1

2024 - 10 - 8

$$\min_x f(x)$$
$$g(x) \leq 0$$
$$h(x) = 0$$

✓ LP
✓ QP
convex

✓ unconstrained
— constrained

nonlinear, nonconvex

$$\min_{x, y \in \mathbb{R}} e^x + y^3 + 4x^2y^2$$

$$2x + y = 7 \rightarrow y = 7 - 2x$$

$$\hookrightarrow \min_{x \in \mathbb{R}} e^x + (7-2x)^3 + 4x^2(7-2x)^2 \rightarrow \text{unconstrained}$$

(2)

$m \times n$

$A\,x = b$ $\qquad x = \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 7$

$\boxed{A x_0 = b}$ $\qquad \boxed{A . \bar{A}^T = 0}$

$x = x_0 + \bar{A}^T . \bar{x} \longrightarrow \underbrace{A x_0}_{b} + \underbrace{A \bar{A}^T}_{0} \bar{x} = b \rightarrow$ is solution

$\bar{x}$ ↳ free vector $\in \mathbb{R}^{n-m}$

$\min_{\bar{x}} f(x_0 + \bar{A}^T . \bar{x})$

↳ unconstrained

$\underline{x_0}$ $\qquad 2x + 1.y = 7$

$n = 2$
$m = 1$

$\bar{x} \in \mathbb{R}$

$x = 0$
$y = 7$ $\qquad x_0 = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{\bar{A}} = 0$

$\alpha = 1 \quad \beta = -2$

$x = \begin{bmatrix} 0 \\ 7 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} . \bar{x} = \begin{bmatrix} \bar{x} \\ 7 - 2\bar{x} \end{bmatrix}$ $\quad \bar{x} \in \mathbb{R}$

$x = \bar{x}$
$y = 7 - 2\bar{x}$

$\min_{\bar{x} \in \mathbb{R}} e^{\bar{x}} + (7 - 2\bar{x})^3 + 4\bar{x}^2 (7 - 2\bar{x})^2$

③ SVD — Singular Value Decomposition

$$\underset{m}{\overset{n}{\boxed{\quad A \quad}}} = \underset{m\times m}{U} \, [\overbrace{\Sigma \;\; 0}^{m \times n}] \, \underset{n\times n}{V^T}$$

Σ → diagonal

diagonal entries ≥ 0

A full rank → $\Sigma_{ii} > 0$

$\mathrm{rank}(A) = m$

$$= U \cdot \underset{m\times m}{[\Sigma \;\; 0]} \begin{array}{c} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \end{array} \begin{array}{l} m \\ n-m \end{array} = U \cdot \Sigma \cdot V_1^T$$

$U \cdot U^T = I = U^T U$

$V \cdot V^T = I = V^T V$

↓

$V_1 V_1^T = I = V_1^T V_1$

$V_2 V_2^T = I = V_2^T V_2$

$V_1 V_2^T = 0 = V_2^T \cdot V_1$

$V_1^T V_2 = 0$

$$\boxed{\begin{aligned} x_0 &= V_1 \Sigma^{-1} U^T b \\ \bar{A} &= V_2^T \end{aligned}}$$

→ $A x_0 \overset{?}{=} b$

$$\underbrace{(U \cdot \underbrace{\Sigma \cdot \underbrace{V_1^T) \cdot (V_1}_{I} \Sigma^{-1}}_{I} U^T}_{I} b) = b \; !$$

$A \cdot \bar{A}^T \overset{?}{=} 0$

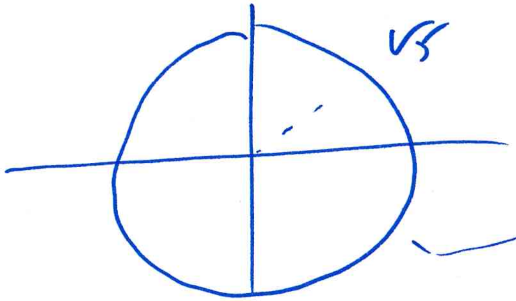
$(U \Sigma \underbrace{V_1^T) \cdot V_2}_{0} = 0$

(4)

$$e^{x} + 2y = 7 \quad \to \quad y = \frac{1}{2}\left(7 - e^{x}\right)$$ eliminate

$$x^2 + y^2 = 5 \quad \to \quad y = \sqrt{5 - x^2}$$ → eliminate

or $y = -\sqrt{5 - x^2}$ → eliminate



$$2^{x+y} + \frac{y^3}{2+x^4} + \frac{\tanh \cosh\left(x^3 + y^2\right)}{e^{x+y}} = 5 \quad \to \quad \begin{matrix} x = F(y) \\ y = G(x) \end{matrix}$$???

