9 \\ & |x_1 - 2x_2 + 3x_3| \leq 7 \\ & x_1 - 3x_2 - x_3 \geq 2 \end{aligned}$$

P3. $$\begin{aligned} \min_{x\in\mathbb{R}^4} \ & 8x_1^2 + 2x_1x_2 + x_2^2 + 4x_3^2 + 4x_3x_4 + 7x_4^2 - 3x_1 - 4x_2 + 8x_3 + 3 \\ \text{s.t.} \ & \exp(x_1) + \exp(x_2) + \exp(x_3) + \exp(x_4) \leq 705 \\ & x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 25 \\ & 3 \leq \max(x_1 + 2x_3 + x_4, 4x_1 - x_2 + 6x_3) \leq 40 \end{aligned}$$

P4. $$\begin{aligned} \max_{x\in\mathbb{R}^3} \ & -5^{x_1^2+2x_2^2+x_3^2-x_1} \\ \text{s.t.} \ & \|x\|_2 \leq 1 \\ & \big(\cosh(x_1+x_2-x_3)\big)^3 \leq 125 \end{aligned}$$

P5. $$\begin{aligned} \min_{x\in\mathbb{R}^5} \ & \exp(x_1^2 + x_2^2 + x_3^2 + x_4) + \max\big(|2x_1 - 5x_2 + 8x_4 - 2|, (4x_1 - x_2 - 2x_3 + x_4)^6\big) \\ \text{s.t.} \ & 8x_1 + 6x_2 - x_3 + x_4 \geq 9 \\ & 2\cosh(x_1^2 + x_2^2) + 4\cosh(x_3 - x_4 + x_5) \leq 10000 \\ & \|x\|_2 \leq 256 \end{aligned}$$

P6. $\min_{x\in\mathbb{R}^5} \sqrt{7|x_1|+3|x_2|+|x_3|+2|x_4|+3|x_5|+1}$

s.t. $(6+3x_1+x_2-x_3+x_4-x_5)^3 \le 91$

$\min(3+2x_1+3x_2, 4x_3+3x_4, 2x_5+8) \ge 1$

$\|x\|_\infty \ge 2$

P7. $\min_{x\in\mathbb{R}^4} x_1^6+9x_2^2-8x_3x_4+2x_3-9$

s.t. $(1+x_1^4+x_2^2)^4+3x_2+x_3^2+3x_4^6+\cosh(x_1+2x_2+4x_3-x_4) = 25$

P8. $\max_{x\in\mathbb{R}^3} \min(4x_1-3x_2+8x_3-5,\ -2x_1+7x_2-x_3+1)$

s.t. $7|x_1|+2|x_2|+5|x_3| \leqslant 10$

P9. $\min_{\theta\in\mathbb{R}^4} \sum_{i=1}^{12} \big(|\alpha_i(\theta)|-1\big)^2$

where $\alpha_1(\theta), \alpha_2(\theta), \ldots, \alpha_{12}(\theta)$ are the (possibly complex) roots of the following parameterized 12-th degree polynomial in the variable y:

$$y^{12}+\theta_1 y^9+\exp(\theta_1-\theta_3\,\theta_4)y^6+\frac{1+\theta_3}{2+\theta_1^2+\sin\theta_2}y^4+\cos(\theta_4)y^2+y-1$$

where θ is the parameter.

P10. $\max_{x\in\mathbb{R}^3} \big(-\exp(x_1^2+2x_2^2+x_3^2-x_1)\big)$

s.t. $\|x\|_\infty \le 2$

$\arctan(x_1+x_2-x_3) \le \frac{\pi}{4}$

Tasks: For each of the problems P1 up to P10:

(a) Give the *most simple type* of optimization problem that the given problem can be reduced to, *irrespective* of the optimization algorithm used later on (So if you have a nonconvex constrained problem that you will solve using, e.g., a barrier or penalty function method in item (b), you should still mark NCC here). Select one of the following types:

- LP: (mixed-integer) linear programming problem
- QP: convex quadratic programming problem
- CP: convex optimization problem
- NCU: nonconvex unconstrained optimization problem
- NCC: nonconvex constrained optimization problem

Note that you may have to perform some extra steps first in order to *simplify* the optimization problem. If you do this, then you have to ***indicate*** it clearly in your reply to item (d).

