① Convex

2024-10-10

f, g convex

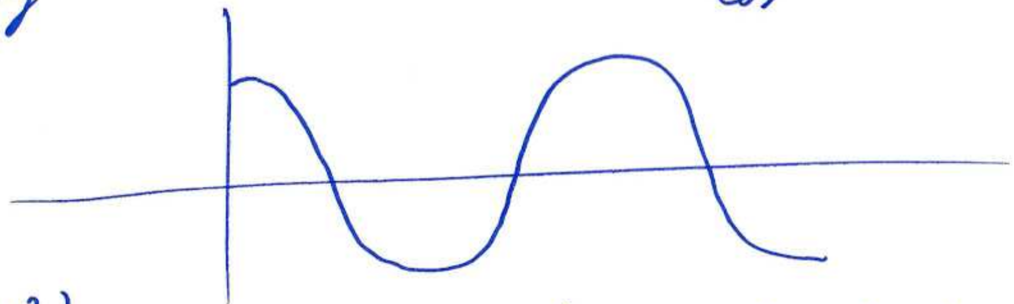
$$\min_x f(x)$$
$$\text{s.t. } g(x) \leq 0$$
$$h(x) = 0 \rightarrow \text{conditions on } h\,?$$

$h(x) \leq 0 \rightarrow h, -h$ both need to be convex

$h(x) \geq 0 \rightarrow -h(x) \leq 0 \rightarrow h$ affine

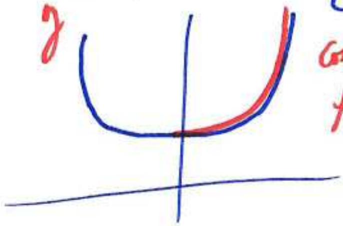
- $v = 0.5x + y$

Cos



not convex

- $\underbrace{\cosh}_{h}(\underbrace{x^2+y^2}_{g})$



cosh

cosh ↑ for positive arg.

$\left.\begin{array}{l} x^2 \text{ convex} \\ y^2 \text{ "} \end{array}\right\} x^2+y^2$ is convex $\downarrow \geq 0$

h, g convex

$h \uparrow$ (for positive arguments)

$\Downarrow$

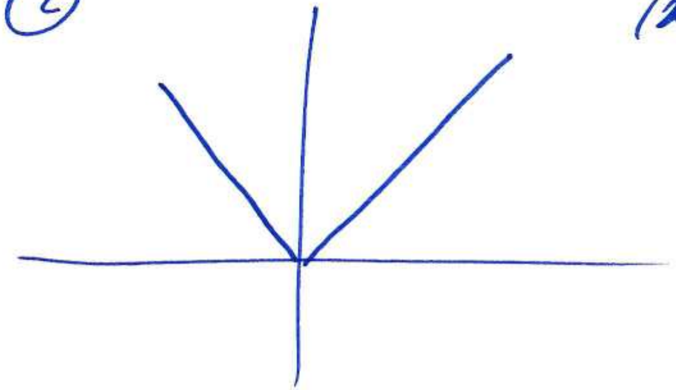
$\cosh(x^2+y^2)$ convex

(2)