→ eliminate if possible

→ unconstrained

→ reduction in number of variables

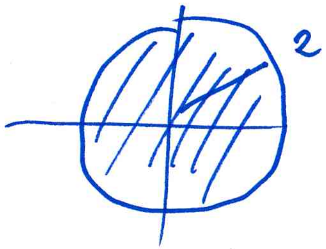
(5)

$x \geq 0$ $\qquad x = e^{u} \qquad u \in \mathbb{R}$

$\qquad\qquad x = v^{2} \qquad v \in \mathbb{R}$

inequality constrained → eliminate if possible

$x^2 + y^2 \leq 4$



polar coordinate

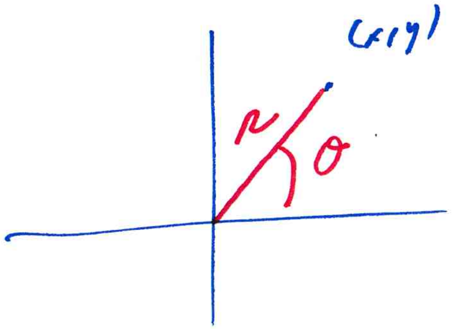
$x = r \cdot \cos\theta$

$y = r \cdot \sin\theta$

$r = 2 \cdot \cos\varphi$

$0 \leq r \leq 2$

$-\pi \leq \theta \leq \pi$



$x = 2\cos\varphi \cos\theta$

$y = 2\cos\varphi \sin\theta$

$0 \leq \varphi \leq \pi/2$

$-\pi \leq \theta \leq \pi$

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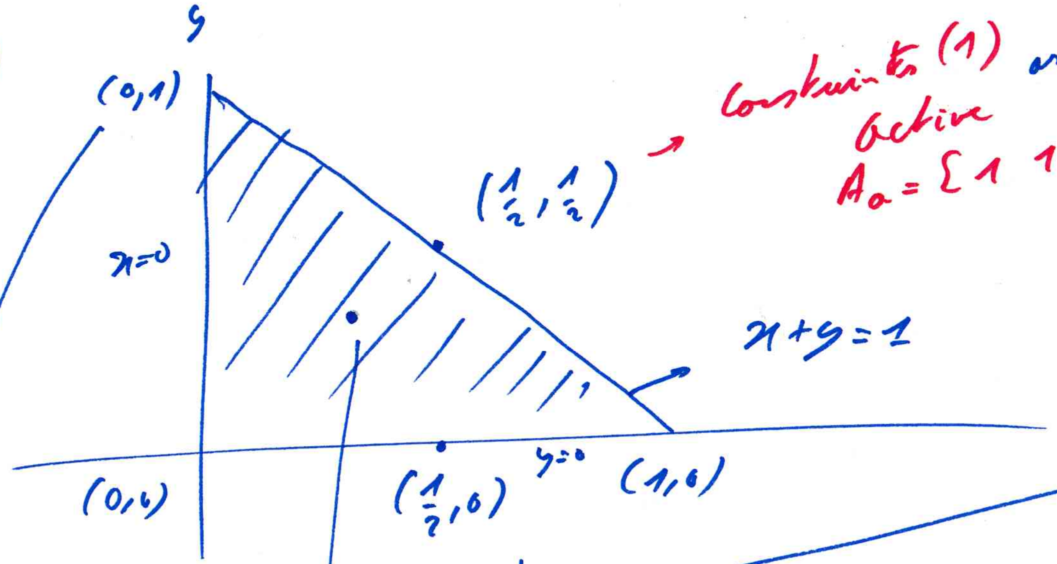


