Resit — November 2024 – Grading template
Optimization for Systems and Control (SC42056)

Important: Please recall the following instructions from the exam procedure:

- Note that – just as in previous years – **correct results without proper and correct motivation will not receive any marks.**

Additional scoring guidelines

S0: correct result without proper and correct motivation: 0
likewise: if in Question 1 the answer for (c) and/or (d) is formally correct, but an error is made in (a) or (b) that affects the result for (c) and/or (d): 0 for (c) and/or (d)

S1: partially incomplete motivation for (non)convexity or simplification: −50%

S2a: small computation error that *does* affect result: −0.5

S2b: small computation error that does *not* affect result: −0.25

S3: missing, wrong, or not properly motivated $N\times$ in Question 1(c): −2 for (c) and (d) together

S4: multi-start listed when it is not needed: −0.5

S5: ∇f as row vector: −0.5

S6: redundant function in stopping criterion: 0

S7: introduction of redundant variables or constraints that are not needed at all: −1

S8: introduction of additional wrong constraints and/or wrong classification of extra/unsimplified constraint: −1

S9: as indicated in instructions: if 2 or more solutions are given, the worst one is assumed to have been selected

CE: Even if the answers to (a)-(d) are wrong, you can sometimes still score marks for (e) if and only if (a)-(d) are internally consistent and if they all result in the answer given in (e) and if (e) is **100 %** correct and complete.

On the next pages an outlines of answers are given with scores marked in red. To earn the indicated score the corresponding answer has to be given completely, including the information inside the brackets; else the score is 0.

QUESTION 1 ($8 \times 9 = 72$ points)

- **P1**

1 (a) First we transform the maximization problem into a minimization problem, which yields

$$\min_{x \in \mathbb{Z}^5} f_s(x) = x_1^2 + x_2^2 + 2x_1x_2 + x_3^3 + 4x_4 + 8x_5$$

For simplifications of the constraints, see (b)

4 (b) Due to the term x_3^3 the simplified objective function is nonconvex: 1.5
As $(\cdot)^4$ is a U-shaped function, constraint (1) can be rewritten as the **intersection** of 2 linear inequality constraints: 0.5

$$-2 \leqslant 7x_1 + 2x_2 + 8x_3 - 9x_4 + 8x_5 \leqslant 2$$

As $\exp(\cdot)$ is an increasing function, constraint (2) can be rewritten as

$$-x_1 + 3x_2 - x_3 + 6x_4 + x_5 \geqslant \log 9$$

which is a linear inequality constraint: 0.5
Constraint (3) can be rewritten as

$$|x_1| + |x_2| + |x_3| + |x_4| + |x_5| + \max(|x_1|, |x_2|, |x_3|, |x_4|, |x_5|) \leqslant 12$$

which is equivalent to the **intersection** of the 5 following constraints:

$$\begin{aligned} 2|x_1| + |x_2| + |x_3| + |x_4| + |x_5| &\leqslant 12 \\ |x_1| + 2|x_2| + |x_3| + |x_4| + |x_5| &\leqslant 12 \\ |x_1| + |x_2| + 2|x_3| + |x_4| + |x_5| &\leqslant 12 \\ |x_1| + |x_2| + |x_3| + 2|x_4| + |x_5| &\leqslant 12 \\ |x_1| + |x_2| + |x_3| + |x_4| + 2|x_5| &\leqslant 12 \end{aligned}$$

which results in $5 \cdot 2^5 = 5 \cdot 32 = 160$ linear inequality constraints:

$$\begin{aligned} \pm 2x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 &\leqslant 12 \\ \pm x_1 \pm 2x_2 \pm x_3 \pm x_4 \pm x_5 &\leqslant 12 \\ \pm x_1 \pm x_2 \pm 2x_3 \pm x_4 \pm x_5 &\leqslant 12 \\ \pm x_1 \pm x_2 \pm x_3 \pm 2x_4 \pm x_5 &\leqslant 12 \\ \pm x_1 \pm x_2 \pm x_3 \pm x_4 \pm 2x_5 &\leqslant 12 \quad : 1.5 \end{aligned}$$