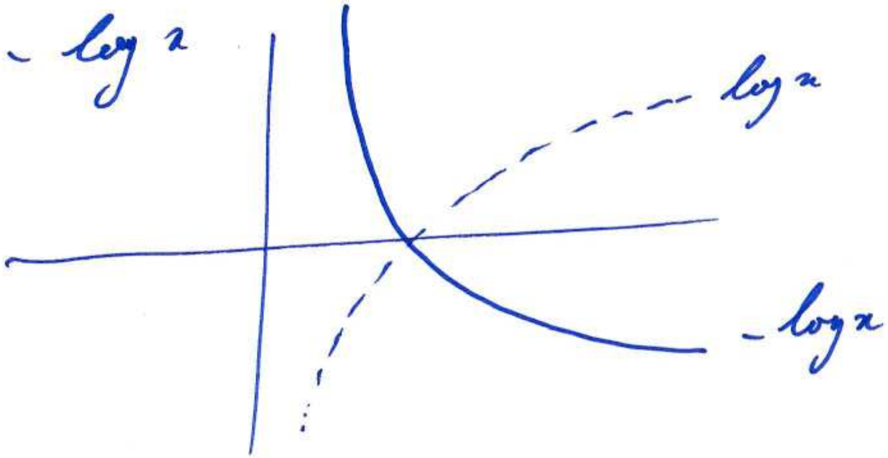
$|x| \to$ convex

$|\underbrace{x+y-5}_{\text{affine}}|$

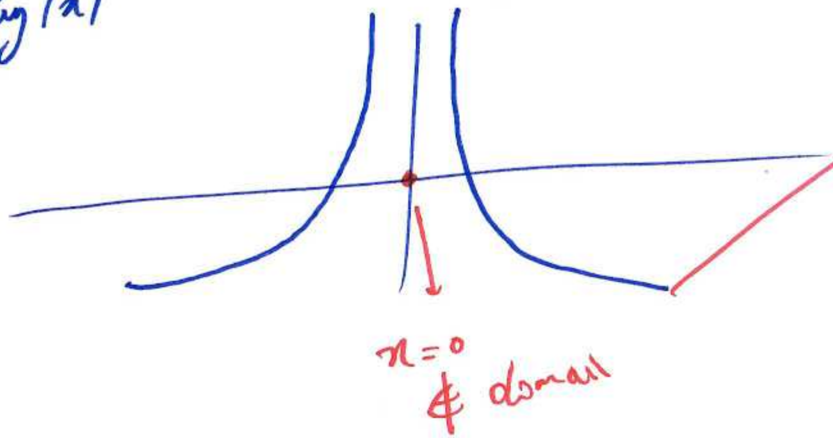
convex function with affine argument $\to$ convex



$-\log x$



$-\log|x|$

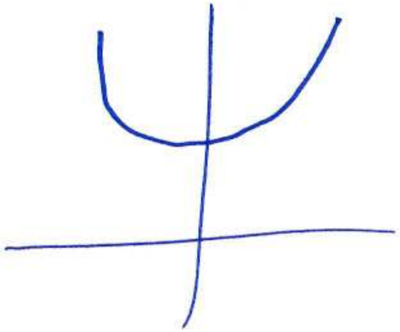


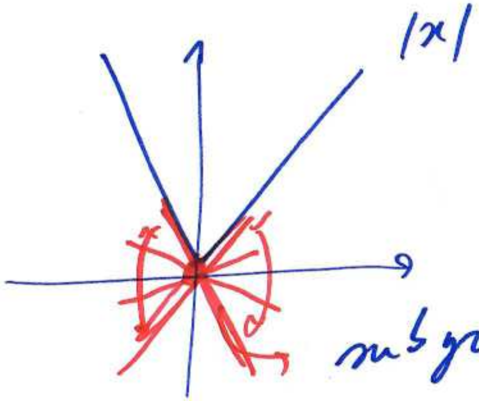
$-\log|x|$ is not convex

$-\log|x+y-5|$ is not convex

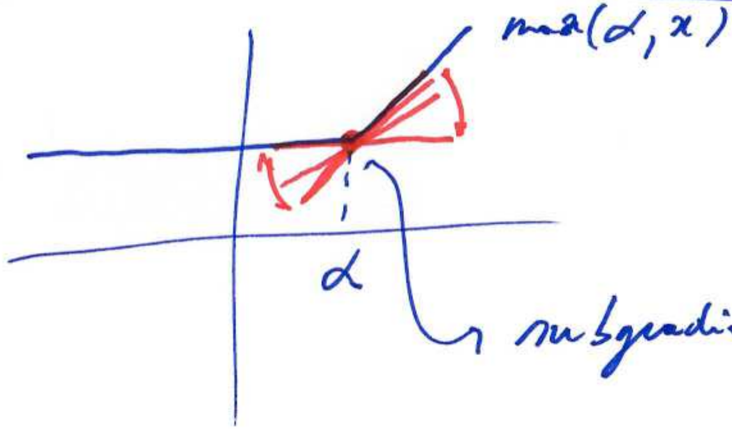
(3) check if H is always positive definite

f scalar is convex if $f'' > 0$





subgradient $\in [-1,1]$



subgradient $\in [0,1]$

$$\boxed{f(x) \geq f(x_0) + \nabla^T f(x_0)(x - x_0)}$$

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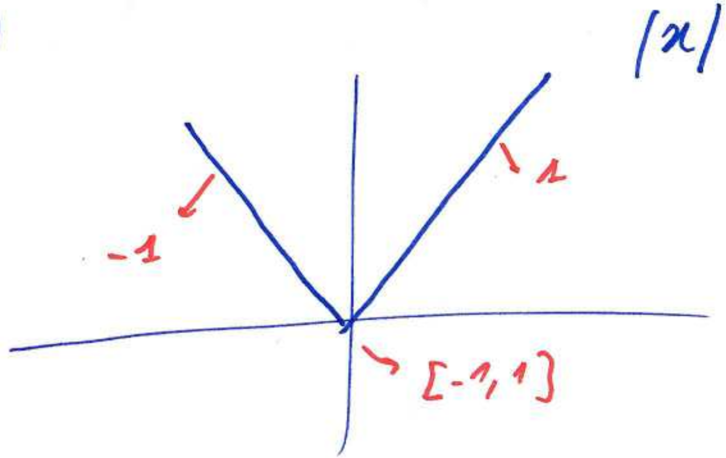

