Exam — November 2024
Optimization for Systems and Control (SC42056)

The exam consists of 2 main questions. For Question 1 the maximal score for each problem is 9; as there are 8 problems, this implies that the maximal total score for Question 1 is 72. The maximal total score for Question 2 is 28. The detailed subscores are marked in red next to each task.

QUESTION 1 ($8 \times 9 = 72$ points)

Consider the following optimization problems:

P1.
$$\max_{x\in\mathbb{Z}^5} 5-\left(x_1^2+x_2^2+2x_1x_2+x_3^3+4x_4+8x_5\right)$$
$$\text{s.t. } (7x_1+2x_2+8x_3-9x_4+8x_5)^4 \leqslant 16$$
$$\exp(-x_1+3x_2-x_3+6x_4+x_5) \geqslant 9$$
$$\|x\|_1+\|x\|_\infty \leqslant 12$$

P2.
$$\min_{x\in\mathbb{R}^4} \frac{2}{(3^{x_1^2+3x_1x_2+4x_2^2}+|x_1-x_2+x_3-5x_4|-2048)^4}$$
$$\text{s.t. } 3^{x_1^2+3x_1x_2+4x_2^2}+|x_1-x_2+x_3-5x_4| \leqslant 2000$$
$$\min(|x_1+5x_2-4x_3+8x_4|, x_1-2x_2+8x_3-2x_4) \geqslant 2$$

P3.
$$\min_{x\in\mathbb{Z}^4} 2^{|5x_1+2x_2-4x_3+8x_4-5|}$$
$$\text{s.t. } (x_1+x_2-x_3-3x_4+1)^4 \geqslant 2^8$$
$$\min\left(5|x_1+20x_2+10x_3-18x_4-6|, x_1+8x_2-x_3-5x_4+2\right) \leqslant 625$$

P4.
$$\min_{x\in\mathbb{R}^3} \exp(x_1+5x_2+x_3-8)+\sinh(2x_1+8x_2+x_3-15)$$
$$\text{s.t. } \|x\|_\infty \leqslant 10$$
$$(5+3x_1+x_2-x_3)^3 \leqslant 1000$$
$$\min(3+2x_1+3x_2, 4x_2+3x_3, 8-x_1+x_2-x_3) \geqslant 2$$

P5.
$$\min_{x\in\mathbb{R}^3} 5^{(4x_1^2-4x_1x_2+2x_2^2+x_1-x_2+8x_3+7)^5}$$
$$\text{s.t. } (1+x_1^2+x_2^4+x_3^2)^2+0.01\exp(x_1+x_2+7x_3)+|x_1-3x_2+x_3| = 512$$

P6. $\max_{x\in\mathbb{R}^4} \ 1-2\exp\Big((x_1+x_2-4x_3-x_4+3)^2+3\log(x_1+2x_2+6x_3+x_4+1)\Big)$

$$\text{s.t. } \min(x_1+2x_2+6x_3,x_4)\leqslant 2$$
$$1\leqslant(x_1+2x_2+3x_3+4x_4)^3\leqslant 125$$

P7. $\min_{x\in\mathbb{R}^4} \ \arctan\left(1+(2x_1^2+4x_1x_2+2x_2^2+2x_3^2-8x_3x_4+16x_4^2)^2+3^{(x_1-8x_2+9x_3-8x_4)^4}\right)$

$$\text{s.t. } 1\leqslant(x_1+2x_2+3x_3+4x_4)^3\leqslant 729$$
$$15-\cosh(x_1+3x_2)-|3x_2-2x_3+5x_4|^3\geqslant 3$$

P8. $\min_{x\in\mathbb{R}^4} \ (1+|x_1+x_2+8x_3-5x_4|)^6$

$$\text{s.t. } 2\leqslant\|x\|_\infty\leqslant 25$$
$$\min(x_1+3x_2-6x_3+2,x_2-8x_3+2x_4+3,x_2-x_4)\leqslant 1$$
$$\max\left((x_1+2x_2+3x_3)^2,(4x_1-7x_2+9x_4)^3\right)\geqslant 3$$

Important for all tasks:

* Give only one solution! If more than one solution is given, the worst one will be taken.
* In your answer do *not* refer to your answer to a previous or a subsequent problem. Such a statement will considered to be a wrong answer.
* A correct answer without the correct motivation will be considered to be a wrong answer.

<u>Tasks:</u> For each of the problems P1 up to P8:

1 (a) Reduce the given optimization problem to the *most simple type* of *minimization* problem that the given problem can be reduced to.
Label the constraints using the **labels** (1), (2), (3), etc.
In your answer you can use f to denote the original objective function (in the original minimization or maximization format), g to refer to the inequality constraint function (in the format $g(x) \leqslant 0$), and h to the equality constraint function (in the format $h(x) = 0$).
Explain in Task (f) how you reduced the problem.