1 (c) NCC: nonconvex constrained optimization problem

2.5 (d) As the variables should be integers, the best choice is:
multi-start: 0.5 + M12: Genetic algorithm: 2

0.5 (e) maximum number of generations reached

- **P2**

1 (a) First we check the sign of the argument inside the brackets in the denominator of the objective function. From the first constraint we find that the term $3^{x_1^2+3x_1x_2+4x_2^2} + |x_1 - x_2 + x_3 - 5x_4|$ is less than or equal to 2000; so after subtracting 2048 we get a strictly negative value. The function $1/(\cdot)^4$ is increasing for strictly negative arguments. So we can **minimize** the argument instead. Moreover, the constant term -2048 can be dropped. This leads to the following simplified optimization problem: 1

$$\min_{x\in\mathbb{R}^4} f_s(x) := 3^{x_1^2+3x_1x_2+4x_2^2} + |x_1 - x_2 + x_3 - 5x_4|$$

For simplifications of the constraint, see (b)

4 (b) The first term of the simplified objective function f_s is convex since $3^{(\cdot)}$ is a convex increasing function and the exponent $x_1^2 + 3x_1x_2 + 4x_2^2$ is convex (as its Hessian is positive definite or alternatively as it can be written as a positive sum of squares: $(x_1 + 1.5x_2)^2 + 1.75x_2^2$). In addition, the second term of f_s is also convex since $|\cdot|$ is convex and its argument is affine. Hence, f_s is convex: 2

Constraint (1) is also convex since function on the left-hand side is convex as shown above: 0.5

Constraint (2) can be rewritten as the **intersection** of 2 constraints:

$$|x_1 + 5x_2 - 4x_3 + 8x_4| \geqslant 2 \qquad \text{and} \tag{2a}$$

$$x_1 - 2x_2 + 8x_3 - 2x_4 \geqslant 2 \tag{2b}$$

Constraint (2b) is an linear inequality constraint. As $|.|$ is a V-shaped function, (2a) is equivalent to

$$x_1 + 5x_2 - 4x_3 + 8x_4 \geqslant 2 \qquad \text{or} \qquad x_1 + 5x_2 - 4x_3 + 8x_4 \leqslant -2$$

which is the **union** of 2 linear inequality constraints: 1.5

1 (c) $2 \times$ CP: convex optimization problem

2.5 (d) M4: Ellipsoid algorithm
If multi-start is checked: -0.5

0.5 (e) $|f_s(x_k) - f_s(x^*)| \leqslant \varepsilon_f$ and/or $\|x_k - x^*\|_2 \leqslant \varepsilon_x$, **and** $g_s(x_k) \leqslant \varepsilon_g$

- **P3**

1 (a) As the function $2^{(\cdot)}$ is a non-decreasing function, we can minimize the argument instead. This leads to the following simplified optimization problem: 0.5

$$\min_{x \in \mathbb{Z}^4} f_s(x) := |5x_1 + 2x_2 - 4x_3 + 8x_4 - 5|$$

Since $|v| = \max(v, -v)$ for a scalar $v \in \mathbb{R}$, we can introduce a dummy variable t and two linear constraints and consider the equivalent problem $\min_{t,x} t$ subject to $t \geqslant 5x_1 + 2x_2 - 4x_3 + 8x_4 - 5$ and $t \geqslant -5x_1 - 2x_2 + 4x_3 - 8x_4 + 5$: 0.5
For simplifications of the constraint, see (b)

4 (b) The simplified objective function and the corresponding constraint introduced in (a) above are all affine: 0.5
Since $(\cdot)^4$ is a U-shaped function the constraint (1) can be rewritten as a **union** of 2 linear inequality constraints : 1.5

$$x_1 + x_2 - x_3 - 3x_4 + 1 \geqslant 2^2 = 4 \qquad \textbf{or} \qquad x_1 + x_2 - x_3 - 3x_4 + 1 \leqslant -2^2 = -4$$