$$(|x|)' = \text{sign}(x) \begin{cases} = 1 & \text{if } x > 0 \\ = -1 & \text{if } x < 0 \\ \in [-1, 1] & \text{if } x = 0 \end{cases}$$

$$\left.\begin{array}{l} \cancel{\text{slide } 13} \\ \cancel{\text{slide } 15,\ 16,\ 17} \end{array}\right\} \text{not considered in } 2024 - 2025$$

(5)

$P = \sum p_{ij}$

P is symmetric

$P > 0$

$$P = \sum_{i \le j} p_{ij} E_{ij}$$

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$=$

$$p_{11} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{11}} + p_{12} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{E_{12}} + p_{22} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{22}}$$

$$\underbrace{F_1 \cdot x_1 + F_2 x_2 + F_3 x_3 > 0}_{LMI}$$

$$x = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix}$$

$A^T P + P A < 0$

$-(A^T P + P A) > 0$

$$\begin{array}{cc} & i \\ E_{ii} \; i & \begin{bmatrix} 0 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix} \end{array}$$

$-(A^T \sum p_{ij} E_{ij} + \sum p_{ij} E_{ij} A) > 0$

$$\begin{array}{cc} & \begin{array}{cc} i & j \end{array} \\ E_{ij} & \begin{bmatrix} 0 & & & 0 \\ & 0 & 1 & \\ & 1 & 0 & \\ 0 & & & 0 \end{bmatrix} \begin{array}{c} \leftarrow i \\ \leftarrow j \end{array} \end{array}$$

$$\underbrace{\sum p_{ij} \underbrace{(-A^T E_{ij} - E_{ij} A)}_{F_{ij} = F_{ij}^T} > 0}_{LMI}$$

(6) A is positive definite $\qquad x^T A x > 0 \quad \forall x \neq 0$

$A > 0$

$\Rightarrow$

- $A^T > 0$
- $-A < 0$
- $A, B > 0 \rightarrow A + B > 0$
- X is invertible

  $X^T A X > 0$

$$y^T (X^T A X) y > 0 \quad \forall y \neq 0$$

$$\underbrace{(y^T X^T)}_{x^T} A \underbrace{Xy}_{x} > 0$$

$$x^T A x > 0 \quad \forall x \neq 0$$

X is invertible $\quad y \neq 0 \rightarrow \underbrace{Xy}_{x} \neq 0$

$y \rightarrow x = Xy$

$x = X^{-1} y \leftarrow$

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$$\underbrace{P^{-1}}_{X^T}\left((A^T+K^TB^T)P+P(A+BK)\right)\underbrace{P^{-1}}_{X}<0$$

X^T
$=$
$(P^{-1})^T$
$=$
P^{-1} since $P=P^T$

$Y^T=(KP^{-1})^T=P^{-T}K^T$
$=P^{-1}K^T$

$$P^{-1}A^T+\underbrace{P^{-1}K^TB^T}_{=Y^T}+AP^{-1}+B\underbrace{KP^{-1}}_{Y}<0$$