→ constraints (1) active as $x+y=1$

$A_a = [1 \;\; 1]$ $\quad b_a = 1$

$x + y \leq 1 \quad (1)$

$x \geq 0 \quad (2)$

$y \geq 0 \quad (3)$

$x+y<1$

$x>0$

$y=0$

Constraint (3) active

$\left(\frac{1}{3}, \frac{1}{3}\right)$ → no constraints active ⇒ $A_a = [\;]$

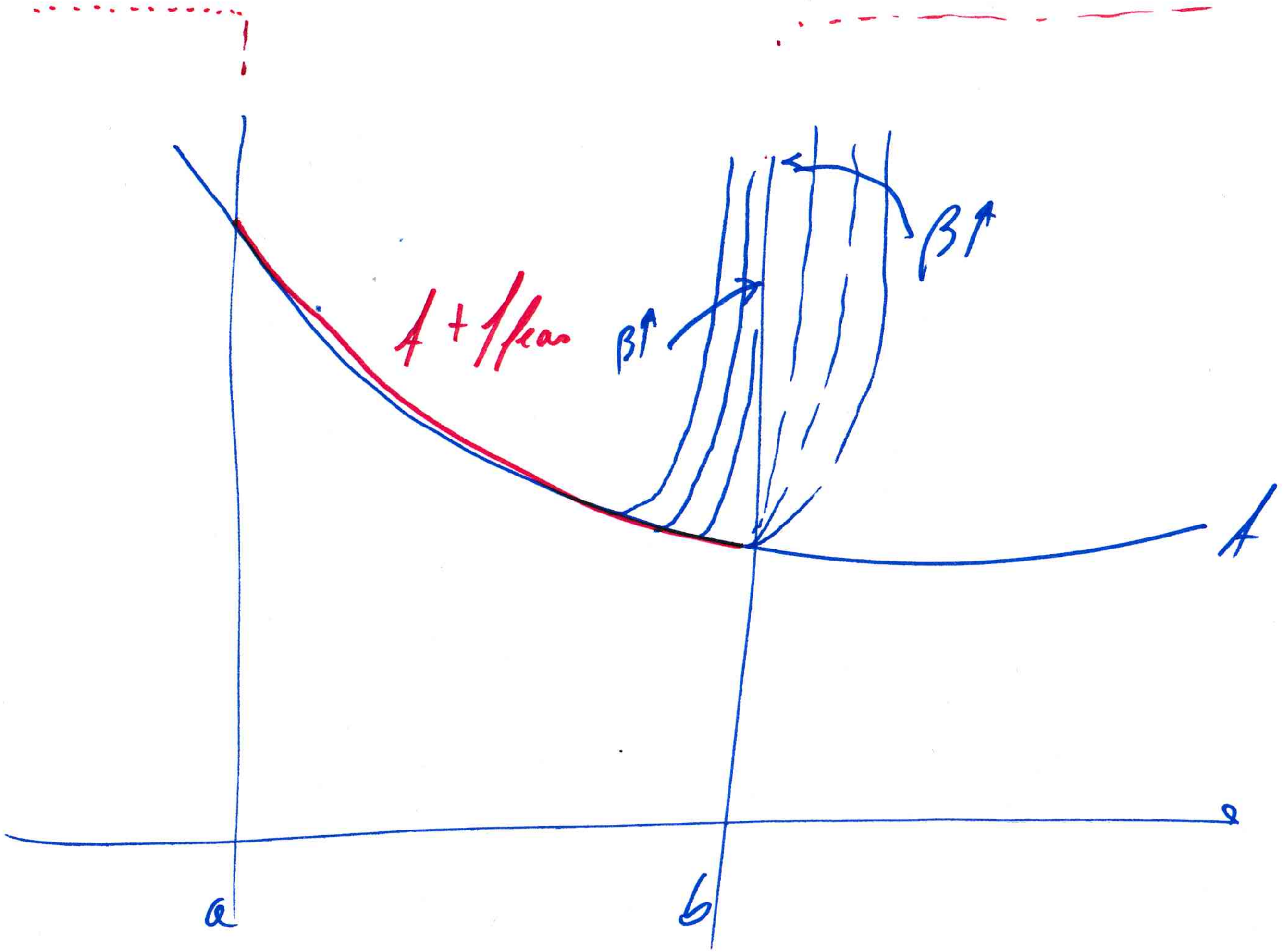
$$A x \leq b$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(1) & (2) active

$A_a = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \quad b_a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\left|\begin{array}{l} x+y=1 \\ x \geq 0 \\ y \geq 0 \end{array}\right.$$



$$\min_x f(x)$$
$$x \in [a,b]$$

$$1_{feas}(x) = 0 \quad x \in [a,b]$$
$$= +\infty \quad x \notin [a,b]$$

$$\min_x f(x) + 1_{feas}(x)$$