Constraint (2) can be written as the **union** of 2 constraints:

$$5|x_1 + 20x_2 + 10x_3 - 18x_4 - 6| \leqslant 625 \qquad \textbf{or} \qquad x_1 + 8x_2 - x_3 - 5x_4 + 2 \leqslant 625$$

which is equivalent to the **union** of the following linear inequality constraints: 2

$$-125 \leqslant x_1 + 20x_2 + 10x_3 - 18x_4 - 6 \leqslant 125 \qquad \textbf{or} \qquad x_1 + 8x_2 - x_3 - 5x_4 + 2 \leqslant 625 \tag{1}$$

1 (c) $2 \times 2 = 4 \times$ MILP: nonconvex constrained optimization problem

2.5 (d) M11 : Branch-and-bound method for mixed-integer linear programming

0.5 (e) Optimum is found once entire tree is explored

- **P4**

1 (a) As $\exp(\cdot)$ and $\sinh(\cdot)$ do not have the same argument, the objective function cannot be simplified: 1
For simplifications of the constraint, see (b)

4 (b) As the function $\sinh(\cdot)$ is nonconvex, the objective function is nonconvex: 1
Constraint (1) can be written as $\max(|x_1|, |x_2|, |x_3|) \leqslant 10$, which is equivalent to the **intersection** of $3 \times 2 = 6$ linear inequality constraints: 1

$$\pm x_1 \leqslant 10 \quad \textbf{and} \quad \pm x_2 \leqslant 10 \quad \textbf{and} \quad \pm x_3 \leqslant 10$$

Since $(\cdot)^3$ is an increasing function, constraint (2) can be rewritten as $5 + 3x_1 + x_2 - x_3 \leqslant 10$, which is a linear inequality constraint: 1
Constraint (3) can be rewritten as the **intersection** of 3 linear inequality constraints: 1

$$3 + 2x_1 + 3x_2 \geqslant 2 \quad \textbf{and} \quad 4x_2 + 3x_3 \geqslant 2 \quad \textbf{and} \quad 8 - x_1 + x_2 - x_3 \geqslant 2$$

1 (c) NCC: nonconvex constrained optimization problem

2.5 (d) The gradient of the objective function can be computed analytically, while the constraints are linear. So the best choice is:
multi-start : 0.5 + M3: Gradient projection method: 2

0.5 (e) KKT conditions with ε (list them!, no h)

- **P5**

1 (a) As the function $5^{(\cdot)}$ is a non-decreasing function, we can minimize the argument instead. Moreover, as $(\cdot)^5$ is a also non-decreasing function, we can minimize its argument instead. After dropping the constant 7, this leads to the following simplified optimization problem: 1

$$\min_{x\in\mathbb{R}^4} f_s(x) := 4x_1^2 - 4x_1x_2 + 2x_2^2 + x_1 - x_2 + 8x_3$$

For simplifications of the constraint, see (b)

4 (b) Note that $4x_1^2 - 4x_1x_2 + 2x_2^2$ is convex as its Hessian is positive definite or as it can be written as a positive sum of squares: $(2x_1 - x_2)^2 + x_2^2$. The other terms of f_s are affine and thus convex. Hence, f_s is a convex quadratic function: 2

The constraint is nonconvex as it is an equality constraint and as $h(\cdot)$ is not affine: 2

1 (c) NCC: nonconvex constrained optimization problem

2.5 (d) The gradient of the objective function and the (sub)gradient of the function on the left-hand side of the constraint can be computed analytically. So an appropriate choice is: multi-start : 0.5 + M7: Lagrange + steepest descent: 2

Alternatively, if it is argued explicitly that the function $|\cdot|$ that appears in the constraint is not differentiable everywhere, and that therefore multi-start + M8: Lagrange + line search method with Powell directions, is selected, then that will also be considered to be a correct reply.