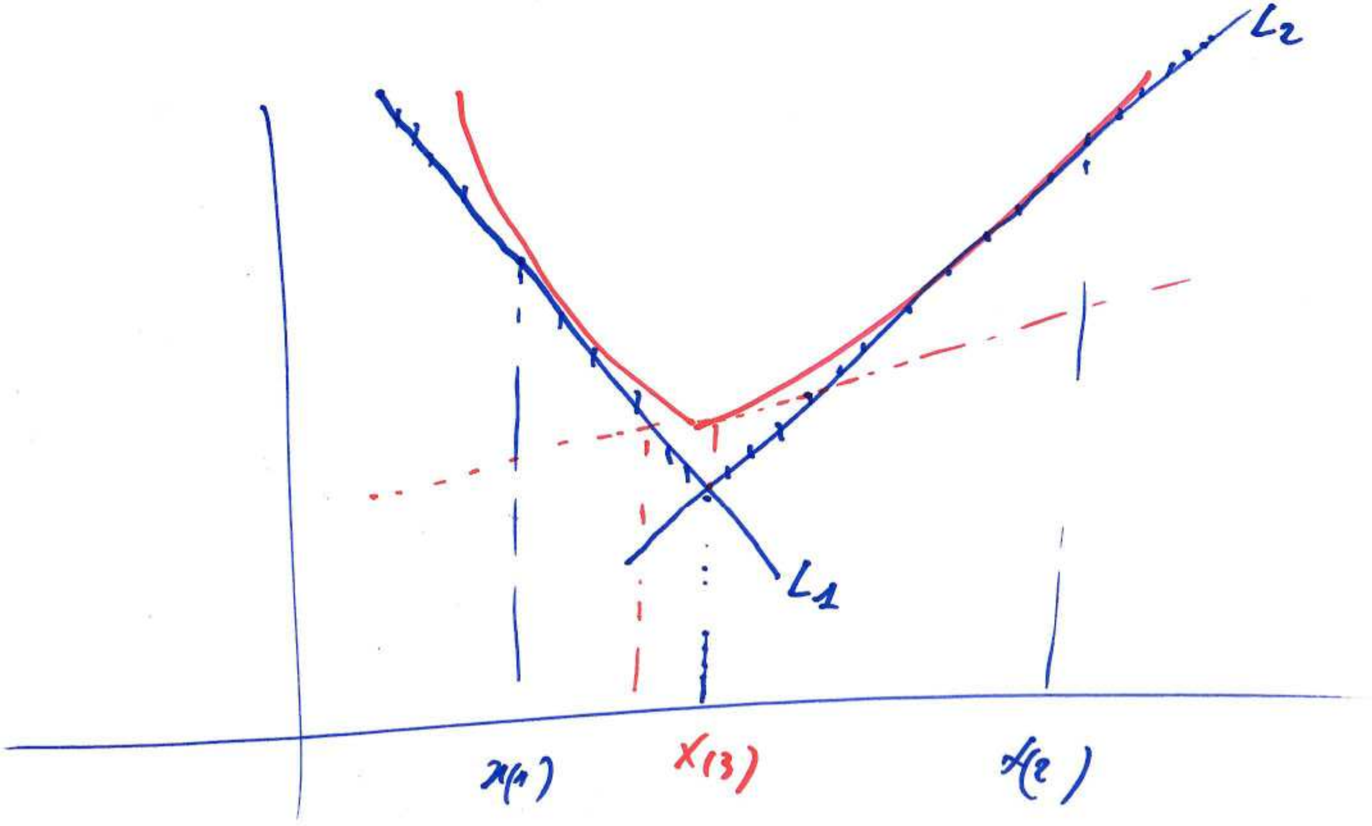
$$XA^T+Y^TB^T+AX+BY<0$$

$\rightarrow$ LMI in X, Y

$\rightarrow$ solve

$P^{-1}=X \rightarrow \boxed{P=X^{-1}}$

$Y=KP^{-1} \rightarrow \boxed{K=YP}$



$$f(x) \geq L_1(x)$$
$$\geq L_2(x)$$

$$f(x) \geq \max(L_1, L_2)$$

V shaped function

$$f(x^*) \geq \min_x \max(L_1, L_2)$$

$$f(x^*) \geq \min_x \max(L_1, L_2, L_3)$$

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- $\min_x \max(\underbrace{L_1(x)}, L_2(x), \dots L_n(x))$

$\qquad \underbrace{L_1 = f(x_{(1)}) + \nabla^T f(x_{(1)})(x - x_1)}_{a_1^T x + b_1}$