4 (b) For the *simplified minimization* problem you obtained in Task (a), indicate whether the objective function is linear, convex quadratic, convex, or nonconvex over the feasible set (***<u>ex</u>cluding*** constraints like $x_i \in \mathbb{Z}$ that restrict the variables to a discrete set, as such constraints would in general make the objective function nonconvex). Select the most restrictive option.
Motivate your answer in Task (f).

Moreover, for ***each*** individual constraint of the *simplified* problem examine whether the constraint is linear/affine, convex, or nonconvex over the feasible set consisting of the other constraints (***<u>ex</u>cluding*** constraints like $x_i \in \mathbb{Z}$ that restrict the variables to a discrete set, as such constraints would in general make the constraint nonconvex). Select the most restrictive option.
List the **labels** (as defined in Task (a)) of the linear, convex, and nonconvex constraints, if any.
Motivate your answer in Task (f).

1 (c) Characterize the type of the *simplified* optimization problem you obtained in Task (a), *irrespective* of the optimization algorithm used later on (so if you have a nonconvex constrained problem that you will solve using a barrier or penalty function method in item (d), you should still mark NCC here). Select one of the following types:

- LP: linear programming problem
- QP: convex quadratic programming problem
- CP: convex optimization problem
- NCU: nonconvex unconstrained optimization problem
- NCC: nonconvex constrained optimization problem
- MILP: mixed-integer linear programming problem

Select the most restrictive option.
The first checkbox serves to indicate that a problem can be recast as multiple, more simple problems. If you obtain, e.g., 4 linear programming problems, check — next to the LP checkbox — also the first checkbox and fill out 4 on the dots: ☒ 4× ☒ LP
Motivate your answer in Task (f).

2.5 (d) Now determine the *best suited optimization method* for the given problem. Select one of the following methods:

M1 : Simplex algorithm for linear programming

M2 : Modified simplex algorithm for quadratic programming

M3 : Gradient projection method with variable step size line minimization

M4 : Ellipsoid algorithm

M5 : Broyden-Fletcher-Goldfarb-Shanno quasi-Newton algorithm

M6 : Line search method with the steepest descent direction and golden section line minimization

M7 : Lagrange method + steepest descent method

M8 : Lagrange method + line search method with Powell directions

M9 : Barrier function approach + steepest descent method

M10 : Penalty function approach + line search method with Powell directions and variable step size line minimization

M11 : Branch-and-bound method for mixed-integer linear programming

M12 : Genetic algorithm

Motivate your answer in Task (f).

* If it is necessary to use the selected algorithm in a *multi-start or multi-run optimization* procedure, you have to ***indicate*** this clearly in your answer. Then also motivate why multi-start or multi-run optimization is needed.

0.5 (e) Give the most appropriate stopping criterion for the selected optimization method.
In your answer you can use f_s to denote the simplified objective function in the minimization format of the reply to Task (a), g_s to refer to the inequality constraint function of Task (a) (in the format $g_s(x) \leqslant 0$), and h_s to the equality constraint function of Task (a) (in the format $h_s(x) = 0$).

(f) Motivate the answers to Tasks (a), (b), (c), and (d).

QUESTION 2 $(11 + 17 = 28$ points)

Question 2.1 (11 points)

Tasks:

8 (a) Explain how the ellipsoid algorithm works in case the number of optimization variables is larger than 1. Illustrate your explanation by drawing one or more pictures.
Note: It is not needed to provide the exact expressions for the iteration formulas. However, the main ingredients of the iteration formulas should be listed.

2 b) What are the two main advantages of the ellipsoid algorithm compared to the cutting-plane algorithm?
Motivate your answer.

1 c) What is the main advantage of the ellipsoid algorithm compared to the interior-point algorithm?
Motivate your answer.

Question 2.2 (17 points)

Consider the following optimization problem:

$$\begin{aligned} \max_{(x,y)\in\mathbb{R}^2} \ & \sinh(7 - 4x^2 - y^2 + 2xy + 32x - 20y) \\ \text{s.t. } & |x| + |2y| \leq 8 \\ & 1 \leq 2^{(y-1)^2} \leq 2^{16} \end{aligned}$$

Tasks:

2 a) Is the given *original* optimization problem convex or not? Motivate your answer.

13 b) Simplify the given optimization problem and apply **two** iteration steps of the gradient projection method on the resulting simplified optimization problem:
Take $(x_0, y_0) = (2, 0)$ as starting point, compute two consecutive search directions for the gradient projection method, and each time perform the corresponding line minimization, where for the line minimization you should determine the *exact* line minimum.
Explain your answer.

2 c) Is the point (x_2, y_2) found as a result of the second iteration of the gradient projection method above, a global optimum, a local optimum, or no optimum at all of the *original* optimization problem?
Motivate your answer.

End of the exam

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