0.5 (e) $\|\nabla f_s(x_k) + \nabla h_s(x_k)\lambda\|_2 \leqslant \varepsilon_1$ and $\|h_s(x_k)\|_2 = |h_s(x_k)| \leqslant \varepsilon_2$

Alternatively, if M8 (Lagrange + Powell) is selected in (d), the stopping criterion should be $|f_s(x_k) - f_s(x_{k-1}| \leqslant \varepsilon_1$ and/or $\|x_k - x_{k-1}\|_2 \leqslant \varepsilon_2$, **and** $|h_s(x)k)| \leqslant \varepsilon_3$

P6

1 (a) The maximization problem is first transformed into a minimization problem. The constant -1 can be then dropped. Since the function $2\exp(\cdot)$ is an increasing function, we can minimize its argument instead, which leads to: 1

$$\min_{x} f_{\mathrm{s}}(x) = (x_1 + x_2 - 4x_3 - x_4 + 3)^2 + 3\log(x_1 + 2x_2 + 6x_3 + x_4 + 1)$$

For simplifications of the constraints, see (b)

4 (b) Although the first term of f_{s} is convex (as $(\cdot)^2$ is convex and as its argument is affine), the second term is not convex as $\log(\cdot)$ is not convex and its argument is affine. Hence, f_{s} is nonconvex: 2
Constraint(1) can be rewritten as the **union** of 2 linear constraints: $x_1 + 2x_2 + 6x_3 \leqslant 2$ **or** $x_4 \leqslant 2$: 1
Since $(\cdot)^3$ is an increasing function, constraint (2) can be rewritten as $1 \leqslant x_1 + 2x_2 + 3x_3 + 4x_4 \leqslant 5$, which is a linear inequality constraint: 1

1 (c) $2 \times$ NCC: nonconvex constrained optimization problem

2.5 (d) The gradient of the objective function can be computed analytically, while the constraints are linear. So the best choice is:
multi-start : 0.5 + M3: Gradient projection method: 2

0.5 (e) KKT conditions with ε (list them!, no h)

P7

1 (a) Since $\arctan(\cdot)$ is increasing, we can minimize the argument instead. Furthermore, the constant 1 can be dropped, which leads to: 1

$$\min_{x\in\mathbb{R}^4} f_s(x) = (2x_1^2+4x_1x_2+2x_2^2+2x_3^2-8x_3x_4+16x_4^2)^2+3^{(x_1-8x_2+9x_3-8x_4)^4}$$

4 (b) The first term of the simplified objective function is of the form $h(g(\cdot))$ with h being convex and increasing for positive arguments (which is the case here, as explained next), and with g being convex and nonnegative, as it can be written as a sum of squares: $(\sqrt{2}x_1+\sqrt{2}x_2)^2+x_3^2+(x_3-4x_4)^2$. So the first term of f_s is convex. The second term of f_s is also of the form $h(g(\cdot))$ with h being convex and increasing, and with g being convex as it is an even power of an affine argument. So the second term of f_s is also convex. Hence, f_s is a convex function: 2

As $(\cdot)^3$ is an increasing function, constraint (1) can be rewritten as

$$1 \leqslant x_1+2x_2+3x_3+4x_4 \leqslant 9$$

which are linear inequality constraints: 0.5

Constraint (2) can be rewritten as

$$\cosh(x_1+3x_2)+|3x_2-2x_3+5x_4|^3 \leqslant 12 \qquad \text{(2bis)}$$

The function $\cosh(\cdot)$ is convex and its argument is affine. So the first term on the left-hand side of (2bis) is convex. The function $(\cdot)^3$ is convex and increasing for nonnegative arguments (which applies here due to the absolute value). Moreover, $|\cdot|$ is convex function and its argument is affine. Hence, the second on the left-hand side of (2bis) is also convex. So (2bis) is a convex constraint: 1.5

1 (c) CP: convex optimization problem

2.5 (d) M4: Ellipsoid algorithm
If multi-start is checked: -0.5

0.5 (e) $|f_s(x_k)-f_s(x^*)| \leqslant \varepsilon_f$ and/or $\|x_k-x^*\|_2 \leqslant \varepsilon_x$, **and** $g_s(x_k) \leqslant \varepsilon_g$