- $\min_{t,x} t \qquad t \geq \max(L_1, L_2, \dots L_n)$

easy to check: $t^* = \max(L_1(x^*), L_2(x^*), \dots L_n(x^*))$

- $\min_{t,x} t$ → linear

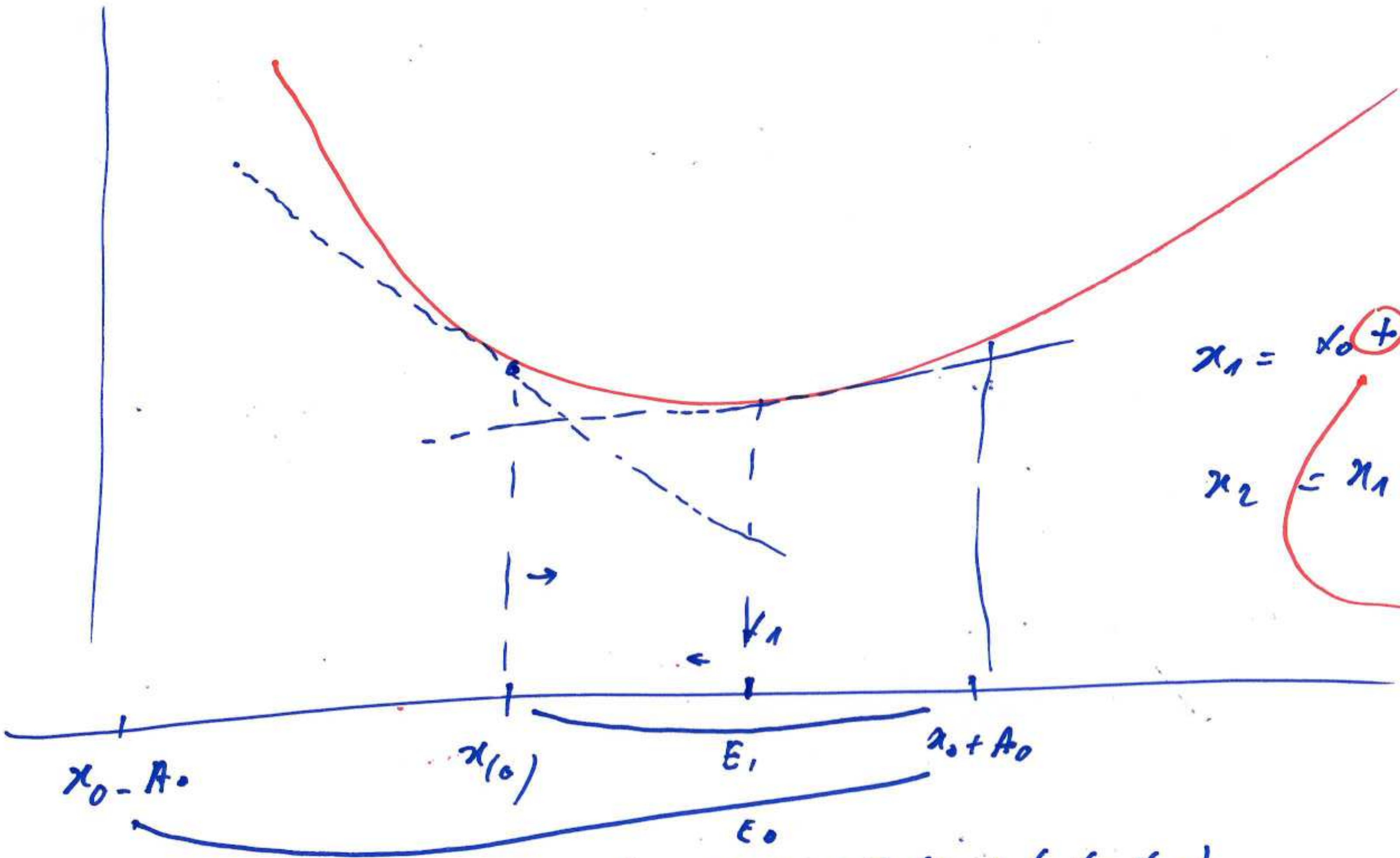
$t \geq L_1 = \underline{a_1^T x + b_1}$ → linear

$t \geq L_2 = a_2^T x + b_2$

$\vdots \qquad \vdots$

$t \geq L_n = a_n^T x + b_n$

= LP!



$x_1 = x_0 + \frac{A_0}{2}$

$x_2 = x_1 - \frac{A_1}{2}$

$f(x) \geq f(x_0) + \nabla^T f(x_0)(x - x_0)$

move in direction where $\nabla^T f(x_0)(x - x_0) < 0$

$-\text{sign}(\nabla^T f(x_0))$

(11)

$\phi = -\sum \log(-g_i)$

$\sum \underbrace{-\log}_{h} (\underbrace{-g_i}_{g})$

g_i is convex

$-g = g_i$ is concave

$-\log \rightarrow$ convex, $\downarrow$

$-\log(-g_i)$ is convex

$\sum$ preserves convexity | $\phi \rightarrow$ convex

$t f + \phi$ is convex

~~slide 29~~, slide 30 only last formula

~~slide 33 - 35~~