(b) Now determine the *best suited optimization method* for the given problem. Select one of the following methods:

M1 : Simplex algorithm for linear programming
M2 : Modified simplex algorithm for quadratic programming
M3 : Gradient projection method with variable step size line minimization
M4 : Steepest descent method
M5 : Line search method with the Broyden-Fletcher-Goldfarb-Shanno direction and cubic line minimization
M6 : Lagrange method + Davidon-Fletcher-Powell quasi-Newton algorithm
M7 : Lagrange method + steepest descent method
M8 : Penalty function approach + steepest descent method
M9 : Barrier function approach + line search method with Powell directions
M10 : Interior point algorithm
M11 : Simulated annealing
M12 : Branch-and-bound method for mixed-integer linear programming

* If it is necessary to use the selected algorithm in a *multi-start optimization* procedure, you have to ***indicate*** this clearly in your reply to item (b) by putting a checkmark in the Multi-start box.

* Give only one solution! If more than one solution is given, the worst one will be taken.

(c) Give the most appropriate stopping criterion for the selected optimization method.
Do *not* reply by referring to your answer to a previous or subsequent problem. Such a statement will considered to be a wrong answer.

(d) ***Explain and motivate*** your reply to items (a) and (b).

QUESTION 2: Optimization methods II (20 points)

Tasks:

1. Explain how the gradient projection method for constrained optimization works. Illustrate your explanation by drawing one or more pictures. Also give the general iteration formula. Note that if you do not know the exact format of the projection formula, you are allowed to use the projection operator P_X to denote projection on the set X — but indicate then yourself what type of projection (oblique, orthogonal, ...) is used and what X is for this algorithm.
Note: It is not required to explain in detail how the line minimization part of the method works, i.e., it is sufficient to briefly indicate how the line minimization part of the approach works ***in general***, but you do not have to select and explain a particular line minimization method.

2. Now consider the following optimization problem:

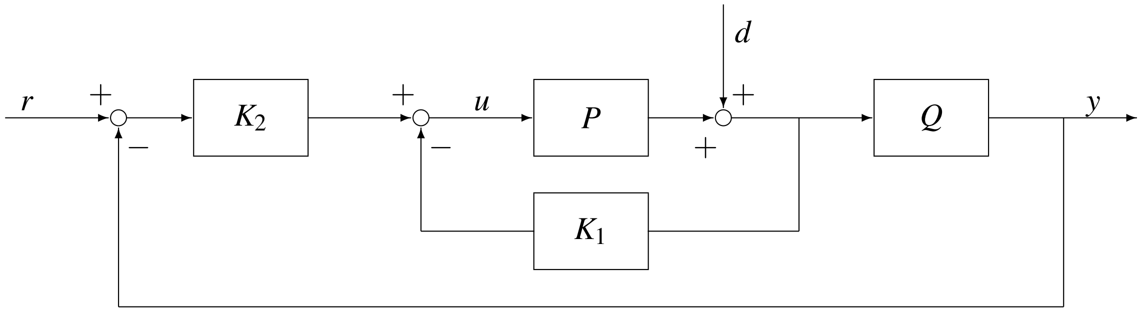
$$\begin{aligned}\min_{(x,y)\in\mathbb{R}^2} \;& x^2+2y^2-xy-4y+12 \\ \text{s.t. } \;& x,y \geqslant 0 \\ & x+y \leqslant 2 \;\; .\end{aligned}$$

   a) Is this optimization problem convex or not? Why?

   b) Now solve the given optimization problem using the gradient projection method and perform two iteration steps:
   Compute the first two search directions for the gradient projection method, and perform the subsequent line minimizations. The starting point is equal to $(x_0, y_0) = (1, 0)$ and the line minimization method is assumed to find the *exact* line minimum.

QUESTION 3: Controller design (20 points)

Consider the following closed-loop system consisting of the stable systems P and Q, and two controllers K_1 and K_2:



The signal r is the reference signal, d is a noise signal, y is the output of the system, and u is the input signal for system P.
The signals r and d are considered to be the input signals of the overall closed-loop system; the signals y and u are considered to be the output signals of the overall closed-loop system.

Tasks:

a) Determine the transfer function matrix M in terms of P, Q, K_1, and K_2 such that

$$\begin{bmatrix} u(k) \\ y(k) \end{bmatrix} = M(q) \begin{bmatrix} r(k) \\ d(k) \end{bmatrix}$$

where q denotes the forward shift operator.

b) Show that any controller parameterized by

$$K_1(q) = \frac{S_1(q)}{1 - P(q)S_1(q) - P(q)Q(q)S_2(q)}$$
$$K_2(q) = \frac{S_2(q)}{1 - P(q)S_1(q) - P(q)Q(q)S_2(q)}$$

where S_1 and S_2 are rational stable transfer functions, will stabilize the overall closed-loop system.

End of the exam