P8

1 (a) Since $(\cdot)^6$ is increasing for positive arguments (which is the case here), we can minimize the argument instead. The constant 1 can be dropped, which leads to $\min_{x\in\mathbb{R}^4} |x_1 + x_2 + 8x_3 - 5x_4|$: 0.5
Since $|v| = \max(v, -v)$ for a scalar $v \in \mathbb{R}$, we can introduce a dummy variable t and two linear constraints and consider the equivalent problem: 0.5

$$\min_{t,x} t \quad \text{subject to } t \geqslant x_1 + x_2 + 8x_3 - 5x_4 \text{ and } t \geqslant -(x_1 + x_2 + 8x_3 - 5x_4)$$

4 (b) The simplified objective function and the corresponding constraint introduced in (a) above are all affine: 0.5
Constraint (1) can be rewritten as $2 \leqslant \max(|x_1|, |x_2|, |x_3|, |x_4|) \leqslant 25$. The first part $2 \leqslant \ldots$ is nonconvex and can be written as the **union** of $2 \cdot 4 = 8$ linear inequality constraints: 1

$$2 \leqslant x_1 \quad \textbf{or} \quad 2 \leqslant -x_1 \quad \textbf{or} \quad \ldots \quad 2 \leqslant x_4 \quad \textbf{or} \quad 2 \leqslant -x_4$$

The second part $\ldots \leqslant 25$ can be written as the intersection of $2 \cdot 4 = 8$ linear inequality constraints: 0.5

$$\pm x_1 \leqslant 25 \quad \textbf{and} \quad \ldots \quad \textbf{and} \quad \pm x_4 \leqslant 25$$

Constraint (2) can be written as the **union** of 3 linear inequality constraints: 0.5

$$\begin{aligned} &x_1 + 3x_2 - 6x_3 + 2 \leqslant 1 \quad \textbf{or} \\ &x_2 - 8x_3 + 2x_4 + 3 \leqslant 1 \quad \textbf{or} \\ &x_2 - x_4 \leqslant 1 \end{aligned}$$

Constraint (3) can be written as the **union** of 2 constraints: (3a) $(x_1 + 2x_2 + 3x_3)^2 \geqslant 3$ **or** (3b) $(4x_1 - 7x_2 + 9x_4)^3 \geqslant 3$. Since $(\cdot)^2$ is a U-shaped function the constraint (3a) results in the **union** of 2 linear inequality constraints: $x_1 + 2x_2 + 3x_3 \geqslant \sqrt{3}$ **or** $x_1 + 2x_2 + 3x_3 \leqslant -\sqrt{3}$. Since $(\cdot)^3$ is an increasing function, constraint (3b) can be written as a linear inequality constraint $4x_1 - 7x_2 + 9x_4 \geqslant \sqrt[3]{3}$. So constraint (3) results in the **union** of 3 linear constraints: 1.5

1 (c) $8 \times 3 \times 3 = 72 \times$ LP: linear programming problem

2.5 (d) M1: Simplex algorithm for linear programming

0.5 (e) The simplex algorithm will always find a global optimum in a finite number of iterations

QUESTION 2 $(11 + 17 = 28$ points)

- **Question 2.1**

8 (a) Mention/provide at least the following:

[1] used for convex problems: 0.5

[2] representation of feasible set with ellipsoids E_k: 0.5

[3] use **<u>sub</u>**gradient inequality to discard points: 0.5

[4] in each step we first intersect ellipsoid with hyper-half-space and next determine smallest ellipsoid that contains the intersection: 0.5

[5] if center point is feasible $\rightarrow$ objective iteration: discard points with higher function value: 1

[6] if center point is infeasible $\rightarrow$ constraint iteration: discard points that are infeasible: 1

[7] update formulas for center and matrix of ellipsoid are based on subgradient: 0.5

[8] stop criterion: volume$(E_k) \leqslant \varepsilon_1$, $|f(x_k) - f(x^*)|_2 \leqslant \varepsilon_2$ with x^* the real but unknown minimum, $g(x_k) \leqslant \varepsilon_3$: 1.5

[9] picture: 2

In case of wrong statements, a penalty of -0.75 applies for each wrong statement.

2 (b) For the final solution the ellipsoid algorithm, we also know how far we are maximally away from the true optimum in the variable space: 1

The size of the LP problems in the cutting-plane and thus also the time needed per iteration increase with each iteration; for the ellipsoid algorithm the computational complexity does not increase as the algorithm progresses: 1

1 (c) For the final solution the ellipsoid algorithm, we also know how far we are maximally away from the true optimum in the variable space: 1

Potential numerical issues due to use of barrier function if optimum is on the boundary of the feasible set is also counted as a correct reply provided the issue is explained properly

- **Question 2.2**

2 (a) If the problem is characterized as convex, the score for the entire subquestion (a) will be 0 Note that formulation of the question indicates that we should not simplify the problem yet.

First of all we transform the maximization problem into a minimization problem, which yields

$$\min_{(x,y)} -\sinh(7-4x^2-y^2+2xy+32x-20y)$$

As the argument of $-\sinh(\cdot)$ can reach positive values and negative values, and as $\sinh(\cdot)$ (and thus also $-\sinh(\cdot)$) is nonconvex on $\mathbb{R}$, the objective function, and thus also the entire problem is nonconvex: 2

13 (b) Now we simplify the problem. As $-\sinh(\cdot)$ is a decreasing function, we can maximize the argument instead. if we transform the maximization problem into a minimization problem and drop the constant 7, we get

$$\min_{(x,y)} 4x^2+y^2-2xy-32x+20y$$

The first constraint can be rewritten as the intersection of 4 linear inequality constraints:

$$\pm x \pm 2y \leq 8$$

As $2^{(\cdot)}$ is an increasing function, the second constraint can be rewritten as $0 \leq (y-1)^2 \leq 16$. Moreover, the part $0 \leq (y-1)^2$ always hold and can thus be dropped. The other part of the constraint can be rewritten as $|y-1| \leq 4$ or equivalently $-3 \leq y \leq 5$. This then results in the following simplified optimization problem: 2

$$\begin{aligned} \min_{(x,y)\in\mathbb{R}^2} \quad & 4x^2+y^2-2xy-32x+20y \\ \text{s.t.} \quad & x+2y \leqslant 8 \\ & x-2y \leqslant 8 \\ & -x+2y \leqslant 8 \\ & -x-2y \leqslant 8 \\ & -3 \leq y \leq 5 \end{aligned}$$

Note that all the constraints are linear inequality constraints. So we can indeed apply the gradient projection method.

First we compute the gradient of the objective function: 1

$$\nabla f(x,y) = \begin{bmatrix} 8x-2y-32 \\ 2y-2x+20 \end{bmatrix}$$

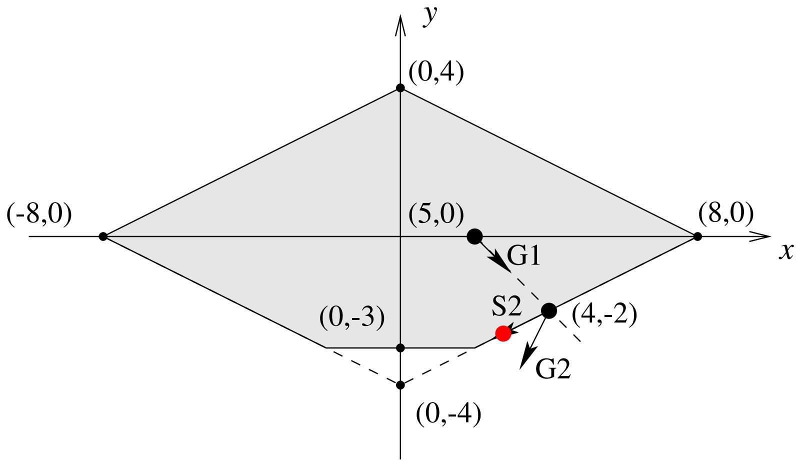
If a sign error is made here or any other computation error later on, marks can still be scored for: use of negative gradient [1], line search equation [1], and the determination of s^* [1], unless s^* is negative or leads to an infeasible point.

In the gradient projection approach the search direction is the **negative** gradient: 1

So we evaluate the negative gradient in the point $(2,0)$: $-\nabla f(2,0) = \begin{bmatrix} 16 & -16 \end{bmatrix}^T$. Since $(2,0)$ is in the interior of the feasible set, no projection is required. So we perform a line search in the direction $(16,-16)$ or equivalently $(1,-1)$, which yield the search line $(x,y) = (2,0) + s(1,-1) = (2+s,-s)$ where s is the step size: 1

If we fill out the values for x and y in the objective function and simplify, we obtain

$\bar{f}(s) = 7s^2 - 32s - 48$. We first take the derivative w.r.t. s and put it equal to 0: 1
The resulting value is of s is $s^* = \dfrac{32}{14} = \dfrac{16}{7} \approx 2.29$. As this value is positive it corresponds to the line minimum. However, this step size would create an infeasible point: $\left(\dfrac{30}{7}, -\dfrac{16}{7}\right) \approx (4.29, -2.29)$. So we also take the constraints into account: $\pm x \pm 2y \leq 8$, $-3 \leq y \leq 5$, which imply that $-\dfrac{10}{3} \leq s \leq 2$. So the largest allowed step size is $s = 2$. This yields the point $(x_1, y_1) = (4, -2)$: 3
The negative gradient in $(x_1, y_1) = (4, -2)$ is $-\nabla f(4, -2) = \begin{bmatrix} -4 & -8 \end{bmatrix}^{\mathrm{T}}$, which is pointing away from the feasible set. So we have to project $(-4, -8)$ on the active boundary, i.e., on the line $x - 2y = 8$: 1
The directional vector of this line is $(2, 1)$. Since $(-4, -8)$ points in the direction of decreasing y values, the projected search direction thus is $(-2, -1)$: 1
The search line is thus defined by $(x, y) = (4, -2) + t(-2, -1) = (4 - 2t, -2 - t)$ where t is the step size. If we fill out the values for x and y in the objective function and simplify, we obtain $\bar{f}(t) = 13t^2 - 16t - 84$. We first take the derivative w.r.t. t and put it equal to 0. The resulting value is of t is $t^* = \dfrac{16}{26} = \dfrac{8}{13} \approx 0.62$; as this value is positive, it corresponds to the line minimum. Moreover, the resulting point $(x_2, y_2) = \left(\dfrac{36}{13}, -\dfrac{34}{13}\right) = \left(2 + \dfrac{10}{13}, -2 - \dfrac{8}{13}\right) \approx (2.77, -2.62)$ is feasible: 2
A graphical representation of the procedure above is given by the following picture, where the gray area represents the feasible set and the red dot corresponds to the point (x_2, y_2); G1 and G2 indicate the direction of the negative gradient, and S2 indicates the direction of the projected gradient:



2 (c) First note that the minima of the original problem and the simplified problem coincide: 0.25

Moreover, the simplified problem is convex as the simplified objective function can be written as a sum of squares ($3x^2 + (x+y)^2$) and an affine term, and it is thus convex. Moreover, all the constraints are all linear inequality constraints: 0.25

We can compute the negative gradient in (x_2, y_2): $\nabla f(x_2, y_2) = \begin{bmatrix} \frac{-60}{13} & \frac{120}{13} \end{bmatrix}^{\mathrm{T}}$ and see that it points away from the feasible region and that it is orthogonal to the active boundary (which has direction vector $s_2 = \begin{bmatrix} -2 & -1 \end{bmatrix}^{\mathrm{T}}$, as indicated above; it is easy to verify that $(\nabla f(x_2, y_2))^{\mathrm{T}} s_2 = 0$): 0.5

This also implies that the projection of the gradient in (x_2, y_2) onto the active boundary will be the zero vector. This in turn implies that the point (x_2, y_2) is a local optimum, and since the simplified problem is convex and its minima coincide with those of the original problem, (x_2, y_2) is thus also the global optimum of the original problem: 1